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Published in:
Journal of Mathematical Physics

Link to article, DOI:
[10.1063/1.1665123](https://doi.org/10.1063/1.1665123)

Publication date:
1970

Document Version
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

Citation (APA):
Johannsen, G. (1970). Electromagnetic Sources in Moving Simple Medium. *Journal of Mathematical Physics*, 11(11), 3251-3253. <https://doi.org/10.1063/1.1665123>

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Electromagnetic Sources in Moving Simple Medium

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(Received 4 May 1970)

A retarded potential tensor (4-vector) is derived in an arbitrary system of inertia for an arbitrary electromagnetic source in a moving homogeneous, isotropic, nondispersive, lossless dielectric. The velocity is uniform, and the result is relativistic correct.

I. INTRODUCTION

The differential equation for the potential tensor is simplest in the system of inertia K_m where the medium is at rest. A 4-dimensional integral representation for the potential tensor can readily be obtained by using the 4-dimensional Green's theorem. One may carry out the integration with respect to the time coordinate to get the space integral representation for the (retarded) potential tensor.

As pointed out in Ref. 1, the space integral representation in K_m is not very utilizable in another system of inertia K which is in uniform motion relative to K_m . This is due to the fact that space-time coordinates mix under the Lorentz transformation from K to K_m .

It is shown that the 4-dimensional integral representation for the potential tensor in K_m can be transformed to an arbitrary system of inertia K . Integrating with respect to the time coordinate then leads to the space integral representation for the (retarded) potential tensor in K . The results found here are in agreement with those in Refs. 2 and 3, where the pertinent differential equation is integrated by using Fourier transformation and an operational method, respectively.

II. INTEGRAL REPRESENTATIONS OF THE POTENTIAL TENSOR

We use Cartesian tensor notation. By a tensor we understand a tensor defined on the Lorentz transformation group. Latin subscripts run from 1 to 4; Greek subscripts run from 1 to 3. The coordinate x_4 is equal to ict , where t is the time and c the speed of light in vacuum; therefore, the metric tensor in 4-space is equal to the Kronecker symbol δ_{ij} (when Cartesian spatial coordinates are used), and we do not distinguish between contravariant and covariant tensors. Repeated subscripts obey the summation convention, and commas in subscripts denote partial differentiation with respect to coordinates (or covariant differentiation, since the metric tensor is independent of the coordinates).

The potential tensor A_i satisfies a differential equation,^{1,4} which in an arbitrary system of inertia K

can be written as a tensor equation:

$$A_{i,nn} - \kappa A_{i,st} U_s U_t = -S_i, \quad \kappa \equiv \frac{n^2 - 1}{c^2}, \quad n \equiv \frac{c}{c'}, \quad c' \equiv \frac{1}{(\mu\epsilon)^{\frac{1}{2}}}, \quad (1)$$

$$S_i \equiv \mu \left(J_i + \frac{\kappa}{n^2} J_r U_r U_i \right).$$

μ and ϵ are the permeability and the dielectric constant of the medium. U_i is the velocity tensor of K_m , and J_i is the current density tensor.^{1,5}

Let primed quantities refer to K_m . Since $U'_i = (0, 0, 0, ic)$, we get from (1)

$$A'_{i,nn} + c^2 \kappa A'_{i,44} = A'_{i,vv} + n^2 A'_{i,44} = -S'_i. \quad (2)$$

Introducing new variables by $y'_\rho \equiv x'_\rho$ and $y'_4 \equiv n^{-1}x'_4$, we get from (2)

$$A'_{i,nn}(y'_r) = -S'_i(y'_r), \quad (3)$$

where $(y'_r) \equiv (y'_1, y'_2, y'_3, y'_4)$.

For the solution of (3) see Ref. 5, p. 146. The result is

$$A'_i(y'_r) = (4\pi^2)^{-1} \iiint_{-\infty}^{\infty} \left(S'_i(z'_r) / \sum_{r=1}^4 (z'_r - y'_r)^2 \right) dV(z'_r). \quad (4)$$

Substituting back to variables x'_r , we obtain

$$A'_i(x'_r) = \iiint_{-\infty}^{\infty} S'_i(z'_r) G'(z'_r - x'_r) dV(z'_r), \quad (5)$$

where

$$G'(z'_r - x'_r) \equiv (4\pi^2 n)^{-1} [u'_r u'_r + (\kappa/n^2)(u'_r U'_r)^2]^{-1}$$

and

$$u'_r \equiv z'_r - x'_r.$$

The poles of the function G' are given by $u'_4 = \tau'_\pm \equiv \pm in(u'_\rho u'_\rho)^{\frac{1}{2}}$. As in Ref. 5 we can deform the contour for the z'_4 integration to circumvent one of the poles. Since advanced potentials are not of interest, we consider the pole τ'_- .

From (5) we get

$$A'_i(x'_r) = \iiint_{-\infty}^{\infty} \oint_{L'} S'_i(z'_r) G'(u'_r) du'_4 du'_3 du'_1 du'_2, \quad (6)$$

where L' is a closed path surrounding τ'_- .

Equation (6) is valid in K_m ; we now transform (6) to an arbitrary system of inertia K . Let $x_r = a_{rs}x'_s$ be the proper Lorentz transformation connecting coordinates in K with coordinates in K_m . Multiplying (6) by a_{ji} and substituting x'_r by x_r , we see that the left side in (6) (which is a tensor) becomes $A_j(x_r)$. a_{ji} is independent of the coordinates, and, since S'_i is a tensor, we have $a_{ji}S'_i(z'_r) = S_j(z_r)$ if z_r and z'_r are connected by the proper Lorentz transformation, i.e., $z_r = a_{rs}z'_s$. Furthermore, G' transforms like an invariant, i.e.,

$$G(u_r) = (4\pi^2 n)^{-1} [u_r u_r + (\kappa/n^2)(u_r U_r)^2]^{-1}. \quad (7)$$

Without loss of generality, we choose a_{rs} so that

$$\begin{aligned} x^1 &= x'_1, & x^2 &= x'_2, \\ x^3 &= \gamma(x'_3 + i\beta x'_4), & x^4 &= \gamma(x'_4 - i\beta x'_3), \end{aligned}$$

where $\gamma \equiv (1 - \beta^2)^{-1/2}$, $\beta \equiv v/c$, and v is the velocity of K relative to K_m (Fig. 1).

To see how the poles to be circumvented are transformed under the Lorentz transformation, we consider the tensor equation

$$u_r u_r + (\kappa/n^2)(u_r U_r)^2 = 0, \quad (8)$$

which is a 3-dimensional hypersurface in Minkowski 4-space. In K_m (8) reduces to $(u'_4/i)^2 - (nu'_3)^2 = (n\rho')^2$ and $\rho' \equiv (u_1'^2 + u_2'^2)^{1/2}$ which in Euclidean 2-space is the equation of a hyperbola (Cartesian coordinates $u'_4/i, u'_3$). In Fig. 2 the situation is illustrated in a way due to Minkowski.⁶

When $n\beta < 1$ and u_3 is given, there is one pole to be circumvented. When $n\beta > 1$ (this situation is shown in Fig. 2), there are two poles to be circumvented if $u_3 > |a|^{1/2} \rho$, $a \equiv [1 - (n\beta)^2]/(1 - \beta^2)$, and no poles are to be circumvented if $u_3 < |a|^{1/2} \rho$ (i.e., A_i is equal to zero).

The roots in (8) are given by

$$\frac{u_4}{i} = \tau_{\pm} \equiv \frac{n\beta}{1 - (n\beta)^2} \times [(n - n^{-1})u_3 \pm (\beta^{-1} - \beta)(u_3^2 + a\rho^2)^{1/2}]. \quad (9)$$

Since $n\beta < 1$ implies that $a > 0$ and $(\beta^{-1} - \beta) > n - n^{-1} > 0$, we actually see that $\tau_+ > 0$ and $\tau_- < 0$. In this case, from (6), we can deduce

$$A_i(x_r) = \iiint_{-\infty}^{\infty} \oint_L S_i(z_r) G(u_r) du_4 du_3 du_1 du_2, \quad (10)$$

FIG. 1. The motion of K relative to K_m .

FIG. 2. Location of poles of the function G in Minkowski space.

where L circumvents τ_- . By the method of residues, we get

$$A_i(x_r) = \frac{1}{4\pi} \iiint_{-\infty}^{\infty} \frac{S_i(z_\rho, x_4 + \tau_-)}{[(z_3 - x_3)^2 + a\rho^2]^{\frac{1}{2}}} dV(z_\rho), \quad n\beta < 1. \quad (11)$$

In the case of $n\beta > 1$, we actually see from (9) that $\tau_{\pm} > 0$ when $u_3 < -|a|^{1/2} \rho$, that $\tau_{\pm}$ are imaginary when $-|a|^{1/2} \rho < u_3 < |a|^{1/2} \rho$, and finally that $\tau_{\pm} < 0$ when $u_3 > |a|^{1/2} \rho$. From (6) we get

$$A_i(x_r) = \iiint_{-\infty}^{\infty} \oint_L S_i(z_r) G(u_r) du_4 du_3 du_1 du_2, \quad (12)$$

where the u_3 integration is to be taken from ∞ to $|a|^{1/2} \rho$ (L surrounding τ_-) and back to ∞ (L surrounding τ_+). It turns out that⁷

$$\begin{aligned} A_i(x_r) = & -\frac{1}{4\pi} \iiint_{-\infty}^{\infty} \frac{S_i(z_\rho, x_4 + \tau_+) + S_i(z_\rho, x_4 + \tau_-)}{[(z_3 - x_3)^2 + a\rho^2]^{\frac{1}{2}}} \\ & \times \theta[z_3 - (x_3 + |a|^{1/2} \rho)] dV(z_\rho), \quad n\beta > 1, \quad (13) \end{aligned}$$

FIG. 3. Conical region in K , where a source point contributes to the field, $n\beta > 1$.

where θ is the unit step function, i.e.,

$$\theta(t) = 0 \quad \text{when } t < 0, \\ = 1 \quad \text{when } t \geq 0.$$

Equations (11) and (13) hold in an arbitrary system of inertia. The step function in (13) implies that a source point (z_ρ) contributes to the field only at points inside a conical region given by $(x_3 - z_3) \leq -|a|^{\frac{1}{2}} \rho$.

III. TIME HARMONIC SOURCES, 3-DIMENSIONAL REPRESENTATION

Let the source be time harmonic in K : $S_i(x_r) = S_i(x_\rho)e^{-kx_4}$, where $k \equiv \omega/c$ and ω is the frequency. Omitting the time factor e^{-kx_4} , we derive from (9)

$$S_i(z_\rho, x_4 + \tau_-) \\ = S_i(z_\rho) \exp[-ikb(z_3 - x_3)] \\ \times \exp\{ik(n/a)[(z_3 - x_3)^2 + a\rho^2]^{\frac{1}{2}}\}, \quad (14)$$

where

$$b \equiv \frac{n\beta}{1 - (n\beta)^2} \left(n - \frac{1}{n} \right) = \frac{\beta\kappa c^2}{1 - (n\beta)^2}.$$

Furthermore, we derive

$$S_i(z_\rho, x_4 + \tau_-) + S_i(z_\rho, x_4 + \tau_+) \\ = 2S_i(z_\rho)e^{-ikb(z_3 - x_3)} \cos\{(kn/a)[(z_3 - x_3)^2 + a\rho^2]^{\frac{1}{2}}\}. \quad (15)$$

From the definition of S_i , and making use of the continuity equation $J_{\rho,\rho} = -J_{4,4} = i\omega\rho = kJ_4$, we get, for the spatial components of S_i ,

$$S_i(z_\rho) = \mu \left(\delta_{\lambda\nu} + \frac{\kappa}{n^2} U_\lambda U_\nu \right) J_\nu(z_\rho) \\ + i \frac{\mu\kappa c^2}{n^2} \gamma U_\lambda J_{\nu,\nu}(z_\rho). \quad (16)$$

Similarly,

$$\frac{c}{i} S_4(z_\rho) = \frac{\mu c^2}{i\omega} \left(1 - \frac{\kappa c^2}{n^2} \gamma^2 \right) J_{\nu,\nu} + \frac{\mu\kappa c^2}{n^2} \gamma U_\nu J_\nu. \quad (17)$$

Substituting (14)–(17) into (11) and (13) leads to expressions which are in agreement with Ref. 1.

The field vector $\vec{E}$ may be obtained by using the equation $\vec{E} = -\nabla\Phi + i\omega\vec{A}$ (Ref. 1; it can be shown that $\vec{E} = -\nabla\Phi - \partial\vec{A}/\partial t$ holds in any system of inertia). By some calculation the results may be transformed to an expression as given in Ref. 8.

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⁷ Equation (10) in Ref. 3 should contain a sum of two δ functions.

⁸ C. T. Tai, *IEEE Trans. Antennas Propagation* **13**, 322 (1965).

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 11, NUMBER 11 NOVEMBER 1970

Expansion of an n -Point Function as a Sum of Commutators*

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(Received 22 October 1969; Revised Manuscript Received 15 April 1970)

We write the n -point function of currents as a sum over nested commutators, a form more suitable for certain current algebra calculations.

I. INTRODUCTION AND RESULT

In this paper we consider the n -point function, defined as

$$T(q_1, \dots, q_{n-1})_{\mu_1 \dots \mu_n} \\ = \iint \dots \int d^4x_1 \dots d^4x_{n-1} e^{-iq_1x_1 - \dots - iq_{n-1}x_{n-1}} \\ \times \langle 0 | T(j_{\mu_1}(x_1) \dots j_{\mu_{n-1}}(x_{n-1}) j_{\mu_n}(0)) | 0 \rangle, \quad (1)$$

where $T(j_1 \dots j_n)$ is the product of the n current operators $j_1, \dots, j_n$, in the order of the time components of their points of evaluation:

$$T(j_1(x_1) \dots j_n(x_n)) \\ = \sum_{\pi \in S_n} \theta(x_{\pi(1)}^0 - x_{\pi(2)}^0) \dots \theta(x_{\pi(n-1)}^0 - x_{\pi(n)}^0) \\ \times j_{\pi(1)}(x_{\pi(1)}) \dots j_{\pi(n)}(x_{\pi(n)}), \quad (2)$$

where the sum is over all permutations π in S_n , the symmetric group of order n . The function $\theta(t)$ is the usual step function. In Eq. (1) we only integrate over $n - 1$ space variables because we are using the translation invariance of the theory to work in a coordinate system where $x_n = 0$ and $q_1 + \dots + q_n = 0$.

It is a straightforward calculation, which is given below, to rewrite Eq. (1) as a linear combination of products of n current operators (or, rather, of their Fourier transforms in momentum space) not involving the step functions θ . However, current algebra treats only commutators of operators rather than arbitrary products, so that it is desirable to express the n -point function as a linear combination of commutators of operators.