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Impossibility of bosonic autonomous entanglement engines in the weak-coupling limit

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Entanglement is a fundamental feature of quantum physics and a key resource for quantum communication, computing, and sensing. Entangled states are fragile and maintaining coherence is a central challenge in quantum information processing. Nevertheless, entanglement can be generated and stabilized through dissipative processes. In fact, entanglement has been shown to exist in the steady state of certain interacting quantum systems subject solely to incoherent coupling to thermal baths. This has been demonstrated in a range of bi- and multipartite settings using systems of finite dimension. Here we focus on the steady state of infinite-dimensional bosonic systems. Specifically, we consider any set of bosonic modes undergoing excitation-number-preserving interactions of arbitrary strength and divided between an arbitrary number of parties that each couple weakly to thermal baths at different temperatures. We show that a unique steady state is always separable.

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I. INTRODUCTION

Entanglement is a central feature of quantum physics and a key resource for many quantum-information-processing tasks [1], such as quantum communication, computing, and sensing [2–5]. The generation and stabilization of entangled states is a challenging task. As entanglement is generally fragile with respect to loss and noise [1], and considerable effort is usually required to isolate the system of interest from the environment. Nevertheless, it has been shown that coupling to an environment can be used to aid entanglement generation [6–13]. Steady-state entanglement can also be generated by dissipation and external driving [14–17]. In fact, dissipative steady-state entanglement generation is even possible without any source of coherence or external control, a setting referred to as autonomous. This has been demonstrated in a number of different contexts for finite-dimensional systems [18–29].

Much less is known for infinite-dimensional systems, although they are widely employed in practice in experiments with quantum optical, optomechanical, and superconducting setups. In particular, the class of so-called Gaussian states and operations in bosonic systems are ubiquitous as they are relatively simple to realize and their theory is well understood [30]. Gaussian processes, e.g., squeezing and linear interferometers, are used extensively in continuous-variable quantum information processing and quantum optics in general. However, it turns out that non-Gaussian resources are necessary in order to perform various important quantum-information-processing tasks including, for example, entanglement distillation [31–33], quantum error correction [34], and universal quantum computation [35,36]. It is therefore of interest to determine the combinations of Gaussian and non-Gaussian resources that can generate steady-state entanglement. In the present work, we demonstrate that autonomous, steady-state entanglement in bosonic systems with passive Gaussian interactions is not possible.

We consider a multimode bosonic system undergoing quadratic, excitation-preserving interactions (i.e., passive Gaussian interactions or linear interferometers) of arbitrary strength, divided between an arbitrary number of parties. The modes of different parties couple weakly to thermal baths at different temperatures, with the energy distribution in each bath following either Bose-Einstein or Fermi-Dirac statistics (both bosonic and spin baths are allowed). The system-bath interactions are also quadratic but not necessarily excitation number preserving. We show that in such a setting, no entanglement can ever be present in the steady state between the parties of the system.

The paper is organized as follows. In Sec. II we introduce the model of our thermal machines. We then present a separability criterion in Sec. III that will be satisfied by the steady states. The master equations that describe the dynamics of our machines are derived in Sec. IV. We focus on two situations: a global approach, which is appropriate when the interactions between subsystems are strong relative to the system-bath couplings; and a local approximation, which describes a situation where the interactions between subsystems are weak relative to the system-bath couplings. We then show that the steady state is separable in both situations. Finally, we conclude in Sec. V.

II. BOSONIC AUTONOMOUS THERMAL MACHINE

We consider a collection of bosonic modes undergoing excitation-preserving interactions of arbitrary strength and distributed between an arbitrary number of parties that each couple weakly to thermal baths at different temperatures (Fig. 1). The total system S consists of d bosonic modes with associated creation and annihilation operators that satisfy the canonical commutation relations $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$, $[\hat{a}_j, \hat{a}_k] = 0$, and $[\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0$. The corresponding quadrature operators for each mode are defined as

$$\hat{q}_j = \frac{1}{\sqrt{2}}(\hat{a}_j^\dagger + \hat{a}_j), \quad \hat{p}_j = \frac{i}{\sqrt{2}}(\hat{a}_j^\dagger - \hat{a}_j), \tag{1}$$

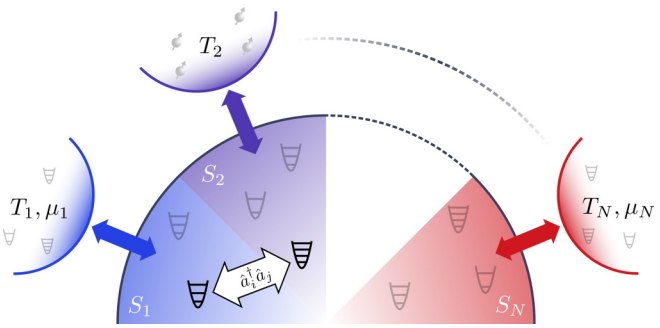

FIG. 1. Sketch of the quantum thermal machine. The system consists of d bosonic modes distributed between N subsystems. The system modes are coupled via excitation-number-preserving interactions. The subsystems are separately coupled to either bosonic or spin baths. Bosonic baths exchange both energy and particles with the system while the spin baths exchange only energy. The system-bath coupling is taken to be weak, while the intersystem couplings can be of arbitrary strength.

and fulfill the commutation relations $[\hat{q}_j, \hat{p}_k] = i\delta_{jk}$, where we use units with $\hbar = 1$ throughout.

The Hamiltonian of the system $\hat{H}_S$ is taken to be a general stable ($\hat{H}_S > 0$), quadratic, and excitation-number-preserving Hamiltonian

$$\hat{H}_S = \sum_{i,j} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j, \tag{2}$$

where $\mathcal{H}_{ij}$ is a Hermitian matrix. There are N subsystems and baths, with the nth subsystem denoted by S_n and coupling to the nth bath E_n. The total environment is described by the continuous noninteracting Hamiltonian

$$\hat{H}_E = \sum_{n=1}^{N} \hat{H}_{E_n} = \sum_{n=1}^{N} \int \epsilon_n(q)\, \hat{c}_n^\dagger(q) \hat{c}_n(q)\, dq, \tag{3}$$

and we assume that the bath spectra are non-negative with $\epsilon_n(q) \geqslant 0$. In general, each bath may be bosonic or consist of spins. For the bosonic baths, the bath creation and annihilation operators satisfy the canonical commutation relations

$$[\hat{c}_n(k), \hat{c}_m^\dagger(q)] = \delta_{nm}\delta(k-q), \tag{4}$$

$$[\hat{c}_n(k), \hat{c}_m(q)] = [\hat{c}_n^\dagger(k), \hat{c}_m^\dagger(q)] = 0, \tag{5}$$

while spins in different spin baths commute and spins within the same bath obey the anticommutation relations

$$\{\hat{c}_n(k), \hat{c}_n^\dagger(q)\} = \delta(k-q), \tag{6}$$

$$\{\hat{c}_n(k), \hat{c}_n(q)\} = \{\hat{c}_n^\dagger(k), \hat{c}_n^\dagger(q)\} = 0. \tag{7}$$

Finally, the system-bath interaction is a quadratic Hamiltonian of the form $\hat{H}_{SE} = \sum_{n=1}^{N} \hat{V}_n$, with

$$\hat{V}_n = \sum_{j \in S_n} \int g_j(q)(\hat{a}_j^\dagger + \eta_j \hat{a}_j)\hat{c}_n(q)\, dq + \text{H.c.} \tag{8}$$

Here the j sum is taken over the modes of subsystem S_n, the complex coefficients $g_j(q)$ quantify the strength of each interaction, and the parameters $\eta_j \in \{0, 1\}$ allow for both excitation-number-conserving ($\eta_j = 0$) and "position-like" ($\eta_j = 1$) coupling to the environment.

The bosonic baths exchange energy and particles with the system, and are thus characterized by an inverse temperature $\beta_n = 1/k_B T_n$ and chemical potential μ_n. On the other hand, the spin baths only exchange energy with the system. We assume that the environment is initially in the thermal state

$$\hat{\rho}_E = \bigotimes_{n=1}^{N} \frac{e^{-(\hat{H}_{E_n} - \mu_n \hat{N}_n)\beta_n}}{\text{tr}[e^{-(\hat{H}_{E_n} - \mu_n \hat{N}_n)\beta_n}]}, \tag{9}$$

where $\hat{N}_n = \int \hat{c}_n^\dagger(q)\hat{c}_n(q)\, dq$ is the number operator of the nth bath and the chemical potential should only be included for bosonic baths. In this case, the two-point correlation functions satisfy

$$\langle \hat{c}_n^\dagger(k)\hat{c}_m(q)\rangle = p_n(\epsilon_n(k))\, \delta_{nm}\, \delta(k-q), \tag{10}$$

$$\langle \hat{c}_n(k)\hat{c}_m(q)\rangle = 0, \quad \langle \hat{c}_n^\dagger(k)\hat{c}_m^\dagger(q)\rangle = 0, \tag{11}$$

where $\langle\cdot\rangle$ is the expectation value in the state $\hat{\rho}_E$ and

$$p_n(\epsilon) = [\xi_n + e^{(\epsilon - \mu_n)\beta_n}]^{-1} \tag{12}$$

is the Fermi-Dirac distribution for spin baths ($\xi_n = 1$) or the Bose-Einstein distribution for bosonic baths ($\xi_n = -1$).

III. A CRITERION FOR SEPARABILITY

Our goal is to derive the steady state of the system in the model above. Before proceeding, however, we first consider a criterion for separability. As the steady state is found to be Gaussian (see Sec. IV C), separability can be determined from the covariance matrix. By definition, a Gaussian state $\hat{\rho}$ has a Wigner function of the form

$$W(x) = \frac{1}{(2\pi)^d} \frac{1}{\sqrt{\det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1}(x-\bar{x})}. \tag{13}$$

Here $x = (q_1, p_1, \ldots, q_d, p_d) \in \mathbb{R}^{2d}$ are canonical phase-space coordinates, $\bar{x}$ is a vector of first moments $\bar{x}_j = \text{tr}[\hat{x}_j \hat{\rho}]$ with $\hat{x} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_d, \hat{p}_d)$, and the covariance matrix Σ is a real, symmetric, and positive-definite matrix with elements

$$\Sigma_{jk} = \frac{1}{2}\langle \hat{x}_j \hat{x}_k + \hat{x}_j \hat{x}_k\rangle - \langle \hat{x}_j\rangle\langle \hat{x}_k\rangle, \tag{14}$$

where $\langle\cdot\rangle = \text{tr}[\cdot\hat{\rho}]$. The covariance matrix of a valid quantum state satisfies the Robertson-Schrödinger uncertainty relation

$$\Sigma + \frac{i}{2}\Omega \geqslant 0, \tag{15}$$

where Ω is the symplectic matrix

$$\Omega = \bigoplus_{j=1}^{d} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{16}$$

As shown by Werner and Wolf [37], a bipartite Gaussian state of systems A and B with covariance matrix Σ_{AB} is separable if and only if there exist valid covariance matrices Σ_A and Σ_B for systems A and B such that $\Sigma_{AB} \geqslant \Sigma_A \oplus \Sigma_B$. We would like to apply this result to our steady states. However, it turns out that the steady-state covariance matrices derived below in Sec. IV are more conveniently expressed in coordinates

different to $(q_1, p_1, \ldots, q_d, p_d)$. Specifically, in the complex coordinates $(q_1 + ip_1, \ldots, q_d + ip_d, q_1 - ip_1, \ldots, q_d - ip_d)/\sqrt{2}$ they have the simple block structure

$$\Sigma' = \begin{pmatrix} \sigma^* & 0 \\ 0 & \sigma \end{pmatrix}, \tag{17}$$

where $\sigma_{jk} = \frac{1}{2}\langle \hat{a}_j^\dagger \hat{a}_k + \hat{a}_k \hat{a}_j^\dagger \rangle - \langle \hat{a}_j^\dagger \rangle \langle \hat{a}_k \rangle$ and the prime denotes the complex form. The transformation to complex coordinates is achieved by first applying the permutation P that maps $(q_1, p_1, \ldots, q_d, p_d)$ to $(q_1, \ldots, q_d, p_1, \ldots, p_d)$, and then the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_d & iI_d \\ I_d & -iI_d \end{pmatrix}, \tag{18}$$

where I_d is the $d \times d$ identity matrix. Thus, the similarity transformation UP yields $\Sigma' = UP\Sigma P^\dagger U^\dagger$.

From the steady-state covariance matrix (17), we observe that $\langle \hat{a}_j \hat{a}_k \rangle - \langle \hat{a}_j \rangle \langle \hat{a}_k \rangle = 0$, and so the corresponding Gaussian steady state is not squeezed. Separability of the steady state then follows from the fact that every entangled Gaussian state is squeezed, see, e.g., Ref. [38]. For completeness, we provide a proof that any Σ' of the form (17) corresponds to a Σ that satisfies the separability condition across any bipartition. In complex coordinates, the physicality condition (15) becomes $\Sigma' + \frac{1}{2}K \geqslant 0$, where

$$K = \begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix}, \tag{19}$$

as may be seen by applying the transformation UP. Since Σ' in (17) comes from a valid steady state, it follows that $\sigma - \frac{1}{2}I_d \geqslant 0$. Taking the complex conjugate, we also have $\sigma^* - \frac{1}{2}I_d \geqslant 0$. These two results then imply that $\Sigma' \geqslant \frac{1}{2}I_{2d}$. Transforming back to the real coordinates we obtain $\Sigma \geqslant \frac{1}{2}I_{2d}$, as the eigenvalues are preserved under the transformation. For any bipartition of the total system we have that $I_{2d} = I_A \oplus I_B$, with I_A and I_B identity matrices corresponding to subsystems A and B, and $\frac{1}{2}I$ is always a valid covariance matrix (the covariance matrix of the vacuum state). Hence, a Gaussian state with covariance matrix of the form (17) is separable across every bipartition.

IV. OPEN-SYSTEM DYNAMICS AND STEADY STATES

In this section we derive master equations for the open-system dynamics of the setup in Fig. 1. It is shown that the steady state is a Gaussian state with a covariance matrix of the form (17), and is therefore separable by the argument of Sec. III. We consider weak system-bath couplings and large baths, and assume that the initial composite state of the system and baths factorizes. In this case, one can apply a perturbative approach in the system-bath coupling to derive a master equation for the time evolution of the system. A time-local (Markovian) description of the system evolution, which does not depend on the entire history of the system state, is obtained provided that correlations in the baths decay much faster than variations of the state of the system $\tilde{\rho}(t)$ in the interaction picture (denoted by a tilde).

Following the standard procedure, one finds (see, e.g., Refs. [39,40] for an in-depth derivation)

$$\frac{d}{dt}\tilde{\rho}(t) = -\int_0^\infty du\, \mathrm{tr}_E([\hat{H}_{SE}(t), [\hat{H}_{SE}(t-u), \tilde{\rho}(t) \otimes \hat{\rho}_E]]). \tag{20}$$

Substituting in the expressions for $\hat{H}_{SE}$ and $\hat{\rho}_E$ leads to

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}(t) = \sum_{n=1}^{N} \sum_{j,k \in S_n} \int_0^\infty du\, (&\mathrm{tr}_E[\hat{B}_{jn}^\dagger(u)\hat{B}_{kn}\hat{\rho}_E] \\ &\times (\hat{A}_j^\dagger(t-u)\tilde{\rho}(t)\hat{A}_k(t) - \hat{A}_k(t)\hat{A}_j^\dagger(t-u)\tilde{\rho}(t)) \\ &+ \mathrm{tr}_E[\hat{B}_{jn}(u)\hat{B}_{kn}^\dagger\hat{\rho}_E](\hat{A}_j(t-u)\tilde{\rho}(t)\hat{A}_k^\dagger(t) \\ &- \hat{A}_k^\dagger(t)\hat{A}_j(t-u)\tilde{\rho}(t)) + \mathrm{H.c.}), \end{aligned} \tag{21}$$

where we have introduced the system operators $\hat{A}_j(t) = \hat{a}_j(t) + \eta_j \hat{a}_j^\dagger(t)$ to simplify the equation, and in the interaction picture the bath operators $\hat{B}_{jn} = \int g_j^*(q)\hat{c}_n(q)\, dq$ have the form

$$\hat{B}_{jn}(u) = \int g_j^*(q)\hat{c}_n(q)\mathrm{e}^{-i\epsilon_n(q)u}\, dq. \tag{22}$$

It is also convenient to define the one-sided Fourier transforms of the bath correlation functions,

$$C_{jk}^{(n,1)}(\omega) = \int_0^\infty \mathrm{tr}_E[\hat{B}_{jn}^\dagger(u)\hat{B}_{kn}\hat{\rho}_E]\mathrm{e}^{-i\omega u}du, \tag{23}$$

$$C_{jk}^{(n,2)}(\omega) = \int_0^\infty \mathrm{tr}_E[\hat{B}_{jn}(u)\hat{B}_{kn}^\dagger\hat{\rho}_E]\mathrm{e}^{i\omega u}du, \tag{24}$$

and split them into their Hermitian and anti-Hermitian parts

$$C_{jk}^{(n,l)}(\omega) = \tfrac{1}{2}\gamma_{jk}^{(n,l)}(\omega) + is_{jk}^{(n,l)}(\omega). \tag{25}$$

The following expressions will also be useful when we consider separability of the steady state,

$$\mathrm{tr}_E[\hat{B}_{jn}^\dagger(u)\hat{B}_{kn}\hat{\rho}_E] = \int g_j(q)g_k^*(q)\mathrm{e}^{i\epsilon_n(q)u}p_n(\epsilon_n(q))\, dq, \tag{26}$$

$$\mathrm{tr}_E[\hat{B}_{jn}(u)\hat{B}_{kn}^\dagger\hat{\rho}_E] = \int g_j^*(q)g_k(q)\mathrm{e}^{-i\epsilon_n(q)u} \times [1 - \xi_n p_n(\epsilon_n(q))]\, dq. \tag{27}$$

In order to make further progress with the evolution equation (21), we need to deal with the u dependence appearing in the interaction-picture creation and annihilation operators. We consider two situations separately and derive a master equation for each case. First we take the global approach, in which the baths interact with the eigenmodes of the total system Hamiltonian. This is appropriate when the interaction between subsystems is strong relative to the system-bath interactions and results in a so-called global master equation [41]. Second, we make a local approximation, where each bath interacts locally with the eigenmodes of the subsystem to which it couples. This is appropriate when the interaction between subsystems is weaker than the system-bath couplings (or comparable, see Ref. [41]), and results in a local master equation. Note that the validity regimes of the local and global

master equations have some overlap [41], and hence treating both enables us to apply our separability result in the entire range from very weak to strong intersystem interactions. In both cases we apply the full secular approximation to obtain a physical (completely positive trace-preserving) master equation. The condition for this approximation to be valid differs between the local and global approaches, and therefore the details are provided separately below.

A. Global master equation

Let us first derive a global master equation. To this end, we follow Ref. [42] and make use of the Bogoliubov transformation. The system Hamiltonian (2) can be written in the form $\hat{H}_S = \hat{\mathbf{a}}^\dagger \mathcal{H} \hat{\mathbf{a}}$, where $\hat{\mathbf{a}}^\dagger = (\hat{a}_1^\dagger, \ldots, \hat{a}_d^\dagger)$ and $\mathcal{H}$ is a Hermitian matrix that can be diagonalized with a unitary matrix U. The transformation $\hat{\mathbf{a}} = U\hat{\mathbf{b}}$ then brings the system Hamiltonian into the form

$$\hat{H}_S = \sum_j \omega_j \hat{b}_j^\dagger \hat{b}_j, \tag{28}$$

where $\omega_j > 0$ are the eigenvalues of $\mathcal{H}$, and the new creation and annihilation operators $\hat{b}_j, \hat{b}_j^\dagger$ satisfy the usual bosonic canonical commutation relations.

Starting from (21) and applying a full-secular approximation yields a Markovian master equation for the system state $\hat{\rho}(t)$ (in the Schrödinger picture) that is completely positive and trace preserving,

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}_S + \hat{H}_{LS}, \hat{\rho}(t)] + \sum_{\lambda=1}^{D} \sum_{u,v\in\Lambda_\lambda} \times \left[\Gamma_{uv}^{(1)}(\omega_\lambda)\mathcal{D}(\hat{b}_u^\dagger, \hat{b}_v^\dagger) + \Gamma_{uv}^{(2)}(\omega_\lambda)\mathcal{D}(\hat{b}_u, \hat{b}_v)\right][\hat{\rho}(t)]. \tag{29}$$

Here we have introduced the superoperator $\mathcal{D}(\hat{X}_i, \hat{X}_j)$ that, given the two operators $\hat{X}_i, \hat{X}_j$, acts on a quantum state $\hat{\rho}$ as

$$\mathcal{D}(\hat{X}_i, \hat{X}_j)[\hat{\rho}] = \hat{X}_i \hat{\rho} \hat{X}_j^\dagger - \tfrac{1}{2}\{\hat{X}_j^\dagger \hat{X}_i, \hat{\rho}\}. \tag{30}$$

The system has D different energy eigenspaces labeled by λ, where Λ_λ is the set of normal modes associated with the λth eigenspace corresponding to eigenvalue ω_λ, and

$$\Gamma_{uv}^{(1)}(\omega) = \sum_{n=1}^{N} \sum_{i,j\in S_n} \gamma_{ij}^{(n,1)}(\omega) U_{iu}^* U_{jv}, \tag{31}$$

$$\Gamma_{uv}^{(2)}(\omega) = \sum_{n=1}^{N} \sum_{i,j\in S_n} \gamma_{ij}^{(n,2)}(\omega) U_{iu} U_{jv}^*, \tag{32}$$

where the $\gamma_{ij}^{(n,k)}(\omega)$ are defined in (25). The unitary part of the dynamics is generated by the system Hamiltonian

$$\hat{H}_S = \sum_{\lambda=1}^{D} \sum_{u\in\Lambda_\lambda} \omega_\lambda \hat{b}_u^\dagger \hat{b}_u \tag{33}$$

and the so-called Lamb-shift Hamiltonian

$$\hat{H}_{LS} = \sum_{\lambda=1}^{D} \sum_{u,v\in\Lambda_\lambda} \varphi_{uv}(\omega_\lambda) \hat{b}_u^\dagger \hat{b}_v, \tag{34}$$

where

$$\varphi_{uv}(\omega) = \sum_{n=1}^{N} \sum_{i,j\in S_n} \left[\left(s_{ij}^{(n,1)}(\omega) + s_{ji}^{(n,2)}(\omega)\right) + \left(s_{ji}^{(n,1)}(-\omega) + s_{ij}^{(n,2)}(-\omega)\right)\eta_i\eta_j\right] U_{iu}^* U_{jv}, \tag{35}$$

and the $s_{ij}^{(n,k)}(\omega)$ are also defined in (25). The Lamb-shift Hamiltonian commutes with $\hat{H}_S$ and so produces a shift in the energy levels of $\hat{H}_S$.

The full-secular approximation, which was required to arrive at the master equation (29), is valid provided that the differences ν of the normal-mode eigenvalues satisfy $\min_{\nu\neq\nu'} |\nu - \nu'| > 1/\tau_B$, where τ_B is the largest correlation time of any of the baths. Furthermore, this approximation has removed all terms from the master equation that could generate squeezing, i.e., only passive terms remain.

B. Local master equation

Now suppose that the N subsystems are weakly interacting with each other. In this case the local approach may be justified, where the coupling between the subsystems is neglected when calculating the effects of the environment. We thus begin by writing the system Hamiltonian in the form $\hat{H}_S = \sum_n \hat{H}_n + \hat{H}_C$, where $\hat{H}_n$ is the local Hamiltonian for subsystem S_n and $\hat{H}_C$ accounts for the coupling between the different subsystems. Each $\hat{H}_n$ is excitation number conserving and, as before, can be put into diagonal form

$$\hat{H}_n = \sum_{j=1}^{M_n} \omega_{nj} \hat{b}_{nj}^\dagger \hat{b}_{nj} \tag{36}$$

with a Bogoliubov transformation

$$\hat{a}_i = \sum_{j=1}^{M_n} U_{ij}^n \hat{b}_{nj}. \tag{37}$$

Here M_n is the number of modes in subsystem S_n, $\omega_{nj} > 0$ are the eigenvalues of $\mathcal{H}_n$, $\hat{b}_{nj}$ are the Bogoliubov operators associated with subsystem S_n, and $i \in S_n$. The components U_{ij}^n can be viewed as the elements of a unitary matrix U^n after a relabeling of the i index.

These transformations do not diagonalize $\hat{H}_S$ and will typically result in a complicated coupling term $\hat{H}_C$. However, this does not complicate the derivation, as the coupling between the subsystems is ignored when calculating the influence of the environment. With this in mind, starting from (21) and applying a full-secular approximation yields a Markovian master equation that is again completely positive and trace preserving,

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}_S + \hat{H}_{LS}, \hat{\rho}(t)] + \sum_{n=1}^{N} \sum_{\lambda=1}^{D_n} \times \sum_{u,v\in n_\lambda} \left[\Gamma_{uv}^{(n,1)}(\omega_{n\lambda})\mathcal{D}(\hat{b}_{nu}^\dagger, \hat{b}_{nv}^\dagger) + \Gamma_{uv}^{(n,2)}(\omega_{n\lambda})\mathcal{D}(\hat{b}_{nu}, \hat{b}_{nv})\right][\hat{\rho}(t)]. \tag{38}$$

Here D_n is the number of energy eigenspaces associated with the subsystem S_n, and n_λ is the set of normal modes associated

with the λth eigenspace corresponding to the eigenvalue $\omega_{n\lambda}$. Furthermore,

$$\Gamma^{(n,1)}_{uv}(\omega) = \sum_{i,j\in S_n} \gamma^{(n,1)}_{ij}(\omega) U^{n*}_{iu} U^{n}_{jv}, \tag{39}$$

$$\Gamma^{(n,2)}_{uv}(\omega) = \sum_{i,j\in S_n} \gamma^{(n,2)}_{ij}(\omega) U^{n}_{iu} U^{n\,*}_{jv}, \tag{40}$$

where the $\gamma^{(n,k)}_{ij}(\omega)$ are defined in (25).

While the mode transformations simplify each local subsystem Hamiltonian $\hat{H}_n$, their effect on the coupling Hamiltonian $\hat{H}_C$ leads to a system Hamiltonian of the form

$$\hat{H}_S = \sum_{n=1}^{N} \sum_{\lambda=1}^{D_n} \sum_{u\in n_\lambda} \omega_{n\lambda} \hat{b}^\dagger_{nu} \hat{b}_{nu} + \hat{H}'_C. \tag{41}$$

The contribution $\hat{H}'_C$ contains terms of the form $\hat{b}^\dagger_{nu}\hat{b}_{mv}$, which involve Bogoliubov operators for different subsystems and energy eigenspaces. However, the full details of the coefficients appearing in this part are not needed in what follows. The coupling of the system to the baths produces an additional contribution to the unitary part of the dynamics

$$\hat{H}_{LS} = \sum_{n=1}^{N} \sum_{\lambda=1}^{D_n} \sum_{u,v\in n_\lambda} \varphi^{(n)}_{uv}(\omega_{n\lambda}) \hat{b}^\dagger_{nu} \hat{b}_{nv}, \tag{42}$$

where

$$\varphi^{(n)}_{uv}(\omega) = \sum_{i,j\in S_n} \Big[\big(s^{(n,1)}_{ij}(\omega) + s^{(n,2)}_{ji}(\omega)\big) + \big(s^{(n,1)}_{ji}(-\omega) + s^{(n,2)}_{ij}(-\omega)\big)\eta_i\eta_j \Big] U^{n*}_{iu} U^{n}_{jv}, \tag{43}$$

and the $s^{(n,k)}_{ij}(\omega)$ are also defined in (25). In contrast to the global approach, this term does not commute with the system Hamiltonian and therefore does not simply shift the energies of the system Hamiltonian.

In the local approach, the validity of the full-secular approximation is determined by the local Hamiltonians of each subsystem. For each subsystem, the differences ν_n of the normal-mode eigenvalues associated with the local Hamiltonian $\hat{H}_n$ must satisfy $\min_{\nu_n\neq\nu'_n} |\nu_n - \nu'_n| > 1/\tau_B$. Again, this approximation results in a master equation containing only passive terms that cannot generate squeezing.

C. The steady-state solution

We now show that whenever the steady state of the global or local master equation is unique, it is a Gaussian state with a covariance matrix of the form (17). Thus, for any initial system state the steady state will be a separable Gaussian state. We first provide a short proof that assumes the steady state is unique, and then derive a condition for the steady state to be unique. Both derivations make use of the fact that the global and local master equations take Gaussian states to Gaussian states. This is because the Hamiltonian part is quadratic in the Bogoliubov operators, and the Lindblad operators in the dissipative part are linear in the Bogoliubov operators.

For the simple proof, note that the right-hand side of both the global (29) and local (38) master equations can be written in terms of a Liouvillian $\mathcal{L}$ as $\mathcal{L}\hat{\rho}(t)$. In each case $\mathcal{L}$ commutes with the superoperator $\mathcal{N}\cdot = [\hat{N}, \cdot]$, where the total excitation-number operator is $\hat{N} = \sum_j \hat{b}^\dagger_j \hat{b}_j$ in the global approach and $\hat{N} = \sum_{n,j} \hat{b}^\dagger_{nj}\hat{b}_{nj}$ in the local approach. Therefore, any state that is initially block diagonal in the $\hat{N}$ eigenbasis, with no coherences between different number sectors, will remain block diagonal for all time. Such a state has no squeezing and thus a covariance matrix of the separable form (17). This proves our result provided that the steady state is unique, because an initial Gaussian state with covariance matrix of the form (17) must then remain a Gaussian state with a covariance matrix of this form for all time.

We now derive a condition for the steady state to be unique. Recall that any initial system state $\hat{\rho}(0)$ can be written as

$$\hat{\rho}(0) = \frac{1}{\pi^{2d}} \int \langle\alpha|\hat{\rho}(0)|\beta\rangle |\alpha\rangle\langle\beta| d^2\alpha d^2\beta, \tag{44}$$

where $|\alpha\rangle$ is a d-mode Glauber coherent state with amplitudes $\alpha = (\alpha_1, \ldots, \alpha_d)$. By the linearity of the master equations, if we obtain the long-time evolution of each term $|\alpha\rangle\langle\beta|$ in the decomposition (44) we can then deduce the form of the steady state.

Let us first deal with the coherent states $|\alpha\rangle\langle\alpha|$ appearing in (44). These are Gaussian states, each with the same covariance matrix $I_{2d}/2$ but different centers. We now show that in the long-time limit every coherent state $|\alpha\rangle\langle\alpha|$ tends to the same Gaussian state $\hat{\rho}_G$ with zero first moments and a covariance matrix of the form (17). Because the master equations map Gaussian states to Gaussian states, the initial coherent states $|\alpha\rangle$ will remain Gaussian for all time. It is then enough to work out the steady-state expectation values of the first and second moments.

We focus on the global master equation (29), as the derivation for the local master equation proceeds along the same lines. From (29) it follows that the time evolution of an operator $\hat{O}$ is determined by

$$\frac{d}{dt}\hat{O} = i[\hat{H}_S + \hat{H}_{LS}, \hat{O}] + \sum_{\lambda=1}^{D} \sum_{u,v\in\Lambda_\lambda} \Gamma^{(1)}_{uv}(\omega_\lambda)\left(\hat{b}_v \hat{O} \hat{b}^\dagger_u - \frac{1}{2}\{\hat{b}_v\hat{b}^\dagger_u, \hat{O}\}\right) + \Gamma^{(2)}_{uv}(\omega_\lambda)\left(\hat{b}^\dagger_v \hat{O} \hat{b}_u - \frac{1}{2}\{\hat{b}^\dagger_v\hat{b}_u, \hat{O}\}\right). \tag{45}$$

The equations of motion for the one- and two-point correlators $\langle\hat{b}_i\rangle$ and $\langle\hat{b}_i\hat{b}_j\rangle$ are thus obtained by setting $\hat{O}$ to $\hat{b}_i$ and $\hat{b}_i\hat{b}_j$, respectively. Now recall that every coherent state $|\alpha\rangle$ has the same covariance matrix, $I_{2d}/2$, and that the Bogoliubov operators are related to the original operators by the unitary transformation $\hat{\mathbf{a}} = U\hat{\mathbf{b}}$. From this we see that $\langle\hat{b}_i\hat{b}_j\rangle - \langle\hat{b}_i\rangle\langle\hat{b}_j\rangle = 0$ for all i, j at time $t = 0$. By plugging these initial conditions into the equations of motion for $\beta_{ij} = \langle\hat{b}_i\hat{b}_j\rangle - \langle\hat{b}_i\rangle\langle\hat{b}_j\rangle$ it is straightforward to check that $\beta_{ij} = 0$ for all time (see Appendix A). Therefore, $\langle\hat{a}_i\hat{a}_j\rangle - \langle\hat{a}_i\rangle\langle\hat{a}_j\rangle = 0$ at every time for all i, j. Because the equations of motion for the (co)variances do not couple to the first moments, the covariance matrix of every initial coherent state $|\alpha\rangle$ tends to the same steady-state covariance matrix of separable form (17).

All that remains is to show that the center of every initial coherent state goes to the same value in the long-time limit.

For this we need the equations of motion for the first moments

$$\frac{d}{dt}\langle\hat{b}_j\rangle = -i\omega_j\langle\hat{b}_j\rangle - i\sum_{u\in\Lambda_j}\varphi_{ju}(\omega_j)\langle\hat{b}_u\rangle + \frac{1}{2}\sum_{u\in\Lambda_j}\big(\Gamma^{(1)}_{ju}(\omega_j) - \Gamma^{(2)}_{uj}(\omega_j)\big)\langle\hat{b}_u\rangle. \tag{46}$$

For simplicity we have redefined ω_j as the eigenvalue associated with the mode $\hat{b}_j$ and Λ_j as the eigenspace to which the eigenvalue ω_j belongs. This set of linear equations can be written in the form $db/dt = Rb$, where $b_i = \langle\hat{b}_i\rangle$ and R is a non-Hermitian matrix. The Hermitian part of R is negative semidefinite (see Appendix B), and therefore, if R is invertible, then $b = 0$ is the only steady-state solution. If this condition is satisfied, it follows that every initial state $|\alpha\rangle$ tends to the same Gaussian state $\hat{\rho}_G$ with zero first moments and a covariance matrix of the form (17).

The long-time dynamics of the cross terms $|\alpha\rangle\langle\beta|$ $(\alpha \neq \beta)$ appearing in (44) require a little more work. The Wigner-Weyl transformation of $|\alpha\rangle\langle\beta|$ is a complex Gaussian in phase space that is highly oscillatory. A familiar example where terms like this appear is in the Wigner function of a cat state. For Lindblad master equations with linear Lindblad operators and quadratic Hamiltonians, the coupling to the environment exponentially damps the oscillatory parts of these phase-space functions. This is a manifestation of decoherence. Therefore, the complex oscillatory parts cannot persist in the steady state. The dynamics of complex Gaussian wave packets generated by equations of Lindblad type were analyzed in Ref. [43]. Using these results, we find that each $|\alpha\rangle\langle\beta|$ tends to $N_{\alpha\beta}\hat{\rho}_G$. Here $\hat{\rho}_G$ is the same Gaussian steady state we found before, and the $N_{\alpha\beta}$ are normalization factors that guarantee the steady state $\hat{\rho}_\infty$ is normalized. We can then conclude that the steady state $\hat{\rho}_\infty$ is the separable Gaussian state $\hat{\rho}_G$.

The same procedure can be followed with the local master equation (38) to obtain the same result. The only difference, barring additional indices, is the coupling term $\hat{H}'_C$ appearing in the system Hamiltonian (41). However, this does not pose any complications when following the steps above, as $\hat{H}'_C$ only includes terms of the form $\hat{b}^\dagger_{nu}\hat{b}_{mv}$ and cannot introduce any squeezing.

V. CONCLUSION

We have shown that autonomous, steady-state entanglement generation is impossible in bosonic systems of arbitrary size undergoing quadratic excitation-number-preserving interactions (i.e., passive Gaussian interactions or linear interferometers) and weakly coupled to thermal bosonic and/or spin baths at different temperatures. Our result applies to both the local and global regimes, which cover the entire range from weak to strong intersystem coupling. Our result holds provided that the system Hamiltonian is stable ($\hat{H}_S > 0$), a secular approximation is justified, and the steady state is unique. This result contrasts with previous findings for finite-dimensional autonomous thermal machines with excitation-preserving interactions, which can generate steady-state entanglement strong enough to exhibit both steering and nonlocality. Passive Gaussian interactions, even combined with nonbosonic baths, are sufficiently restrictive to preclude any steady-state entanglement. We note that the condition for the steady state to be unique (R invertible) will most likely be satisfied in all practical situations, where the model parameter values will never be completely symmetric.

Having ruled out steady-state entanglement in this case, it is interesting to ask whether bosonic steady-state entanglement can be generated in more complicated settings. One natural next step would be to include higher-order intersystem interactions, such as three- or four-wave mixing. The challenge here may be to determine whether the (generally non-Gaussian) steady state is separable or not. One could also attempt to go beyond the weak-coupling limit by applying, for example, the reaction coordinate method [44].

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APPENDIX A: TIME EVOLUTION OF THE COVARIANCE MATRIX ELEMENTS β_{jk}

The equations of motion for the covariance matrix elements $\beta_{jk} = \langle\hat{b}_j\hat{b}_k\rangle - \langle\hat{b}_j\rangle\langle\hat{b}_k\rangle$ can be obtained from the operator evolution equation (45). We first compute the equations for $\langle\hat{b}_j\rangle$ and $\langle\hat{b}_j\hat{b}_k\rangle$, and then combine the results to find

$$\frac{d}{dt}\beta_{jk} = -i(\omega_j+\omega_k)\beta_{jk} + \sum_{u\in\Lambda_j} f_{ju}(\omega_j)\beta_{ku} + \sum_{u\in\Lambda_k} f_{ku}(\omega_k)\beta_{ju}. \tag{A1}$$

For simplicity we have redefined ω_j as the eigenvalue associated with mode $\hat{b}_j$ and Λ_j as the eigenspace to which ω_j belongs. The functions $f_{iu}(\omega)$ are defined as

$$f_{iu}(\omega) = \tfrac{1}{2}\big[\Gamma^{(1)}_{iu}(\omega) - \Gamma^{(2)}_{ui}(\omega)\big] - i\varphi_{iu}(\omega), \tag{A2}$$

in terms of functions appearing in the main text. The key result of this Appendix is that when each element $\beta_{jk} = 0$ at $t = 0$, then each β_{jk} is zero for all time. This follows immediately from (A1).

APPENDIX B: THE HERMITIAN PART OF R IS NEGATIVE SEMIDEFINITE

Let the Hermitian part of R be denoted by $R_H = (R + R^\dagger)/2$. From the equations of motion for the first moments in the main text (46) we obtain the matrix elements

$$[R_H]_{ij} = \tfrac{1}{2}\big(\Gamma^{(1)}_{ij}(\omega_i) - \Gamma^{(2)}_{ji}(\omega_i)\big)\delta_{\omega_i,\omega_j}, \tag{B1}$$

where ω_i is the eigenvalue associated with the Bogoliubov mode $\hat{b}_i$. We permute the rows and columns of R_H to bring it into block diagonal form, where each block corresponds to the same eigenvalue. After a relabeling of indices, the block B corresponding to the eigenvalue ω has the elements

$$B_{ij} = \tfrac{1}{2}\big(\Gamma^{(1)}_{ij}(\omega) - \Gamma^{(2)}_{ji}(\omega)\big). \tag{B2}$$

We now show that this block is negative semidefinite.

Making use of the expressions (23)–(27), (31), and (32) we find that the matrix elements in (B2) can be written as

$$B_{ij} = \pi \sum_{n=1}^{N} \int \tilde{g}_{in}(q)\tilde{g}^{*}_{jn}(q)\delta(\epsilon_n(q) - \omega) \times [(1+\xi_n)p_n(\epsilon_n(q)) - 1]\, dq, \tag{B3}$$

where we have defined

$$\tilde{g}_{in}(q) = \sum_{j \in S_n} U^{*}_{ji} g_j(q) \tag{B4}$$

in terms of the unitary matrix U that diagonalizes the system Hamiltonian and the coupling parameters between the system and environment $g_j(q)$. For every complex vector x

$$x^{\dagger} B x = \pi \sum_{n=1}^{N} \int |x_n(q)|^2 \delta(\epsilon_n(q) - \omega) \times [(1+\xi_n)p_n(\epsilon_n(q)) - 1]\, dq, \tag{B5}$$

with $x_n(q) = \sum_j x_j \tilde{g}^{*}_{jn}(q)$. For the bosonic baths, $\xi_n = -1$ and the contribution to (B5) is less than or equal to zero. For the spin baths, $\xi_n = +1$ but $p_n(\omega) < 1/2$ since $\omega > 0$. The contribution from the spin baths is therefore also less than or equal to zero, and it follows that $x^{\dagger} B x \leqslant 0$. This holds for every block of R_H, and so the Hermitian part of R is negative semidefinite.

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