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Publication date:
2024

Document Version
Peer reviewed version

Twelfth International Conference on Computational Fluid Dynamics (ICCFD12), Kobe, Japan, July 14-19, 2024

ICCFD12-2024-xxxx

An unfitted high-order spectral element method for incompressible Navier-Stokes equations with a free surface: The pressure problem

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Abstract: We propose a novel unfitted high-order spectral element solver for nonlinear water wave simulations based on an incompressible Navier-Stokes formulation with a free surface. The solver achieves its unfitted high-order accuracy from a basis of polynomial functions that are combined with the recently developed shifted boundary method. This unfitted method allows for a natural curvature representation, circumvents complex meshing and re-meshing, and avoids the small-cut-cell problem. For water wave problems, there are multiple boundaries that are complex, curved, moving, and deforming in time. Hence, they are suitable for unfitted methods like the shifted boundary approach. For this work, we solely consider the dynamic pressure problem of the complete Navier-Stokes formulation.

Keywords: spectral element method, shifted boundary method, incompressible free surface Navier-Stokes, high-order, unfitted mesh.

1 Introduction

Accurate high-fidelity numerical modeling of fluid flows is of key importance across many engineering disciplines. Conventional element/volume/cell-based numerical methods rely on body-fitted meshes, where the discrete representation of the fluid domain conforms to its boundaries. However, meshing and re-meshing serves as a major computational burden when boundaries are complex or move/deform in time. To circumvent this, unfitted/embedded/immersed techniques can be adopted, where the problem is modeled on a simple, structured, and linear mesh.

One promising unfitted approach is the shifted boundary method (SBM), originally proposed by Main & Scovazzi [1, 2] for finite elements. The *true* domain, Ω, is embedded onto a simple background mesh. Then, a *surrogate* domain, $\overline{\Omega}_h$, is formed by discarding elements outside Ω. The problem is now solved on $\overline{\Omega}_h$, where the boundary conditions for $\overline{\Gamma}_h = \partial\overline{\Omega}_h$ is *shifted* from $\Gamma = \partial\Omega$ by the use of a simple mapping and Taylor series expansions. Thus, the SBM presents itself as an unfitted method for solving boundary value problems that naturally incorporates complex geometry and curvature. This is done while retaining optimal convergence rates and circumventing the well-known problem of small-cut-cells. See Figure 1 for a conceptualization of the SBM domain.

We present a new unfitted high-order accurate spectral element method (SEM) for solving the dynamic pressure problem in an incompressible Navier-Stokes formulation with a free surface. This serves as our initial contribution towards precise and high-fidelity modeling of wave-wave and wave-structure interactions in the field of offshore engineering.

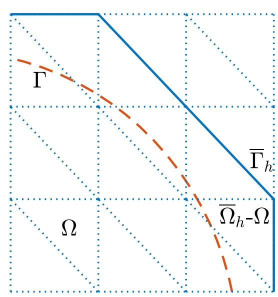

Figure 1: Concept of the SBM domain with $\Omega \subseteq \overline{\Omega}_h$. Also, $\overline{\Omega}_h \subseteq \Omega$ is a possibility.

2 Mathematical Problem and Numerical Approach

Consider a fluid domain, Ω, in the xz-plane, bounded from above by the time-dependent free surface, Γ^η, and from below by the bathymetry, Γ^{b}. From the incompressible conservation equations of mass and momentum, a Poisson problem for the dynamic pressure, p_D, is derived as

$$\begin{aligned} \boldsymbol{\nabla}^2 p_D &= -\boldsymbol{\nabla}^2 p_S + \mu \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}^2\boldsymbol{u} - \rho\boldsymbol{\nabla}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}, \quad \text{in} \quad \Omega, \\ p_D &= 0, \quad \text{on} \quad \Gamma^\eta, \\ \boldsymbol{n}\cdot\boldsymbol{\nabla} p_D &= -\boldsymbol{n}\cdot\boldsymbol{\nabla} p_S + \mu\boldsymbol{n}\cdot\boldsymbol{\nabla}^2\boldsymbol{u} - \rho\boldsymbol{n}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{n}\cdot\boldsymbol{F}, \quad \text{on} \quad \Gamma^{\mathrm{b}}, \end{aligned} \tag{1}$$

where $\boldsymbol{n} = (n_x, n_z)^T$ is the outward facing normal, $\boldsymbol{\nabla} = (\partial_x, \partial_z)^T$ is the Cartesian gradient operator, $\boldsymbol{F} = (0, -\rho g)^T$ is the external force vector, and $\boldsymbol{u} = (u, w)^T$ is the velocity vector. Moreover, ρ is the density, g is the gravitation acceleration constant, μ is the kinematic viscosity, and $p_S = \rho g(\eta - z)$ is the static pressure with η being the free surface elevation measured from $z = 0$.

To solve (1), we adopt the SBM combined with an arbitrary-order accurate SEM discretization as presented in [3]. See e.g. [4] for more on the SEM. Previous work on the use of the SBM for the incompressible Navier-Stokes equations has been reported in [5, 2]. In this contribution, we focus on the use of a high-order SEM polynomial representation and on an efficient solution to the dynamic pressure Poisson problem. The governing equations are solved on a simple structured mesh of quadrilateral elements, $\overline{\Omega}_h$, upon which boundary information is projected on $\overline{\Gamma}^\eta$ and $\overline{\Gamma}^{\mathrm{b}}$ from Γ^η and Γ^{b}, respectively.

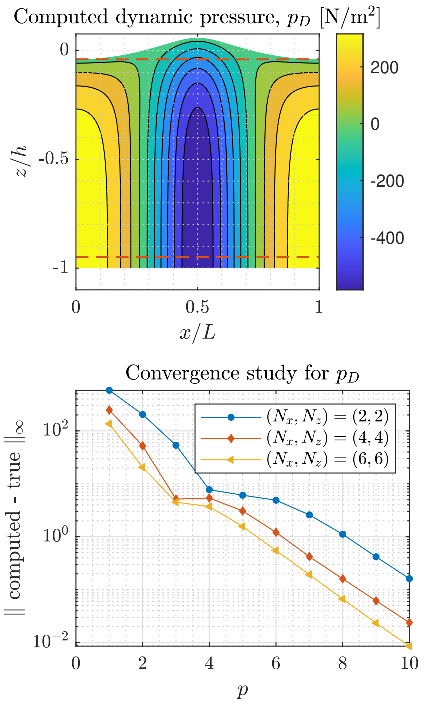

Figure 2: Top: The computed dynamic pressure distribution for a numerical solution with fifth-order basis functions on a mesh with $(N_x, N_z) = (6, 6)$ elements. Bottom: Convergence study (p) for p_D on three different meshes.

3 Preliminary Results and Perspectives

As preliminary results, we show the computed dynamic pressure distribution in Figure 2 (top) for a stream function wave solution due to Fenton [6]. For the dynamic pressure equations to be valid, we assume $\mu = 0$, from where a true solution can be derived accordingly. Moreover, the domain is periodic in the x-direction, and the depth, h, is assumed constant. The computational properties are: wave number: $kh = 2\pi$, maximum steepness: $(H/L)_{\max} = 70\%$, polynomial order of basis function: $p = 5$, and mesh: 6 elements in each direction, $(N_x, N_z) = (6, 6)$. From the figure, a contour plot shows the pressure distribution. Despite the simple mesh (indicated by the red dashed lines), the wave dynamics are still captured due to the SBM technique.

For the same wave setup, a p-convergence study is performed as shown in Figure 2 (bottom). The study is carried out on three meshes, where the polynomial order (p) of the basis functions are varied as $p \in \{1, ..., 10\}$. From this, an exponential decrease in the error is observed when increasing the polynomial order of the basis functions. Some irregularities around $p = 3$ are noted which are subject to further investigation.

For the final paper, we aim to present more elaborate results and verifications of the unfitted pressure solver. In particular, attention will be given to the smoothness of the pressure. Moreover, the influence of high-order approximations will be investigated for the fluctuating behavior highlighted in [5]. This solver will serve as a key component in the final free surface Navier-Stokes formulation.

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