DTU Library

A structure theorem for fundamental solutions of analytic multipliers in R^n

Winterrose, David Scott

Published in:
Journal of Pseudo-Differential Operators and Applications

Link to article, DOI:
10.1007/s11868-024-00586-2

Publication date:
2024

Document Version
Publisher's PDF, also known as Version of record

J. Pseudo-Differ. Oper. Appl. (2024) 15:17
https://doi.org/10.1007/s11868-024-00586-2

A structure theorem for fundamental solutions of analytic multipliers in $\mathbb{R}^n$

David Scott Winterrose[1]

Received: 23 July 2023 / Revised: 14 December 2023 / Accepted: 2 January 2024

Abstract
Using a version of Hironaka's resolution of singularities for real-analytic functions, any elliptic multiplier $\mathrm{Op}(p)$ of order $d > 0$, real-analytic near $p^{-1}(0)$, has a fundamental solution μ_0. We give an integral representation of μ_0 in terms of the resolutions supplied by Hironaka's theorem. This μ_0 is weakly approximated in $H^t_{\mathrm{loc}}(\mathbb{R}^n)$ for $t < d - \frac{n}{2}$ by a sequence from a Paley-Wiener space. In special cases of global symmetry, the obtained integral representation can be made fully explicit, and we use this to compute fundamental solutions for two non-polynomial symbols.

Keywords Fundamental solutions · Pseudo-differential equations · PDE

Mathematics Subject Classification 35A08 · 35E05 · 35C05 · 35A17

1 Introduction

Let $\mathcal{S}'(\mathbb{R}^n)$ denote the space of tempered distributions on $\mathbb{R}^n$. A fundamental solution of $\mathrm{Op}(p) = \mathcal{F}^{-1} p \mathcal{F}$ is a $\mu_0 \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\mathrm{Op}(p)\mu_0 = \delta_0 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

where δ_0 is the unit measure at 0, $\mathcal{F}$ is the Fourier transform, and p is the symbol. The study of these is classical, and most results are recorded in standard texts [7, 8]. The Hörmander–Lojasiewicz theorem [6, 9] ensures existence when p is a polynomial, and provides a way to construct a μ_0, at least in principle, explicitly from the symbol. But the situation becomes nebulous when p is not a polynomial or globally smooth.

✉ David Scott Winterrose
dawin@dtu.dk

1 Department of Applied Mathematics and Computer Science, Technical University of Denmark, Kgs. Lyngby, Denmark

Published online: 26 February 2024

We address this problem when $p^{-1}(0)$ is compact and p is real-analytic near $p^{-1}(0)$, and obtain an integral representation that clearly shows the structure of μ_0.

In order to do so, we use a variant of Hironaka's resolution of singularities [1]. Generally, the local charts that are supplied by Hironaka's theorem are unknowable, but they allow us to build an integral representation out of the geometry of $p^{-1}(0)$. Occasionally, a diffeomorphism that brings p into a resolved form can replace them, and such a diffeomorphism can often be constructed when there is global symmetry. Important examples with this property include any sum of powers of the Laplacian, or any sum of powers of certain elliptic self-adjoint second-order differential operators. However, the novel and most interesting case for us here is when p is *not* a polynomial, and p is not necessarily globally smooth, but with dimension $n > 1$ and order $d > 0$. We give examples showing the utility of this approach.

A lot of research has been devoted to the construction of explicit representations. See e.g. Ortner and Wagner [10] and Camus [3, 4] for a broad class of operators. Usually, it is very difficult to find explicit representations of fundamental solutions, and the study is often focused on a particular operator of fixed order and dimension. In the case of general homogeneous elliptic and some types of non-elliptic operators, Camus [3, 4] obtained explicit representations valid for any number of dimensions. Apart from the base practical value of constructing general solutions via convolution, an explicit form may find application in proofs of mapping properties of its operator. See e.g. Rabier [11], where the solution obtained in [3], implicit in [7], is used.

2 Notation

Let $p \in C(\mathbb{R}^n)$ be real-analytic in a neighbourhood of $p^{-1}(0) \neq \emptyset$. It must be smooth outside an open ball $B(0, R)$ centered at 0 of some radius $R > 0$. Putting $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ for $\xi \in \mathbb{R}^n$, it must satisfy

$$\sup_{\xi \in \mathbb{R}^n \setminus B(0,R)} \langle \xi \rangle^{-d} |\partial_\xi^\alpha p(\xi)| < \infty \quad \text{for all} \quad \alpha \in \mathbb{N}_0^n,$$

and the ellipticity constraint

$$\inf_{\xi \in \mathbb{R}^n \setminus B(0,R)} \langle \xi \rangle^{-d} |p(\xi)| > 0.$$

Definition 2.1 (*The Paley-Wiener spaces*) Let $K \subset \mathbb{R}^n$ be compact and convex. Define $\mathrm{PW}_K^d(\mathbb{R}^n)$ to be the space of entire functions u satisfying

$$\sup_{x \in \mathbb{C}^n} \exp\Big(- \sup_{\xi \in K} \operatorname{Im}(x) \cdot \xi \Big) \langle x \rangle^{-d} |u(x)| < \infty.$$

If $\{K_j\}_{j=1}^\infty$ is an exhaustion of $\mathbb{R}^n$ by compact convex sets, we put

$$\mathrm{PW}^d(\mathbb{R}^n) = \cup_{j=1}^\infty \mathrm{PW}_{K_j}^d(\mathbb{R}^n).$$

Moreover, we put $\mathrm{PW}^{-\infty}(\mathbb{R}^n) = \cap_{d\in\mathbb{Z}} \mathrm{PW}^d(\mathbb{R}^n)$ and $\mathrm{PW}^{+\infty}(\mathbb{R}^n) = \cup_{d\in\mathbb{Z}} \mathrm{PW}^d(\mathbb{R}^n)$.

These spaces are related to $\mathcal{E}'(\mathbb{R}^n)$, the compactly supported distributions on $\mathbb{R}^n$. We write $H^t_{\mathrm{loc}}(\mathbb{R}^n)$ for the Frechet space of distributions locally belonging to $H^t(\mathbb{R}^n)$, and $H^{-t}_{\mathrm{comp}}(\mathbb{R}^n)$ for its dual space of compactly supported distributions in $H^{-t}(\mathbb{R}^n)$. Finally, $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space, $u \in \mathcal{S}(\mathbb{R}^n)$ decays faster than any polynomial, and we put $\mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,dx$ and $\mathcal{F}^{-1}u(x) = \frac{1}{(2\pi)^n}\mathcal{F}u(-x)$ for $x, \xi \in \mathbb{R}^n$.

3 Solution Operator

Our main tool is Hironaka's resolution of singularities. Often it is stated abstractly [2], but we need a local embedded version.

Theorem 3.1 (Local embedded version of Hironaka's theorem. From Atiyah [1]) *Let $U \subset \mathbb{R}^n$ be an open neighbourhood of 0, and let f be a function $0 \not\equiv f \in C^\omega(U)$. Then there is an open $0 \in V \subset U$, a real-analytic manifold M, and a map*

$$\Psi : M \to V.$$

It has the following properties:

1. $\Psi : M \to V$ *is proper and real-analytic.*
2. $\Psi : M\backslash(f \circ \Phi)^{-1}(0) \to V\backslash f^{-1}(0)$ *is a real-analytic diffeomorphism.*
3. $(f \circ \Psi)^{-1}(0)$ *is a hypersurface in M with normal crossings.*

As p is real-analytic near its zero-set, it is compact with Lebesgue measure zero. The resolution theorem implies that p can be written locally in normal crossing form. Fix an open cover $\{V_j\}_{j=1}^N$ of $p^{-1}(0)$ and open $\{U_j\}_{j=1}^N$ such that

1. $\Psi_j : U_j\backslash(p \circ \Psi_j)^{-1}(0) \to V_j\backslash p^{-1}(0)$ is a real-analytic diffeomorphism,
2. $(p \circ \Psi_j)(x) = c_j(x)x^{\alpha_j}$ for all $x \in U_j$ for some $\alpha_j \in \mathbb{N}_0^n$,

where each c_j is a complex-valued, but nowhere zero, real-analytic function on U_j. Also, we put $m = \max\{\alpha_j\}_{j=1}^N$.

Theorem 3.2 *Let $\{\chi_j\}_{j=1}^N$ be any partition of unity subordinate to $\{V_j\}_{j=1}^N$. There is a fundamental solution μ_0, smooth away from $x = 0$, of the form*

$$\mu_0(x) = \mathcal{F}^{-1}\Big(\frac{\chi}{p}\Big)(x) + \sum_{j=1}^N \int_{\mathbb{R}^n} I_j(z)\partial_z^{\alpha_j}\Big[e^{ix\cdot\Psi_j(z)}\frac{(\chi_j \circ \Psi_j)(z)}{c_j(z)}|\det d\Psi_j(z)|\Big]\,dz,$$

where $\chi = 1 - \sum_{j=1}^N \chi_j$, and the I_j are given a.e. by

$$I_j(z) = \frac{1}{(2\pi)^n}\prod_{\alpha_{j,k}\neq 0}\frac{-\ln|z_k|}{(\alpha_{j,k}-1)!}.$$

It is weakly approximated in $H^t_{\mathrm{loc}}(\mathbb{R}^n)$ for $t < d - \frac{n}{2}$ by a sequence in $\mathrm{PW}^m(\mathbb{R}^n)$. Finally, if $p^{-1}(0)$ is embedded, any $v \in \ker \mathrm{Op}(p) \subset \mathrm{PW}^\infty(\mathbb{R}^n)$ is of the form

$$v(x) = \sum_{j=1}^{N} \sum_{k \le k_j - 1} \left\langle (\Psi_j^*)^{-1}(u_{j,k} \otimes \partial_{z_n}^k \delta_0)(\xi), e^{ix\cdot\xi} \right\rangle,$$

where $u_{j,k} \in \mathcal{E}'(U_j^0)$ are supported in the $z_n = 0$ slice $U_j^0 = \{z \in \mathbb{R}^{n-1} \mid (z, 0) \in U_j\}$, and each Ψ_j is arranged so that

$$(p \circ \Psi_j)(x) = c_j(x) x_n^{k_j}.$$

In this case, all other fundamental solutions differ from μ_0 by such a v.

The first step is to prove a lemma about principal value integrals with log kernel. It is used here in a way similar to Björk [2, Chapter 6, Theorem 1.5].

Lemma 3.3 *Let $\psi \in C^\infty(\mathbb{R})$ be either rapidly decaying or compactly supported. Then, for any $k \in \mathbb{N}$, we have*

$$\int_{-\infty}^{\infty} \psi(r)\, dr = \frac{-1}{(k-1)!} \int_{-\infty}^{\infty} \ln(|r|) \frac{d^k}{dr^k}\left(r^k \psi(r)\right) dr.$$

Proof The proof of this is a routine exercise in repeated integration by parts. Observe that we can write $\psi(r) = r^{-k} r^k \psi(r)$, and

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi(r)\, dr &= \frac{-1}{k-1} r^1 \psi(r)\Big|_{-\infty}^{\infty} + \frac{1}{k-1} \int_{-\infty}^{\infty} r^{-k+1} \frac{d}{dr}\left(r^k \psi(r)\right) dr \\
&\cdots \\
&= \frac{-1}{(k-1)!} r^{k-1} \psi(r)\Big|_{-\infty}^{\infty} + \frac{1}{(k-1)!} \int_{-\infty}^{\infty} r^{-1} \frac{d^{k-1}}{dr^{k-1}}\left(r^k \psi(r)\right) dr \\
&= \frac{-1}{(k-1)!} \int_{-\infty}^{\infty} \ln(|r|) \frac{d^k}{dr^k}\left(r^k \psi(r)\right) dr,
\end{aligned}$$

where all boundary terms at 0 in the final integration vanish, because

$$\lim_{r \to 0\pm} \ln(|r|) \frac{d^{k-1}}{dr^{k-1}}\left(r^k \psi(r)\right) = 0,$$

and boundary terms at $\pm\infty$ vanish by the hypothesis on ψ. □

Lemma 3.4 *Define $Q : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by*

$$Qv(x) = \sum_{j=1}^{N} \int_{\mathbb{R}^n} I_j(z) \partial_z^{\alpha_j} \left[e^{ix\cdot\Psi_j(z)} \frac{(\chi_j \mathcal{F} v) \circ \Psi_j(z)}{c_j(z)} |\det d\Psi_j(z)| \right] dz.$$

Then $P = \mathrm{Op}(\frac{\chi}{p}) + Q$ *satisfies both*

$$\mathrm{Op}(p)Pv = v \quad \text{and} \quad P\mathrm{Op}(p)v = v \quad \text{for all} \quad v \in \mathcal{S}(\mathbb{R}^n).$$

Proof The proof is an application of Lemma 3.3 and the Fubini–Tonelli theorem. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Using Lemma 3.3 coordinate-wise, we compute

$$\begin{aligned}
\langle \mathrm{Op}(p)Qv, \psi\rangle &= \int_{\mathbb{R}^n} I_j(z)\partial_z^{\alpha_j}\Big[\langle \mathrm{Op}(p)e^{i(\cdot)\cdot\Psi_j(z)}, \psi\rangle \frac{(\chi_j\mathcal{F}v)\circ\Psi_j(z)}{c_j(z)}|\det d\Psi_j(z)|\Big]\,dz\\
&= \int_{\mathbb{R}^n} I_j(z)\partial_z^{\alpha_j}\Big[z^{\alpha_j}\langle e^{i(\cdot)\cdot\Psi_j(z)}, \psi\rangle(\chi_j\mathcal{F}v)\circ\Psi_j(z)|\det d\Psi_j(z)|\Big]\,dz\\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\backslash(p\circ\Psi_j)^{-1}(0)} \langle e^{i(\cdot)\cdot\Psi_j(z)}, \psi\rangle(\chi_j\mathcal{F}v)\circ\Psi_j(z)|\det d\Psi_j(z)|\,dz\\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\backslash p^{-1}(0)} \langle e^{i(\cdot)\cdot\xi}, \psi\rangle(\chi_j\mathcal{F}v)(\xi)\,d\xi\\
&= \langle \mathrm{Op}(\chi_j)v, \psi\rangle,
\end{aligned}$$

and we then get

$$\langle \mathrm{Op}(p)Qv, \psi\rangle = \sum_{j=1}^{N}\langle \mathrm{Op}(\chi_j)v, \psi\rangle = \langle v - \mathrm{Op}(\chi)v, \psi\rangle.$$

Point-wise in $x \in \mathbb{R}^n$, we compute

$$\begin{aligned}
Q\mathrm{Op}(p)v(x) &= \int_{\mathbb{R}^n} I_j(z)\partial_z^{\alpha_j}\Big[e^{ix\cdot\Psi_j(z)}\frac{(\chi_j\mathcal{F}\mathrm{Op}(p)v)\circ\Psi_j(z)}{c_j(z)}|\det d\Psi_j(z)|\Big]\,dz\\
&= \int_{\mathbb{R}^n} I_j(z)\partial_z^{\alpha_j}\Big[z^{\alpha_j}e^{ix\cdot\Psi_j(z)}(\chi_j\mathcal{F}v)\circ\Psi_j(z)|\det d\Psi_j(z)|\Big]\,dz\\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\backslash(p\circ\Psi_j)^{-1}(0)} e^{ix\cdot\Psi_j(z)}(\chi_j\mathcal{F}v)\circ\Psi_j(z)|\det d\Psi_j(z)|\,dz\\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\backslash p^{-1}(0)} e^{ix\cdot\xi}(\chi_j\mathcal{F}v)(\xi)\,d\xi\\
&= \mathrm{Op}(\chi_j)v(x),
\end{aligned}$$

which shows that

$$Q\mathrm{Op}(p)v = \sum_{j=1}^{N}\mathrm{Op}(\chi_j)v = v - \mathrm{Op}(\chi)v.$$

Note that the properties of Ψ_j ensure that all the above integrals are well-defined. The determinant of $d\Psi_j$ on each component of $U_j\backslash(p\circ\Psi_j)^{-1}(0)$ never becomes zero. This completes the proof. $\square$

Lemma 3.5 *$P : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is continuous.*

Proof The proof is just estimating $C^\infty(\mathbb{R}^n)$ semi-norms of Q in those of $\mathcal{S}(\mathbb{R}^n)$. By the chain rule, if $v \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\partial_{z_k}(\mathcal{F}v \circ \Psi_j) = -i(\mathcal{F}(x_1 v) \circ \Psi_j, \ldots, \mathcal{F}(x_n v) \circ \Psi_j) \cdot \partial_{z_k}\Psi_j.$$

Using the Leibniz rule, we get for any $\alpha \in \mathbb{N}_0^n$ some $C'_\alpha, C_\alpha > 0$ such that

$$\begin{aligned}|\partial_x^\alpha Qv(x)| &\le C'_\alpha \sum_{j=1}^N \int_{\operatorname{supp}(\chi_j)} |I_j(z)| \sum_{\beta \le \alpha_j} \langle x\rangle^{\alpha_j - \beta} |\partial_z^\beta(\mathcal{F}v \circ \Psi_j)(z)|\, dz \\ &\le C_\alpha \sum_{j=1}^N \Big(\int_{\operatorname{supp}(\chi_j)} |I_j(z)|\, dz\Big) \langle x\rangle^m \max_{|\beta| \le m} \sup_{z \in \mathbb{R}^n} |\mathcal{F}(x^\beta v)(z)|,\end{aligned}$$

which by the continuity of $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ implies the lemma. □

Lemma 3.6 *$P : \mathrm{PW}^{-\infty}(\mathbb{R}^n) \to \mathrm{PW}^m(\mathbb{R}^n)$ is well-defined.*

Proof Because each map Ψ_j is proper, each $(\chi_j \mathcal{F}v) \circ \Psi_j$ is compactly supported. By the well-known [7, Paley–Wiener–Schwartz Theorem 7.3.1], $\mathrm{Op}(\frac{\chi}{p})v \in \mathrm{PW}^{-\infty}(\mathbb{R}^n)$. A simple estimate gives $C', C > 0$ such that

$$\begin{aligned}|Qv(x)| &\le C' \sum_{j=1}^N \int_{\operatorname{supp}(\chi_j)} |I_j(z)| \sum_{\beta \le \alpha_j} |\partial_z^\beta [e^{ix\cdot\Psi_j(z)}]|\, dz \\ &\le C \sum_{j=1}^N \Big(\int_{\operatorname{supp}(\chi_j)} |I_j(z)|\, dz\Big) \langle x\rangle^m \exp\Big(\sup_{\xi \in K} \mathrm{Im}(x) \cdot \xi\Big),\end{aligned}$$

where K is a compact and convex set so large that

$$-\cup_{j=1}^N \operatorname{supp}(\chi_j) \subset K,$$

and so Qv is entire with $Qv \in \mathrm{PW}^m(\mathbb{R}^n)$. □

Define the reflection map A by $A\psi(x) = \psi(-x)$ for all $x \in \mathbb{R}^n$ on $\psi \in C^\infty(\mathbb{R}^n)$. It takes $\mathcal{S}(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n)$ continuously to themselves. The transpose of Q is AQA. Using this fact and Lemma 3.5, we extend Q, hence P, by duality:

Definition 3.7 Define $Q : u \mapsto Qu : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle Qu, \psi\rangle = \langle u, AQA\psi\rangle \quad \text{for all} \quad \psi \in \mathcal{S}(\mathbb{R}^n).$$

Lemma 3.8 $\mathrm{Op}(p)Ps = s$ *holds for any* $s \in \mathcal{E}'(\mathbb{R}^n)$.

Proof By Lemma 3.4, if $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}\langle \mathrm{Op}(p)Ps, \psi\rangle &= \langle s, APA\mathrm{Op}(p)^*\psi\rangle \\ &= \langle s, APA\mathcal{F}(p\mathcal{F}^{-1}\psi)\rangle \\ &= \langle s, AP\mathrm{Op}(p)A\psi\rangle \\ &= \langle s, A^2\psi\rangle \\ &= \langle s, \psi\rangle.\end{aligned}$$

□

Applying Lemma 3.8, we get the fundamental solution $\mu_0 = P\delta_0$ for the operator. Using e.g. [12, Theorems 5.2 and 7.1], or similar in [5], it is smooth in $x \neq 0$, and

$$\mu_0 \in H^t_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{if} \quad t < d - \frac{n}{2}.$$

Lemma 3.9 *μ_0 is weakly approximated in $H^t_{\mathrm{loc}}(\mathbb{R}^n)$ by a $\mathrm{PW}^m(\mathbb{R}^n)$ sequence.*

Proof Take a bump function $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ holds for all $|x| < 1$. Put $\eta_k(x) = \eta(\frac{x}{k})$ for all $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. By Lemma 3.6, $P\mathcal{F}^{-1}\eta_k \in \mathrm{PW}^m(\mathbb{R}^n)$. Given any $u \in H^{-t}_{\mathrm{comp}}(\mathbb{R}^n)$, then for k large enough, we get

$$\begin{aligned}|\langle \mu_0 - P\mathcal{F}^{-1}\eta_k, u\rangle|^2 &= \left|\left\langle \mathcal{F}^{-1}\left(\frac{\chi}{p}(1-\eta_k)\right), u\right\rangle\right|^2 \\ &= \left|\left\langle \frac{\chi}{p}(1-\eta_k), \mathcal{F}^{-1}u\right\rangle\right|^2 \\ &\leq \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2t}\left|\frac{\chi}{p}(1-\eta_k)\right|^2 d\xi\right)\left(\int_{\mathbb{R}^n}\langle\xi\rangle^{-2t}|\mathcal{F}^{-1}u(\xi)|^2\, d\xi\right)\end{aligned}$$

and so $P\mathcal{F}^{-1}\eta_k \to \mu_0$ weakly in $H^t_{\mathrm{loc}}(\mathbb{R}^n)$ as $k \to \infty$. □

Lemma 3.10 *Suppose that $p^{-1}(0)$ is embedded as a real-analytic submanifold. Then* $\ker \mathrm{Op}(p)$ *consists of functions $v \in \mathrm{PW}^\infty(\mathbb{R}^n)$ of the form*

$$v(x) = \sum_{j=1}^{N}\sum_{k\leq k_j-1}\left\langle (\Psi_j^*)^{-1}(u_{j,k}\otimes \partial^k_{z_n}\delta_0)(\xi), e^{ix\cdot\xi}\right\rangle,$$

where $u_{j,k} \in \mathcal{E}'(U^0_j)$ and Ψ_j are precisely as stated in Theorem 3.2.

Proof Observe that $p\mathcal{F}v = 0$ implies $\mathrm{supp}\,\mathcal{F}v \subset p^{-1}(0)$ so that $\mathcal{F}v$ is compact. Again by [7, Theorem 7.3.1], $v \in \mathrm{PW}^\infty(\mathbb{R}^n)$. Observe then that

$$0 = \Psi_j^*(\chi_j p\mathcal{F}v) = c_j z_n^{k_j}\Psi_j^*(\chi_j\mathcal{F}v),$$

and since c_j is never zero, by [7, Theorem 2.3.5], we must have

$$\Psi_j^*(\chi_j \mathcal{F} v) = \sum_{k \le k_j - 1} (2\pi)^n u_{j,k} \otimes \partial_{z_n}^k \delta_0,$$

where $u_{j,k} \in \mathcal{E}'(\mathbb{R}^{n-1})$ are some distributions supported inside the $z_n = 0$ slice of U_j. It follows that

$$\begin{aligned} v(x) &= \mathcal{F}^{-1}\Big(\sum_{j=1}^{N} \chi_j \mathcal{F} v\Big)(x) \\ &= \sum_{j=1}^{N} \sum_{k \le k_j - 1} \mathcal{F}^{-1}(\Psi_j^{-1})^* \Big((2\pi)^n u_{j,k} \otimes \partial_{z_n}^k \delta_0\Big)(x) \\ &= \sum_{j=1}^{N} \sum_{k \le k_j - 1} \Big\langle (\Psi_j^*)^{-1}(u_{j,k} \otimes \partial_{z_n}^k \delta_0)(\xi), e^{i x \cdot \xi} \Big\rangle. \end{aligned}$$

□

The main Theorem 3.2 is finally obtained by combining the above partial results. Unfortunately, it is impossible to obtain Ψ_j explicitly for any given multiplier symbol. But if, for example, $p^{-1}(0)$ is the real-analytic boundary of some star-convex domain, we can replace the charts by a single deformation Ψ of the boundary onto a sphere. Given p, we look for Ψ so that $\Psi^* p$ factorizes. Our main theorem gives

$$\mu_0(x) = \mathcal{F}^{-1}\Big(\frac{\chi}{p}\Big)(x) + Q\delta_0(x),$$

where χ appropriately suppresses a region surrounding $p^{-1}(0)$ on which Ψ is defined (Fig. 1).

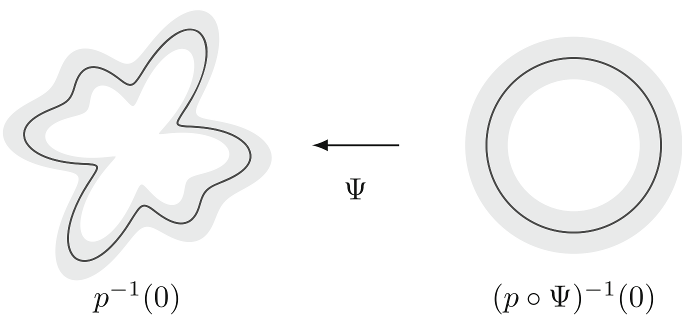

Fig. 1 Deformation of a star-convex zero-set onto a circle

3.1 Sums of powers of Δ_g

Let g be a positive-definite symmetric matrix. Consider for $d \in \mathbb{N}$ and $\{c_j\}_{j=0}^d \subset \mathbb{C}$ with $c_d = 1$ and $c_0 \neq 0$ the multiplier

$$\mathrm{Op}(p) = \sum_{j=0}^{d} c_j \Delta_g^{\frac{j}{2}}.$$

The symbol p is taken into a polynomial form by the map

$$\Psi : (0, \infty) \times \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus 0 : (r, \omega) \mapsto r g^{-\frac{1}{2}} \omega.$$

Pulling back, we find that

$$(p \circ \Psi)(r, \omega) = c(r) \prod_{j=1}^{m} (r - r_j)^{m_j},$$

where c is a polynomial with no root in $[0, \infty)$, and $r_j > 0$ are the positive real roots. Let $C_{j,k}$ be the unique coefficients in $\prod_{j=1}^m (r-r_j)^{-m_j} = \sum_{j=1}^m \sum_{k=1}^{m_j} C_{j,k}(r-r_j)^{-k}$. Pick $\chi \in C^\infty(\mathbb{R}^n)$ so that $1-\chi \circ \Psi \in C_0^\infty((0,\infty)\times\mathbb{S}^{n-1})$ is independent of $\omega \in \mathbb{S}^{n-1}$, and equal to 1 in a neighbourhood of $\cup_{j=1}^m \{r_j\} \times \mathbb{S}^{n-1}$, all of the spheres of radius r_j. If the multiplicities satisfy $m_j < n$, we have

$$Q\delta_0(x) = \sum_{j=1}^{m} \sum_{k=1}^{m_j} B_{j,k} \int_0^\infty \ln|r - r_j|\, \partial_r^k \Big[(1 - \chi \circ \Psi)(r) \frac{r^{\frac{n}{2}}}{c(r)} \frac{J_{\frac{n}{2}-1}(r|g^{-\frac{1}{2}}x|)}{|g^{-\frac{1}{2}}x|^{\frac{n}{2}-1}} \Big] dr,$$

where $J_{\frac{n}{2}-1}$ is the cylindrical Bessel function of order $\frac{n}{2} - 1$, and

$$B_{j,k} = -\frac{\det\, g^{-\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}(k-1)!} C_{j,k}.$$

3.2 A perturbation of Δ_g

Let $\arg\xi$ be the multi-valued argument of $\xi \in \mathbb{R}^2$. Consider instead the multiplier symbol

$$p : \mathbb{R}^2 \to \mathbb{R} : \xi \mapsto |\xi|_g^2 - \Big(1 + a\cos(n \arg\xi)\Big),$$

where $n \in \mathbb{N}$ and $a < \frac{1}{2}$. It is certainly real-analytic near its star-shaped zero-set. This p is taken into normal crossing form by the map

$$\Psi : (-\tfrac{1}{2}, \tfrac{1}{2}) \times (0, 2\pi) \to \mathbb{R}^2 : (r, \theta) \mapsto \Big(r + 1 + a\cos(n\theta)\Big)^{\frac{1}{2}} g^{-\frac{1}{2}}(\cos\theta, \sin\theta).$$

It is clear that Ψ is a diffeomorphism onto its image, not covering the whole zero-set, as depicted in Fig. 2. But a representation using only Ψ is still possible, because it misses just a single point. Pulling back, we find that

$$(p \circ \Psi)(r, \theta) = r,$$

and we compute

$$\det d\Psi(r, \theta) = \frac{1}{2} \det g^{-\frac{1}{2}}.$$

Pick χ such that $1 - \chi \circ \Psi \in C^\infty((-\frac{1}{2}, \frac{1}{2}) \times (0, 2\pi))$ does not depend on $\theta \in (0, 2\pi)$, and is compactly supported in $(-\frac{1}{4}, \frac{1}{4})$ and equal to 1 in a neighbourhood of $r = 0$. We tacitly extend χ by one to all of $\mathbb{R}^2$. In that case, we have

$$Q\delta_0(x) = -\frac{\det g^{-\frac{1}{2}}}{8\pi^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln|r|\, \partial_r\Big[(1 - \chi \circ \Psi)(r) \int_0^{2\pi} e^{ix\cdot\Psi(r,\theta)}\, d\theta\Big]\, dr,$$

and $\ker \mathrm{Op}(p)$ consists of v of the form

$$v(x) = \Big\langle u(\theta), e^{ix\cdot\Psi(0,\theta)}\Big\rangle,$$

where $u \in \mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$ is a distribution on the space of 2π-periodic smooth functions. We could replace $|\xi|_g^2$ in p by any integer power of $|\xi|_g^2$ and still get a similar result,

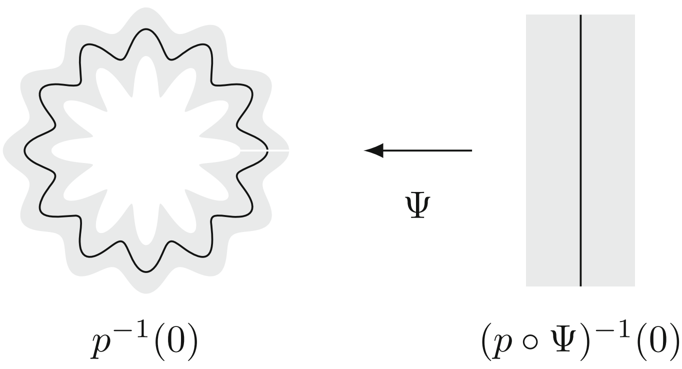

Fig. 2 Covering the zero-set of p except for a point. Here $a = \frac{1}{4}$ and $n = 12$

provided that we adjust the fractional power $\frac{1}{2}$ in Ψ in accordance with this change. A similar technique can be applied to sums of powers of such multipliers too.

Acknowledgements The author wishes to thank the anonymous reviewer. Their useful comments and suggestions helped to improve this paper.

Funding Open access funding provided by Technical University of Denmark.

References

1. Atiyah, M.: Resolution of singularities and division of distributions. Commun. Pure Appl. Math. **XXIII**, 145–150 (1970)
2. Björk, J.E.: Rings of Differential Operators. North Holland, Amsterdam (1979)
3. Camus, B.: Fundamental solutions of homogeneous elliptic differential operators. Bull. Sci. Math. **130**, 264–268 (2006)
4. Camus, B.: Fundamental solutions for a class of non-elliptic homogeneous differential operators. Bull. Sci. Math. **132**, 486–499 (2008)
5. Grubb, G.: Distributions and Operators, vol. 252. Springer, Berlin (2009)
6. Hormander, L.: On the division of distributions by polynomials. Ark. Mat. **3**, 555–568 (1958)
7. Hörmander, L.: The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis, Classics in Mathematics. Springer, Berlin (2003)
8. Hörmander, L.: The Analysis of Linear Partial Differential Operators II: Differential Operators with Constant Coefficients, Classics in Mathematics. Springer, Berlin (2005)
9. Lojasiewiez, S.: Sur le probleme de division. Stud. Math. **18**, 87–136 (1959)
10. Ortner, N., Wagner, P.: A survey on explicit representation formulae for fundamental solutions of linear partial differential operators. Acta Appl. Math. **47**, 101–124 (1997)
11. Rabier, P.: l^p regularity of homogeneous elliptic differential operators with constant coefficients on $\mathbb{R}^n$. Rev. Mat. Iberoam. **34**, 423–454 (2015)
12. Shubin, M.A.: Pseudodifferential Operators and Spectral Theory. Springer, Berlin (2001)