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Triangulations Admit Dominating Sets of Size $2n/7$.*

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Abstract

We show that every planar triangulation on $n > 10$ vertices has a dominating set of size $2n/7 = n/3.5$. This approaches the $n/4$ bound conjectured by Matheson and Tarjan [12], and improves significantly on the previous best bound of $17n/53 \approx n/3.117$ by Špacapan [18].

From our proof it follows that every 3-connected n -vertex near-triangulation (except for 3 sporadic examples) has a dominating set of size $n/3.5$. On the other hand, for 3-connected near-triangulations, we show a lower bound of $3(n-1)/11 \approx n/3.666$, demonstrating that the conjecture by Matheson and Tarjan [12] cannot be strengthened to 3-connected near-triangulations.

Our proof uses a penalty function that, aside from the number of vertices, penalises vertices of degree 2 and specific constellations of neighbours of degree 3 along the boundary of the outer face. To facilitate induction, we not only consider near-triangulations, but a wider class of graphs (*skeletal* triangulations), allowing us to delete vertices more freely. Our main technical contribution is a set of *attachments*, that are small graphs we inductively attach to our graph, in order both to remember whether existing vertices are already dominated, and that serve as a tool in a divide and conquer approach. Along with a well-chosen potential function, we thus both remove and add vertices during the induction proof.

We complement our proof with a constructive algorithm that returns a dominating set of size $\leq 2n/7$. Our algorithm has a quadratic running time.

1 Introduction

A *dominating set* in an n vertex graph G is a subset S of the vertices of G such that every vertex in G either is in S or neighbours a vertex in S . When studying dominating sets, one is typically interested in making them as small as possible. The minimum size of a dominating set in G is denoted by $\gamma(G)$. Instead of studying the minimum dominating set for a particular graph, Matheson and Tarjan [12] originally asked if one can determine an upper bound on γ for classes of graphs. In particular, they studied two classes of graphs: plane *triangulations* and plane *near-triangulations*. Here a plane graph refers to a planar graph, i.e. a graph that may be embedded in the plane in such a way that no two edges cross, together with such a crossing-free embedding in the plane. A plane graph is *internally triangulated* if every bounded face is bounded by a triangle. A *near-triangulation* is a 2-connected internally triangulated plane graph, and a *triangulation* is a near-triangulation with exactly three boundary vertices. Matheson and Tarjan [12] showed that for any plane near-triangulation G , it holds that $\gamma(G) \leq \frac{n}{3}$. They also showed that this result is tight in the sense that there exists an infinite family of plane near-triangulation such that for every graph in the infinite family, the minimum dominating set has size exactly a third of the number of vertices in the graph, i.e. $\gamma(G) = \frac{n}{3}$. However, for triangulations they were only able to prove an upper bound of $\gamma(G) \leq \frac{n}{3}$ and provide an infinite family where every graph in the family required $\frac{n}{4}$ points to dominate. Aside from some small sporadic examples, they conjectured that asymptotically $\gamma(G) \leq \lfloor n/4 \rfloor$ when G is a sufficiently large triangulation. This problem has proved difficult to approach, and for over 20 years, there were no improvements that applied to all triangulations. Recently, Špacapan [18] gave the first improved bound for general triangulations, when he showed that, in every large enough triangulation, $\gamma(G) \leq \lfloor 17n/53 \rfloor \approx n/3.117$.

In broad terms, the problem has been approached in two different ways. Either 1) papers have tried to find combinatorial objects – like a colouring or a Hamiltonian cycle – with certain properties that allows one to extract

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Reference	Class of graphs	Size of dom. set	Comment
Matheson & Tarjan [12]	Near-triangulations	$\frac{n}{3}$	
Campos & Wakabayashi [2] Tokunaga [17]	Maximal outer-planar graphs	$\lceil \frac{n+t}{4} \rceil$	Here t is the no. of degree-2 vertices.
King & Pelsmayer [8]	Plane triangulations with max-degree 6	$\frac{n}{4}$	
Liu & Pelsmayer [10]	Plane triangulations with max-degree 6	$\frac{n}{6} + c$	For some constant c .
Plummer, Ye & Zha [13, 14]	Hamiltonian triangulations	$\frac{5n}{16}$	For $n \geq 23$.
Špacapan [18]	Triangulations	$\frac{17n}{53}$	
New	Triangulations	$\frac{2n}{7}$	
New	3-connected near-triangulations	$\frac{2n}{7}$	

Table 1: Upper bounds for the size of a minimum dominating set for various graph classes.

a small dominating set, or 2) one has attempted some sort of inductive or reduction based approach in order to try and iteratively reduce the problem complexity until it can be handled directly. The problem is elusive, as the above approaches has to deal with two obstructions: Firstly, the bound does not hold for small values of n as there are small, sporadic counter examples, which means that one has to be careful when reducing the problem. Secondly, it seems difficult to pin-point enough structure in general triangulations to guarantee a combinatorial object with strong enough properties. This has motivated researchers to either restrict the problem to sub-classes of (near-)triangulations containing more structure like for instance triangulations with maximum degree 6 [8, 10], Hamiltonian triangulations [14] or maximal outerplanar graphs [2, 17], or to consider broader classes of graphs in which it is easier to reduce the problem to one of smaller complexity [18]. See Table 1 for an overview of known upper bounds.

More specifically, in the first line of research: King and Pelsmayer [8] confirmed the conjecture for graphs of maximum degree 6, and Liu and Pelsmayer [10] strengthened this result to show that in fact for these graphs $\gamma(G) \leq \frac{n}{6} + c$ for some constant c . Plummer, Ye and Zha [13] studied first 4-connected plane triangulations, which in particular are Hamiltonian [16] and have minimum degree at least 4, and showed the existence of a dominating set of size $\leq \max\{\lceil \frac{2n}{7} \rceil, \lfloor \frac{5n}{16} \rfloor\}$. Then in [14], they showed that for Hamiltonian triangulations of size at least 23 it holds that $\gamma(G) \leq \frac{5n}{16}$. Finally, in maximal outerplanar graphs, even more fine-grained results are known: Campos and Wakabayashi [2] showed $\gamma(G) \leq \lfloor (n+t)/4 \rfloor$ where t is the number of degree-2 vertices. In the three last results, a good understanding of the obstructions to achieving an $n/4$ bound, such as degree-2 vertices, is key. Tokunaga [17] gave an elegant proof of this bound via a coloring method.

On the other hand, Špacapan [18] considered a more general class of graphs that he denoted *weak near-triangulations*. He showed how to reduce weak near-triangulations while staying inside the graph class, until one eventually ends up with an irreducible weak near-triangulation. These irreducible weak near-triangulations contained enough structure for Špacapan to subsequently construct a small dominating set, if one begins with a triangulation. However, in Špacapan's framework one has to argue this in a manual fashion separately from the arguments that handle the reductions.

In our approach, we extend the reduction step to make the construction of the small dominating set automatic. Similarly to Špacapan, we consider a more general class of graphs, however, in our case, we consider what we call *skeletal triangulations*. In order to avoid having to extract the dominating set manually, we employ a penalty function that in a more fine-grained manner accounts for the cost of performing certain reductions. This penalty function not only penalises degree-2 vertices (more specifically degree-2 cut vertices and 'ears'), but also a new type of attachment that we call *facial bad 5-wheels*. To illustrate the importance of penalising these 5-wheels, we show an infinite family of near-triangulations with no degree-2 vertices in which the smallest dominating set has size $\frac{3n}{10}$. Furthermore, we show that our analysis using this penalty function is tight in the sense that there

Reference	Class of graphs	Size of dom. set
Matheson & Tarjan [12]	Near-triangulations.	$\frac{n}{3}$
Matheson & Tarjan [12]	Triangulations.	$\frac{n}{4}$
New	Near-triangulations with minimum degree 3.	$\frac{3n}{10}$
New	3-connected near-triangulations.	$\frac{3n}{11}$
New	Near-triangulations with neither bad 5-wheels nor degree 2 vertices.	$\frac{2n}{7}$
New	Eulerian triangulations.	$\frac{n}{4}$

Table 2: Lower bounds for the size of a minimum dominating set for various graph classes.

exists an infinite family of near-triangulations which contain none of the penalised attachments and admit no dominating sets with fewer than $\frac{2n}{7}$ vertices. Finally, we show that only penalising attachments arising from a 2-cut is not sufficient to achieve an $\frac{n}{4}$ bound for non-penalised near-triangulations, as we provide an infinite family of 3-connected near-triangulations with $\gamma(G) = 3n/11 - O(1)$. Interestingly, this indicates a big difference between what is conjectured for triangulations and what holds for 3-connected near-triangulations. In Table 2 we give an overview over known lower bounds and the new lower bounds we introduce in this paper. We introduce the lower bound constructions in Section 1.1.

Since we conduct our inductive argument over a broader class of graphs, we can reduce very aggressively while staying in the same class of graphs, but we now have the added difficulty of also carrying the penalty function along, as we reduce. In order to be able to do so, we apply two techniques. Firstly, we show how to encode the fact that some vertices might already be dominated in our candidate dominated set, while staying in the same graph class. To do so, we fuse small attachments to the graph and thus increase the number of vertices and create small cuts. Secondly, in order to be able to handle this broader class of graphs, we study small cuts of size ≤ 2 , and show that we may replace one side of the cut by one of a finite list of examples that ‘acts as’ the cut that was just replaced on the rest of the graph. This allows us to assume that G is ‘almost’ 3-connected, which makes a deletion-based induction proof feasible. We elaborate further on this in the proof-overview section. In the next section, we sum up our contributions.

1.1 Our contributions The following generalization of near-triangulations allows for cut vertices, which gives some added flexibility when deleting vertices.

DEFINITION 1.1. (SKELETAL TRIANGULATION) *A skeletal triangulation is a connected internally triangulated planar graph in which every vertex has degree ≥ 2 .*

Every near-triangulation is a skeletal triangulation. In fact, every connected weak-near triangulation [18] is a skeletal triangulation, but not vice-versa.

DEFINITION 1.2. (PROBLEMATIC CONFIGURATIONS) *Let G be a skeletal triangulation. An ear in G is a facial triangle with at least one vertex of degree 2 in G . A bad 5-wheel is a subgraph $H \subseteq G$ isomorphic to the 5-wheel such that the outer 4-cycle in H contains at least two consecutive G -boundary vertices of degree 3, called a 3-pair. (See Figure 1.)*

Upper bounds. Our main result is the following

THEOREM 1.1. *Let G be a skeletal triangulation on $n > 10$ vertices. Let e , f and t be the number of ears, bad 5-wheels and degree-2 cut vertices in G , respectively. Then, $\gamma(G) \leq \lfloor \frac{n+e/2+f/2+t/2}{3.5} \rfloor$.*

COROLLARY 1.1. *Let G be a triangulation or 3-connected near-triangulation on $n > 10$ vertices. Then, $\gamma(G) \leq \lfloor \frac{n}{3.5} \rfloor$.*

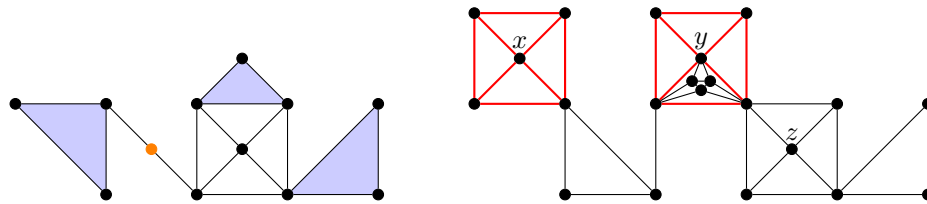


Figure 1: (Left) a skeletal triangulation with three ears (blue) and a degree-2 cut vertex (orange). (Right) A skeletal triangulation with two bad 5-wheels (red), centered at x and y . Note that there is no bad 5-wheel centered at z , as there are no two consecutive boundary vertices of degree 3 on that 5-wheel.

Proof. As G is 3-connected, there are no ears or cut vertices and at most one bad 5-wheel. If there is a bad 5-wheel, then G has exactly four boundary vertices, two of which have degree at least four. Deleting the boundary edge between the latter destroys the 5-wheel without creating any new problematic configurations. Finally, Theorem 1.1 gives the result with $e = f = t = 0$. \square

Lower bounds. The following (infinite) families of examples motivate our definition of skeletal triangulations and our selection of problematic configurations. Matheson and Tarjan [12] constructed near-triangulations with $\gamma(G) = n/3$ and triangulations with $\gamma(G) = n/4$, see Figure 2. The limiting factor in these examples are vertices of degree 2 and 3, respectively. We construct near-triangulations with (a) no degree-2 vertices, no bad 5-wheels and $\gamma(G) = 2n/7$ and (b) no-degree 2 vertices and $\gamma(G) = 3n/10$, see Figure 3. This shows that $n/3.5$ is best possible given our choice of problematic configurations and that penalizing both degree-2 vertices and bad 5-wheels is necessary to achieve the $n/3.5$ bound.

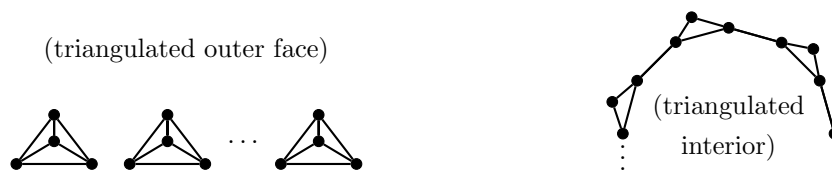


Figure 2: (Left) copies of K_4 with the outer face triangulated arbitrarily shows that $n/4$ is needed [12]. (Right) an outer-planar near-triangulation where every third vertex of the outer face has degree 2 motivates penalising ears [12].

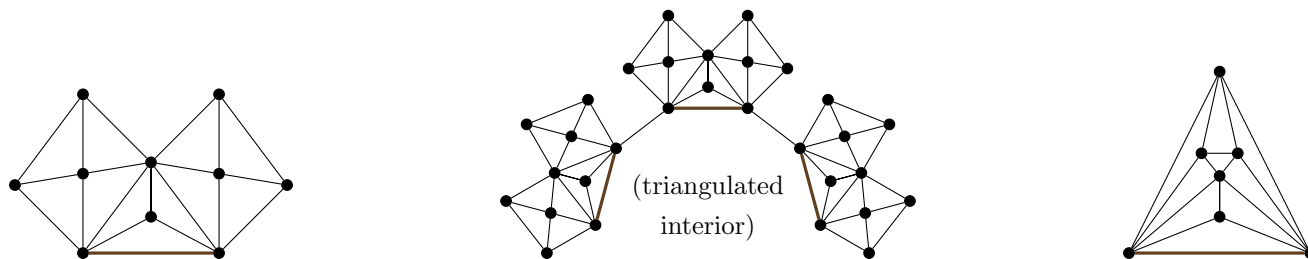


Figure 3: (Left) a 10-vertex graph that requires a dominating set of size 3. (Middle) attaching this to every second edge of a triangulated polygon yields a class with $10k$ vertices and $\gamma = 3k$. (Right) a 7 vertex graph demanding 2 vertices to dominate. By a similar construction as in (middle), this gives a class with $7k$ vertices and $\gamma = 2k$.

Requiring skeletal triangulations to (a) be connected avoids disjoint unions of octahedra, with $\gamma(G) = n/3$, and (b) have minimum degree 2 avoids caterpillars with $\gamma(G) = n/2$. A penalty of $\frac{1}{2}$ on degree-2 cut vertices is motivated by the example shown in Figure 4.

So far, our lower bounds for near-triangulations describe classes of graphs with many chords. It is natural to think that chords, or two-cuts, are the sole reason $n/4$ does not suffice for these graphs. In [17], Tokunaga

conjectured that every 3-connected near-triangulation satisfies $\gamma(G) \leq \lfloor (n + 2)/4 \rfloor$. We construct 3-connected triangulations with $\gamma(G) = 3n/11 - O(1)$, refuting this conjecture, see Figure 5 (left and middle). To our surprise, this either shows a stark difference between triangulations and 3-connected near-triangulations, or is a counter-indication of the $n/4$ conjecture. In particular, a proof of the $n/4$ conjecture might have to approach triangulations via 4-connected triangulations and separating triangles, in order to break through this $3n/11$ barrier.

Finally, we construct a triangulation with no odd-degree vertices and $\gamma(G) = n/4$, see Figure 5 (right). Placing disjoint copies of this graph and carefully triangulating the outer face (similar to Figure 2 (left)) yields an infinite class of even graphs with $\gamma(G) = n/4$. In particular, the conjectured $n/4$ bound is best possible even in the absence of degree-3 vertices.



Figure 4: (Left) A gadget G on 10 vertices with $\gamma(G) = 3$. (Right) Attaching copies of this gadget to each vertex of a triangulated polygon yields a skeletal triangulation with k degree-2 cut vertices and $\gamma = (n + \frac{k}{2})/3.5$.

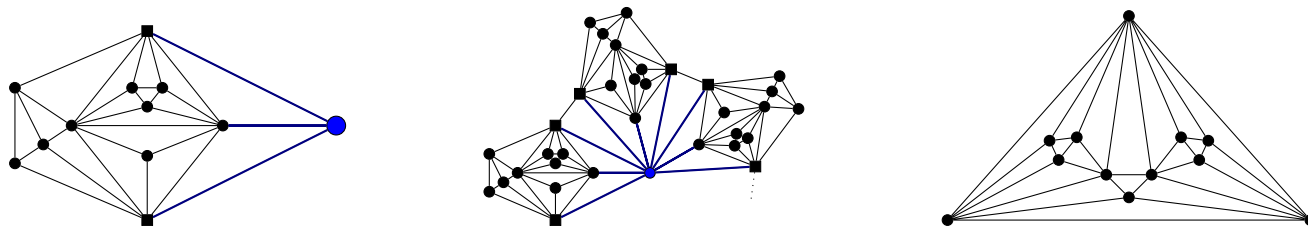


Figure 5: (Left) Even if the rightmost large vertex is added to the dominating set for free, it still requires 3 vertices to dominate the remaining 11. (Middle) Identifying several copies of these by the large vertex and adding edges between the rectangular vertices (■) to make it 3-connected yields a graph class with $n = 11k + 1$ and $\gamma = 3k$. (Right) An even graph with $\gamma = n/4$.

Algorithm We complement our upper bound of Theorem 1.1 and Corollary 1.1 by a quadratic-time algorithm. The algorithm takes as input an n -vertex skeletal triangulation G with $\Phi = n + (e + f + t)/2$, the algorithm outputs a dominating set of size $\leq 2\Phi/7$. Particularly, if G is a triangulation, it outputs a dominating set of size $\leq 2n/7$.

1.2 Further related work The original bound due to Matheson and Tarjan [12] has been extended to other surfaces than the sphere. In [7] and [15] it is shown that the $\frac{n}{3}$ bound holds for a larger class of graphs, including those embedded on a torus, the Klein bottle, and the projective plane. In [5], these results are further extended to all triangulations embedded on a closed surface. A related question is to upper bound the domination number of planar graphs with small diameter [6, 11]. Here, it is shown that all sufficiently large planar graphs with diameter 3 can be dominated by at most 6 vertices.

Total domination of a graph differs from domination in that every vertex must have a neighbour in the totally dominating set, regardless of whether the vertex belongs to the set itself. Lemanska, Zuazua, and Zylinski [9] study the total domination number of maximal outerplanar graphs, and show that $2n/5$ vertices suffice to totally dominate this class of graphs, as is also shown in [4]. Similarly to our problem, they also have to consider reducing one side of a two-cut. Since maximal outerplanar graphs allow induction over the dual tree, this allows them to provide a simple and elegant proof. In [3], these bounds for maximal outerplanar graphs serve as a stepping stone for improving the bound for general triangulations; namely via what in retrospect can be interpreted as a form of *attachment* as the ones introduced in the paper at hand.

2 Main Techniques

In this section, we develop the techniques needed for our proof. Our proof is by induction and, roughly speaking, consists of two main parts. In the first part, we deal with any small cuts, such as bridges, cut vertices or chords. Here, a general classification scheme for dominating sets along small cuts allows us to replace one side of the cut by a finite list of “minimal attachments”, which can then be checked by hand. In the second part, G is 3-connected, and we want to add a vertex to our dominating set and delete many of its neighbors without creating too many problematic configurations. Here, some of our minimal attachments help keep track of which vertices are in the dominating set or are already dominated. Also, in many cases, these deletions create bridges or cut vertices, which necessitates working with skeletal triangulations.

2.1 Skeletal triangulations with small vertex cuts

Fusing Let $u \in G$ be a cut vertex in a skeletal triangulation, whose removal splits G into two components C_1, C_2 . For $i \in \{1, 2\}$, let G_i be the graph induced by $C_i \cup \{u\}$ and let $u_i \in G_i$ correspond to $u \in G$. See Figure 8 on page 12. If $\deg_{G_i}(u) = 1$, then G_i need not be a skeletal triangulation, but this is the only obstruction.

DEFINITION 2.1. A rooted skeletal triangulation (G, u) with root u is a connected triangulated planar graph in which every vertex except possibly u has degree ≥ 2 .

In the above setting, (G_1, u_1) and (G_2, u_2) are both rooted skeletal triangulations. The following operation reconstructs G from G_1 and G_2 :

DEFINITION 2.2. (FUSING) Let (G_1, u_1) and (G_2, u_2) be rooted skeletal triangulations. We fuse (G_1, u_1) to (G_2, u_2) by taking the disjoint union $G_1 \sqcup G_2$ and identifying u_1 with u_2 .

In the above setting, G is the graph obtained by fusing (G_1, u_1) to (G_2, u_2) .

Classifying dominating sets In the same setting, let $S \subseteq G$ be a dominating set. Put $S_1 = S \cap G_1$, then S_1 dominates all vertices in $G_1 - u_1$, and u_1 is either (a) contained in S_1 , (b) dominated by S_1 , or (c) not dominated by S_1 . Intuitively speaking, extending S_1 to a small dominating set in G is easiest in case (a) and most difficult in case (c). In fact, this case distinction perfectly describes which vertices in G_2 still have to be dominated. This motivates the following definition.

DEFINITION 2.3. (ACTS AS) A rooted dominating set S in (G, u) is a set that dominates every vertex except maybe u . $\gamma(G, u)$ denotes the size of a minimum rooted dominating set. In the following, each case excludes all the previous ones. We say G acts as

- AB* if G has a rooted dominating set of size $\gamma(G, u)$ that contains u ,
- LR* if G has a rooted dominating set of size $\gamma(G, u)$ that dominates u , and
- Nope* otherwise.

Figure 12 on page 16 depicts the smallest rooted skeletal triangulation of each act-as type. The following lemma illustrates how small ABs and small LRs can be used to “remember” that certain vertices are required to be in S or are already dominated.

LEMMA 2.1. (FORCING AND COVERING) Let G be a skeletal triangulation, with boundary vertex u . Let $s \in \mathbb{N}$ be arbitrary.

1. Let H_1 be obtained by fusing a small AB to (G, u) . Then, H_1 has a dominating set of size s if and only if G has a dominating set of size s that contains u .
2. Let H_2 be obtained by fusing a small LR to (G, u) . Then, H_2 has a dominating set of size $s + 1$ if and only if G has a set of size s that dominates all vertices except maybe u .

Proof. Straight-forward. \square

The following lemma illustrates that rooted skeletal triangulations of the same act-as type are essentially interchangeable. This enables us to use a divide-and-conquer approach later on.

LEMMA 2.2. (FUSING REPLACEMENT) *Let $(G_1, u_1), (G_2, u_2)$ be skeletal triangulations. Let H be obtained by fusing (G_2, u_2) to (G_1, u_1) . Then $\gamma(H) - \gamma(G_2, u_2)$ depends only on (G_1, u_1) and on the act-as type of (G_2, u) (but not on the precise graph-structure of G_2).*

Proof. Looking at suitable rooted dominating sets shows that

$$\gamma(H) = \gamma(G_1, u_1) + \gamma(G_2, u_2) + c$$

where $c = -1$ if both G_1 and G_2 act as AB, $c = 1$ if both act as Nope, and $c = 0$ otherwise. □

Near-triangulations and chords The above machinery allows us to deal with bridges and cut vertices. For chords (=2-vertex-cuts) in near triangulations, we use similar techniques:

DEFINITION 2.4. *A rooted near-triangulation (G, u, v) is a near-triangulation G with boundary edge $\{u, v\}$. A rooted dominating set $S \subseteq G$ is a set that dominates every vertex except maybe u and v . $\gamma(G, u, v)$ denotes the size of a minimum rooted dominating set. If (G_1, u_1, v_1) and (G_2, u_2, v_2) are rooted near-triangulation, then we attach the latter to the former by taking the disjoint union $G_1 \sqcup G_2$ and identifying u_1 with u_2 and v_1 with v_2 .*

The attaching operation creates a chord, see Figure 7 on page 11. Having two root vertices greatly increases the number of acts-as types.

DEFINITION 2.5. *Let (G, u, v) be a rooted near-triangulation. Let $\gamma = \gamma(G, u, v)$. In the following, each case excludes all the previous ones. We say G acts as*

- A+B if G has a dominating set of size γ that contains u and v ,*
- OR if G has two dominating sets of size γ , one that contains u and one that contains v ,*
- A if G has a dominating set of size γ that contains u ,*
- B if G has a dominating set of size γ that contains v ,*
- AND if G has a dominating set of size γ and a dominating set of size $\gamma + 1$ that contains both u and v ,*
- L+R if G has a dominating set of size γ ,*
- OCTA if G has two rooted dominating set of size γ , one that dominates u and one that dominates v , plus a dominating set of size $\gamma + 1$ that contains u and v .*
- L OR R if G has two rooted dominating sets of size γ , that, respectively, dominate u and v ,*
- L if G has a rooted dominating set of size γ that dominates u ,*
- R if G has a rooted dominating set of size γ that dominates v , and*
- None if otherwise.*

Here, the list of cases considered is tailored to our proof and non-exhaustive. For example, in the L, R, None cases, we could also distinguish whether G has a dominating set of size $\gamma + 1$ that contains both u and v . Figure 13 on page 17 gives an example of each case.

General k -vertex cuts The idea of considering “rooted” instances is a technical contribution that we hope has applications in other classes of graphs. It can be generalized to k -vertex cuts for any $k \geq 1$: Pick k distinguished vertices. For each of those vertices, we may (a) require it to be in the dominating set, (b) require it to be dominated, or (c) not require anything. This yields 3^k combinations of restrictions in total. The acts-as type is the $3 \times \dots \times 3$ tensor that describes how much each restriction increases the size of a minimum rooted dominating set. One can show that the entries in such a tensor decrease along each dimension and decrease by at most 1 at a time, and that the number of such tensors is $\leq 3^{k \cdot 2^{(k-1)}}$.

2.2 Penalty functions To facilitate a divide-and-conquer approach that deals with bridges, cut vertices and chords, we want to generalize Theorem 1.1 to the rooted setting. There, we should only count problematic configurations that remain even after a fusing operation.

DEFINITION 2.6. (PENALTY FUNCTION) *If G is a skeletal triangulation on n vertices, define $\Phi(G) = n + e/2 + f/2 + t/2$ where e, f and t are the number of ears, bad 5-wheels and degree-2 cut vertices in G , respectively.*

If (G, u) is a rooted skeletal triangulation on $n + 1$ vertices, define $\phi(G, u) = n + e/2 + f/2 + t/2 + r/2$ where e is the number of ears containing a degree-2 vertex other than u , f is the number of bad 5-wheels with a

3-pair disjoint from u , t is the number of degree-2 cut vertices not equal to u , and r is 1 if $\deg_G(u) = 1$ and zero otherwise.

If (G, u, v) is a rooted near-triangulation on $n + 2$ vertices, define $\phi(G, u, v) = n + e/2 + f/2 + t/2$ where e is the number of ears containing a degree-2 vertex other than u or v , f is the number of bad 5-wheels with a 3-pair disjoint from $\{u, v\}$, and t is the number of degree-2 cut vertices not equal to u or v .

The following properties follow immediately from the definitions:

$$\begin{aligned} \phi(G, u) + 1 &\leq \Phi(G) \leq \phi(G, u) + 1.5 \\ \phi(G, u, v) + 2 &\leq \Phi(G) \leq \phi(G, u, v) + 2.5 \end{aligned}$$

If G is obtained by attaching (G_2, u_2, v_2) to (G_1, u_1, v_1) , then

$$\Phi(G) = \phi(G_1, u_1, v_1) + \phi(G_2, u_2, v_2) + 2.$$

If G is obtained by fusing (G_2, u_2) to (G_1, u_1) , then

$$\Phi(G) \leq \phi(G_1, u_1) + \phi(G_2, u_2) + 1,$$

with equality if $\deg_{G_1}(u_1) \neq 1 \neq \deg_{G_2}(u_2)$.

Theorem 1.1 states that, for any skeletal triangulation G on $n > 10$ vertices, $\gamma(G) \leq \lfloor \Phi(G)/3.5 \rfloor$. Using Theorem 1.1, we can show the following:

COROLLARY 2.1. (SKELETAL TRIANGULATION ACTS-AS BOUNDS) *Let (G, u) be a rooted skeletal triangulation. If (G, u) acts as*

$$AB \text{ then } \phi(G, u) \geq 3.5 \cdot \gamma(G, u) - 1,$$

$$LR \text{ then } \phi(G, u) \geq 3.5 \cdot \gamma(G, u),$$

$$Nope \text{ then } \phi(G, u) \geq 3.5 \cdot \gamma(G, u) + 1.5.$$

Proof. If G has ≤ 10 vertices, check by hand. In practice, only three specific triangulations have to be checked. Suppose (G, u) acts as AB. Let H be obtained by attaching a small LR to (G, u) . Then $\Phi(H) \leq \phi(G, u) + 3.5 + 1$ and $\gamma(H) = \gamma(G, u) + 1$. Theorem 1.1 yields

$$\phi(G, u) \geq \Phi(H) - 4.5 \geq 3.5\gamma(H) - 4.5 = 3.5\gamma(G, u) - 1.$$

Suppose (G, u) acts as LR. Let H be obtained by attaching a small AB to (G, u) . Then $\Phi(H) \leq \phi(G, u) + 2.5 + 1$ and $\gamma(H) = \gamma(G, u) + 1$. Theorem 1.1 yields

$$\phi(G, u) \geq \Phi(H) - 3.5 \geq 3.5\gamma(H) - 3.5 = 3.5\gamma(G, u).$$

Suppose (G, u) acts as Nope. If $\deg_G(u) \neq 1$, then G is a skeletal triangulation, with $\Phi(G) \leq \phi(G, u) + 1.5$ and $\gamma(G) = \gamma(G, u) + 1$. Theorem 1.1 yields

$$\phi(G, u) \geq \Phi(G) - 1.5 \geq 3.5\gamma(G) - 1.5 = 3.5\gamma(G, u) + 2$$

If $\deg_G(u) = 1$, then let v be the neighbor of u and let $H = G - u$. Then (H, v) acts as LR, $\phi(H, v) \leq \phi(G, u) - 1.5$ and $\gamma(H, v) = \gamma(G, u)$. The LR case yields

$$\phi(G, u) \geq \phi(H, v) + 1.5 \geq 3.5\gamma(H, v) + 1.5 = 3.5\gamma(G, u) + 1.5.$$

□

Note that these bounds are tight in the examples in Figure 12 on page 16. For rooted near-triangulations, there are analogous bounds, but those are not tight in all cases.

COROLLARY 2.2. (NEAR-TRIANGULATION ACTS-AS BOUNDS) *Let (G, u, v) be a rooted near-triangulation. If (G, u, v) acts as*

$$\begin{aligned}
& A+B, \text{ OR then } \phi(G, u, v) \geq 3.5 \cdot \gamma(G, u, v) - 2, \\
& \quad A, B \text{ then } \phi(G, u, v) \geq 3.5 \cdot \gamma(G, u, v) - 1, \\
& \text{AND, } L+R, \text{ OCTA, } L \text{ OR } R \text{ then } \phi(G, u, v) \geq 3.5 \cdot \gamma(G, u, v), \\
& \quad L, R \text{ then } \phi(G, u, v) \geq 3.5 \cdot \gamma(G, u, v) + 0.5, \\
& \quad \text{None then } \phi(G, u, v) \geq 3.5 \cdot \gamma(G, u, v) + 1.5.
\end{aligned}$$

Proof. Similar to the proof of Corollary 2.1. We omit the details. \square

Lower-bound examples These bounds suggest that the most efficient building blocks for lower-bound examples typically act as A+B or OR. Indeed, the building blocks in Figure 3 both act as A+B. The left building block in Figure 5 acts as A on the bottom edge and as B on the top edge, and was found by enumerating 3-connected near-triangulations with Plantri [1] and filtering for large domination numbers and interesting combinations of acts-as types. Filtering for acts-as types enables us to find this building block at $n = 12$ already, even though the constructed example only exceeds the $\lfloor n/4 \rfloor$ bound at $n = 21$.

2.3 The divide-and-conquer technique Consider a skeletal triangulation. The machinery we introduced so far allows us to assume that, for any bridge, cut vertex, or 2-vertex-cut in G , one side of the cut has constant size. We illustrate this in the case of cut vertices.

Let G be a skeletal triangulation obtained by fusing (G_2, u_2) to (G_1, u_1) . Suppose, for example, that (G_2, u_2) acts as AB. Let H be obtained by fusing a small AB, denoted (H_2, v) , to (G_1, u_1) . Then, by Lemma 2.2,

$$\gamma(G) = \gamma(H) + \gamma(G_2, u_2) - \gamma(H_2, v).$$

The small AB satisfies $\phi(H_2, v) = 3.5 \cdot \gamma(H_2, v) - 1$. By Corollary 2.1, $\phi(G_2, u_2) \geq 3.5 \cdot \gamma(G_2, u_2) - 1$. Therefore,

$$\begin{aligned}
\Phi(G) - \Phi(H) &\geq \phi(G_1, u_1) + \phi(G_2, u_2) + 1 - \left(\phi(G_1, u_1) + \phi(H_2, v) + 1 \right) \\
&= \phi(G_2, u_2) - \phi(H_2, v) \\
&\geq 3.5 \left(\gamma(G_2, u_2) - \gamma(H_2, v) \right) = 3.5 \left(\gamma(G) - \gamma(H) \right)
\end{aligned}$$

In particular, if H satisfies Theorem 1.1, i.e. if $\Phi(H) \geq 3.5\gamma(H)$, then so does G .

In the actual proof, some care has to be taken to avoid circular arguments inside the induction step. For example, in order to use Corollary 2.1, G_2 should not be a small AB / LR / Nope.

2.4 Dealing with the 3-connected case Once G is 3-connected, we manually pick specific (high-degree) vertices to be in the dominating set and then delete the picked vertices and sufficiently many of their neighbors. Intuitively, this should always be possible by looking at a large enough section of the graph, given that we expect 3-connected near-triangulations to satisfy $\gamma \leq 3n/11$, which is a bit stronger than the $2n/7$ bound we are aiming for.

The main difficulty is that deleting a high-degree vertex may yield many problematic configurations, increasing Φ . Even worse, the graph might get separated into many small components, for which Theorem 1.1 no longer holds. We deal with these issues in three different ways: (1) delete edges instead of vertices. Deleting an edge only affects the two incident vertices, which is much easier to handle than a vertex deletion. For example, a (non-bridge) boundary edge between two vertices of degree ≥ 5 may always be deleted, as this never creates any problematic configurations. (2) whenever we delete vertices, fuse a small LR to any vertex that is already dominated. This gets rid of any problematic configurations caused by that vertex. (3) when picking a vertex to be in the dominating set, instead of deleting that vertex, fuse a small AB to it. Fusing a small AB increases Φ by 2.5 while (often) increasing γ by 1. This has essentially the same effect as decreasing Φ by 1 by deleting the vertex, but avoids the aforementioned issues around vertex deletions.

Nevertheless, this part of our proof contains many cases. This is likely unavoidable: Since Theorem 1.1 does not hold for $n = 10$, our proof needs to look at a large enough piece of the graph to avoid a specific 10-vertex example, see Theorem 3.1.

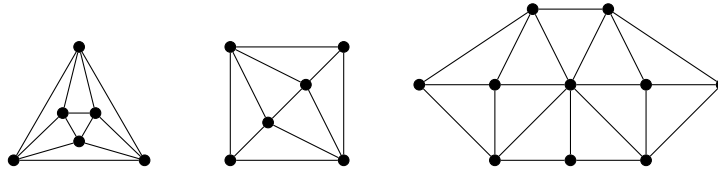


Figure 6: From left to right: octahedron, 3-bifan, special 4343434-heptagon.

3 A sketch of the full proof

The precise version of Theorem 1.1 is the following.

THEOREM 3.1. *Let G be a skeletal triangulation that is not the 3-bifan, octahedron or the special 4343434-heptagon (see Figure 6). Then $\gamma(H) \leq \lfloor \frac{\Phi(G)}{3.5} \rfloor$.*

We prove this via induction, using a carefully chosen partial ordering on skeletal triangulations.

DEFINITION 3.1. *Let G and H be skeletal triangulations. We say G is smaller than H if, in decreasing order of importance: (1) G has fewer interior vertices (than H), (2) G has fewer bridges, (3) G has smaller Φ , (4) G has fewer blocks (2-connected components), (5) G has fewer vertices, (6) G has fewer degree-2 vertices.*

Here is a rough sketch of how we prove the theorem.

1. If G has a bridge, apply Corollary 2.1 to both sides of the bridge, then check the 9 combinations of AB / LR / Nope. The corollary may be used as both sides have fewer bridges (and at most as many interior vertices) as G , hence Theorem 1.1 holds for these graphs by induction. Conclusion: G has no bridges.
2. If G has a cut vertex, use Section 2.3 to replace one side by a small AB / LR / Nope. Then, replace the AB by an A attachment, delete the Nope and LR. In the LR case, delete the cut vertex too if there is a problematic configuration. This is justified by Lemma 2.1. Conclusion: G has no cut vertices.
3. If G has a chord and one side acts as AND, L+R, OCTA, L OR R, L, R, None, use Corollary 2.2 to bound that side and delete it, possibly together with one of the endpoints of the chord. Some of the bounds in Corollary 2.2 are not tight on any small example, so we cannot just replace these attachments by small ones.
4. If G has a chord, then one side acts as A+B, OR, A, B. Replace that side by a small OR, OR, A, B. (Here, Corollary 2.2 is tight.)
5. Handle small As and Bs by (a) deleting boundary edges leading to high-degree vertices and (b) deleting neighboring low-degree vertices that are dominated by the “forced” vertex in the A / B.
6. Handle small ORs by considering many cases. After this step, we may conclude: G is 3-connected as there are no chords.
7. Try deleting any boundary edge without creating problematic configurations. If this does not work, then the boundary of G consists of problematic configurations that are “covered” by a single edge. After this step, we conclude: G has many degree-3 boundary vertices. Moreover, the degrees on the boundary of G follow one of the following patterns: 345^+43 , 345^+3 , 35^+43 , 35^+3 , 34443 , 3443 , 343 , 33 .
8. Handle the degree patterns 345^+43 , 345^+3 , 35^+43 and 34443 , followed by 33 , followed by 35^+3 and 3443 . This involves checking many cases by hand. Using the techniques from Section 2.4, this is not difficult, but it is a bit tedious. Conclusion: only the degree patterns 3443 and 343 remain.
9. Handle the remaining cases while avoiding the 3-bifan, octahedron and special 4343434-heptagon. If G has many boundary vertices, this is easy, but if G has few vertices, we have to be careful to avoid these examples.

4 Full Proof in Detail

In the following, we will provide the details sketched in Sections 2 and 3.

4.1 Attaching “Attaching” small near-triangulations to a boundary edge of a given planar graph turns out to be a useful tool for manipulating dominating sets. The natural way of doing this is by creating a 2-cut.

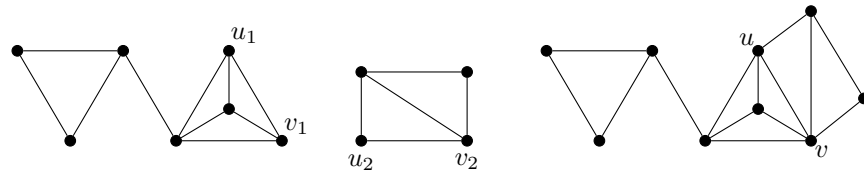


Figure 7: A skeletal triangulation with boundary edge $\{u_1, v_1\}$, a rooted near triangulation with base u_2, v_2 and the result of attaching the later to u_1, v_1 .

DEFINITION 4.1. (ROOTED NEAR-TRIANGULATION) A rooted near-triangulation $G = (G, u, v) = (V, E, u, v)$ with base u, v is a near-triangulation G with a boundary edge u, v . The base vertices are u, v and the base edge is $\{u, v\}$.

DEFINITION 4.2. (ATTACHING) Let G_1 be a skeletal triangulation with boundary edge u_1, v_1 . Let $G_2 = (G_2, u_2, v_2)$ be a rooted near-triangulation. We can attach G_2 to u_1, v_1 as follows: Consider the disjoint union $G_1 \sqcup G_2$ and identify $u := u_1 = u_2$ and $v := v_1 = v_2$.

See Figure 7 for an example. The resulting graph G is a (unrooted) skeletal triangulation with chord u, v , with one “side” (including u, v) being isomorphic to G_1 and the other side isomorphic to G_2 . If G_1 was a near-triangulation, then so is G . The following generalization of dominating sets behaves well with regards to attaching.

DEFINITION 4.3. (ROOTED DOMINATING SET) A rooted dominating set of a rooted near-triangulation with base u, v is a subset $S \subseteq V$ such that $N[S] \supseteq V \setminus \{u, v\}$, i.e. such that every vertex except maybe u, v is either in S or has a neighbor in S . We denote the size of a minimum rooted dominating set by $s(G, u, v)$.

LEMMA 4.1. In the setting of Definition 4.2, let S_1 be a dominating set in G_1 and let S_2 be a rooted dominating set in G_2 . Then $S_1 \cup S_2$ is a dominating set in the graph G obtained by attaching G_2 to u_1, v_1 . In particular, $s(G) \leq s(G_1) + s(G_2, u, v)$.

Proof. Straight-forward. \square

4.2 Fusing “Fusing” is the analog of attaching, but now both graphs are skeletal triangulations and we identify only a single vertex. The natural way of doing this is by creating a 1-cut. We want to entertain the possibility of creating a bridge or degree-2 cut vertex this way, so we allow the “fused” vertex to have degree one.

DEFINITION 4.4. (ROOTED SKELETAL TRIANGULATION) A rooted skeletal triangulation $G = (G, u) = (V, E, u)$ with root u is a connected planar graph in which every bounded face is a triangle and every vertex except maybe u has degree ≥ 2 .

DEFINITION 4.5. (FUSING) Let G_1 be a skeletal triangulation with boundary vertex u_1 or a rooted skeletal triangulation with root u_1 . Let G_2 be a rooted skeletal triangulation with root u_2 . We can fuse G_2 to u_1 as follows: Consider the disjoint union $G_1 \sqcup G_2$ and identify $u := u_1 = u_2$.

See Figure 8 for an example. The resulting graph G is a skeletal triangulation with cut vertex u , with each block in G corresponding to a block in exactly one of G_1, G_2 . The following generalization of dominating sets behaves well with regards to fusing.

DEFINITION 4.6. (ROOTED DOMINATING SET) A rooted dominating set in a rooted skeletal triangulation G with root u is a subset $S \subseteq V$ such that $N[S] \supseteq V \setminus \{u\}$, i.e. such that every vertex except maybe u is either in S or has a neighbor in S . We denote the size of a minimum rooted dominating set by $s(G, u)$.

LEMMA 4.2. In the setting of Definition 4.5, let S_1 be a dominating set in G_1 and let S_2 be a rooted dominating set in G_2 , then $S_1 \cup S_2$ is a dominating set in the graph G obtained by fusing G_2 to u_1 . In particular, $s(G) \leq s(G_1) + s(G_2, u_2)$.

Proof. Trivial. \square

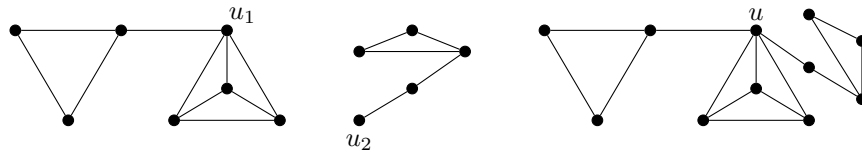


Figure 8: A skeletal triangulation with boundary vertex $\{u_1\}$, a rooted skeletal triangulation with base u_2 and the result of fusing the later to u_1 .

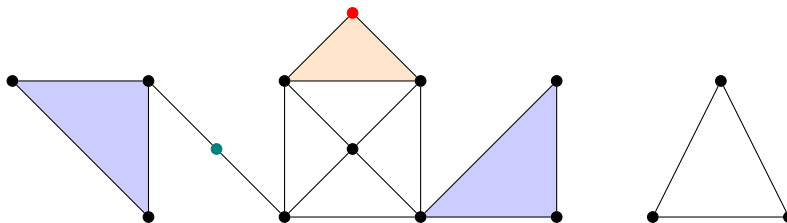


Figure 9: On the left: A skeletal triangulation with an ear (orange) with ear tip (red), two pivoting triangles (blue) and a degree-2 cut vertex (teal). On the right: An isolated triangle.

4.3 The Penalty Function Let G be a skeletal triangulation, rooted skeletal triangulation or rooted near-triangulation.

DEFINITION 4.7. (CLUSTER) Let G be a graph and let P be some property that a vertex in G may or may not have, e.g. being adjacent to a fixed vertex u or having a certain degree. A cluster in G is a maximal connected subgraph consisting only of vertices that satisfy property P .

DEFINITION 4.8. (EARS AND PIVOTING TRIANGLES) An Ear in G is a facial triangle with exactly one vertex of degree two, called the ear tip. Equivalently, an ear tip is a size-1 cluster of degree-2 non-cut vertices.

A pivoting triangle in G is a facial triangle with exactly two vertices of degree two. Equivalently, a pivoting triangle is a size-2 cluster of degree-2 non-cut vertices + their shared neighbor.

An isolated triangle in G is a facial triangle with exactly three vertices of degree two. This implies the whole graph is a triangle. (See Figure 9.)

DEFINITION 4.9. (BAD 5-WHEEL) A bad 5-wheel in G is a subgraph $H \subseteq G$ isomorphic to the 5-wheel such that the outer 4-cycle in H contains two consecutive G -boundary vertices of degree 3, called a 3-pair. A bad 5-wheel contains 1 – 4 such 3-pairs. (See Figure 10.)

DEFINITION 4.10. (PENALTY FUNCTION) Let G be a skeletal triangulation. The penalty function is $\Phi = \Phi(G) = n + e/2 + f/2$ where n is the number of vertices, e is the number of ears, pivoting triangles, isolated triangles and degree-2 cut vertices and f is the number of bad 5-wheels.

Let (G, u, v) be a rooted near-triangulation. The penalty function is $\phi = \phi(G, u, v) = n + e/2 + f/2$ where n is the number of non- u, v vertices, e is the number of ears with tip $\neq u, v$, and f is the number of bad 5-wheels that contain 3-pair disjoint from $\{u, v\}$.

Let (G, u) be a rooted skeletal triangulation. The penalty function is $\phi = \phi(G, u) = n + e/2 + f/2 + r/2$ where n is the number of non- u vertices, e is the number of ears with tips $\neq u$, pivoting triangles (which may include u), isolated triangles and non- u degree-2 cut vertices, f is the number of bad 5-wheels that contain a 3-pair disjoint from $\{u\}$ and r is 1 if $\deg(u) = 1$ and 0 otherwise.

DEFINITION 4.11. A low-degree problem in a (rooted) near triangulation or (rooted) skeletal triangulation is anything that contributes to Φ or ϕ other than the n term. A vertex is involved in a low-degree problem if it is

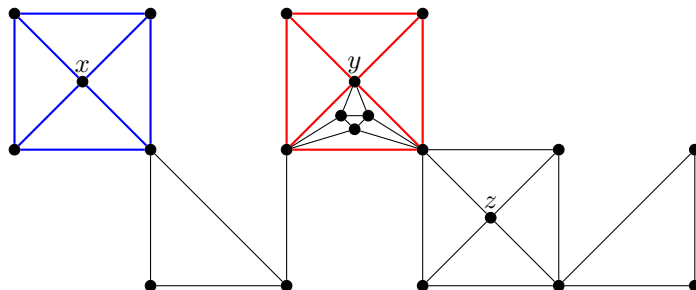


Figure 10: A skeletal triangulation with two bad 5-wheels: the blue one centered at x and the red one centered at y . Note that there is no bad 5-wheel centered at z , as there are no two consecutive boundary vertices of degree 3 on that 5-wheel.

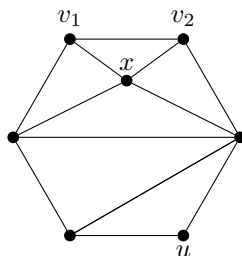


Figure 11: A near-triangulation G with an ear tip u and a bad 5-wheel $N[x]$, in which v_1, v_2 are consecutive boundary vertices of degree 3, i.e. a 3-pair. Note that $\{v_1, v_2\}$ is a cluster of x -adjacent degree-3 vertices. In terms of the penalty function, $n = 7$, $e = 1$, $t = 1$ and $\Phi = 9$. If we instead consider this as a rooted near-triangulation with base v_1, v_2 , then $n = 5$, $e = 1$, $t = 0$ and $\phi(G, v_1, v_2) = 6$. If we consider it as a rooted skeletal triangulation with root u , then $n = 6$, $e = 0$, $t = 1$ and $\phi(G, u) = 7$.

a degree-2 vertex in an ear / pivoting triangle / isolated triangle / degree-2 cut vertex or if it is in a 3-pair in a bad 5-wheel.

REMARK 4.1. In a skeletal triangulation, a vertex that is a cut vertex is never involved in any low-degree problems. In a near-triangulation a vertex that is incident to a chord is never involved in any low-degree problems.

Note that (rooted) near-triangulations do not contain pivoting triangles; all their low-degree problems are ears or bad 5-wheels. See Figure 11 for an example involving one of each. The three penalty functions are closely related.

LEMMA 4.3. (De-rooting) If G is a skeletal triangulation with boundary vertex u then

$$\phi(G, u) + 1 \leq \Phi(G) \leq \phi(G, u) + 1.5.$$

Moreover, $\Phi(G) = \phi(G, u) + 1.5$ if and only if $u \in G$ is an ear tip, a degree-2 cut vertex, or is contained in every 3-pair of a bad 5-wheel in G . Otherwise, $\Phi(G) = \phi(G) + 1$.

If (G, u, v) is a rooted near-triangulation, then

$$\phi(G, u, v) + 2 \leq \Phi(G) \leq \phi(G, u, v) + 2.5$$

Moreover, $\Phi(G) = \phi(G, u, v) + 2.5$ if and only if one or both of $u, v \in G$ is an ear tip or u, v is a 3-pair of a bad 5-wheel and G is not an isolated 5-wheel. Otherwise, i.e. if neither case happens, then $\Phi(G) = \phi(G, u, v) + 2$.

Proof. Trivial, but this lemma is very important, so you should check it. \square

At first, it might seem a bit weird to have the +1 and +2 here, but this leads to nicer formulas when fusing and attaching:

LEMMA 4.4. (Detaching and defusing) Let G_1 be a skeletal triangulation with boundary vertex u_1 and let (G_2, u_2) be a rooted skeletal triangulation. Let G be the graph obtained by fusing G_2 to u_1 . Then

$$\Phi(G) \leq \Phi(G_1) + \phi(G_2, u_2) \leq \Phi(G) + 0.5.$$

If moreover $\deg_{G_2}(u_2) \neq 1$, i.e. if G_2 is a skeletal triangulation, then

$$\Phi(G) = \phi(G_1, u_1) + \phi(G_2, u_2) + 1$$

Let H_1 be a skeletal triangulation with boundary edge u_1, v_1 and let (H_2, u_2, v_2) be a rooted near-triangulation. Let H be the graph obtained by attaching H_2 to u_1, v_1 . Then

$$\Phi(H) \leq \Phi(H_1) + \phi(H_2, u_2, v_2) \leq \Phi(H) + 0.5.$$

If moreover H_1 is a near-triangulation, then

$$\Phi(H) = \phi(H_1, u_1, v_1) + \phi(H_2, u_2, v_2) + 2.$$

Proof. We make the following crucial observation: If there is a low-degree problem in G (or H), then the same low-degree problem occurs in exactly one of G_1 and G_2 . This needs exactly the right definition of a bad 5-wheel. The rest is straight-forward. \square

5 Setting up

5.1 The Main Result The main result in this document is the following;

THEOREM 5.1. Let G be a skeletal triangulation, then G has a dominating set of size $\left\lceil \frac{\Phi(G)}{3.5} \right\rceil$ unless G is one of the following:

- octahedron

- 3-bifan (= octahedron minus one edge)
- special 4343434 heptagon

We call these the sporadic examples. They are depicted in Figure 6.

In Section 6, we prove Theorem 5.1 via induction. This requires a very particular ordering on skeletal triangulations.

DEFINITION 5.1. (SMALLER) Let G_1, G_2 be skeletal triangulations. We say G_1 is smaller than G_2 if

- G_1 has fewer interior vertices than G_2 , or the same number and
- G_1 has fewer bridges than G_2 , or the same number and
- G_1 has smaller Φ than G_2 , or the same number and
- G_1 has fewer blocks (i.e. 2-connected components) than G_2 , or the same number and
- G_1 has fewer vertices than G_2 , or the same number and
- G_1 has fewer degree-2 vertices than G_2 .

Formally, we show the following

PROPOSITION 5.1. (INDUCTION STEP) Let G be a skeletal triangulation that is not one of the sporadic examples. Suppose every G' that is smaller than G satisfies the following:

- (Induction Hypothesis) If G' is not one of the sporadic examples, then it has a dominating set of size $\left\lfloor \frac{\Phi(G')}{3.5} \right\rfloor$.

Then G has a dominating set of size $\left\lfloor \frac{\Phi(G)}{3.5} \right\rfloor$.

REMARK 5.1. (PITFALLS) The conditions in the induction hypothesis might look innocuous, but we have to be very careful when applying the induction hypothesis to some graph G' we constructed. Here are some common pitfalls and how we might deal with them:

- If G' is disconnected, then it is not a skeletal triangulation. Solution: Handle cut vertices, chords or shared interior neighbors in earlier cases. This allows for stronger connectivity assumptions in later cases.
- If G' contains a leaf, then it is not a skeletal triangulation. Solution: When deleting things, pay special attention to vertices that lose two or more neighbors. A vertex of degree ≥ 3 can only turn into a leaf if it loses at least two neighbors.
- G' might be a sporadic example. Solution: The sporadic examples are all 3-connected. If G' is the result of an attaching or fusing operation, then G' is not 3-connected and hence not a sporadic example.

5.2 Acts as Intuitively speaking, if a rooted skeletal triangulation (or rooted near triangulation) has a minimum rooted dominating set that contains the root (or base), this makes it “easier” to find small dominating sets in the graph obtained by fusing (or attaching). Our goal is to establish a precise relation between this “easier” and the minimum possible penalty ϕ .

DEFINITION 5.2. (ACTS AS) Let (G, u) be a rooted skeletal triangulation. We say (G, u) acts as

AB if G has a minimum rooted dominating set that contains u ,

LR if G has a minimum rooted dominating set that dominates u (and G does not act as $A+B$), and

Nope otherwise.

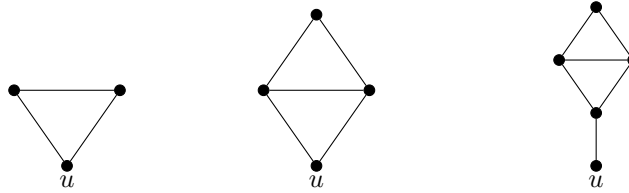


Figure 12: From left to right: small AB, small LR, small Nope, each with root u .

The sporadic examples, rooted at any boundary vertex, all act as Nope. If we know what G acts as, we can make Theorem 5.1 more specific.

THEOREM 5.2. *Let (G, u) be a rooted skeletal triangulation. Let $s = s(G, u)$ and let $\phi = \phi(G, u)$. If (G, u) acts as*

AB then $\phi \geq 3.5s - 1$.

LR then $\phi \geq 3.5s$.

Nope then $\phi \geq 3.5s + 1.5$.

Proof. Follows from Theorem 5.1, see Section 4. □

Figure 12 depicts a small example for each case. The bounds in Theorem 5.2 are tight in those examples. Next, we consider a similar notion and bound for rooted near-triangulations.

DEFINITION 5.3. (ACTS AS) *Let (G, u, v) be a rooted near-triangulation. Let $s = s(G, u, v)$. In the following, each case excludes the preceding ones. We say (G, u, v) acts as*

A+B if G has a rooted dominating set of size s that contains both u and v .

OR if G has a two rooted dominating sets of size s with one containing u and one containing v .

A if G has a rooted dominating set of size s that contains u .

B if G has a rooted dominating set of size s that contains v .

AND if G has a dominating set of size s and a rooted dominating set of size $s + 1$ that contains both u and v .

L+R if G has a dominating set of size s .

OCTA if G has two rooted dominating sets of size s with one dominating u and one dominating v , plus a rooted dominating set of size $s + 1$ that contains both u and v .

L OR R if G has two rooted dominating sets of size s with one dominating u and one dominating v .

L if G has a rooted dominating set of size s that dominates u .

R if G has a rooted dominating set of size s that dominates v .

None if otherwise.

THEOREM 5.3. *Let (G, u, v) be a rooted near-triangulation. Let $s = s(G, u, v)$ and let $\phi = \phi(G, u, v)$. If (G, u, v) acts as*

A+B, OR then $\phi \geq 3.5s - 2$.

A, B then $\phi \geq 3.5s - 1$.

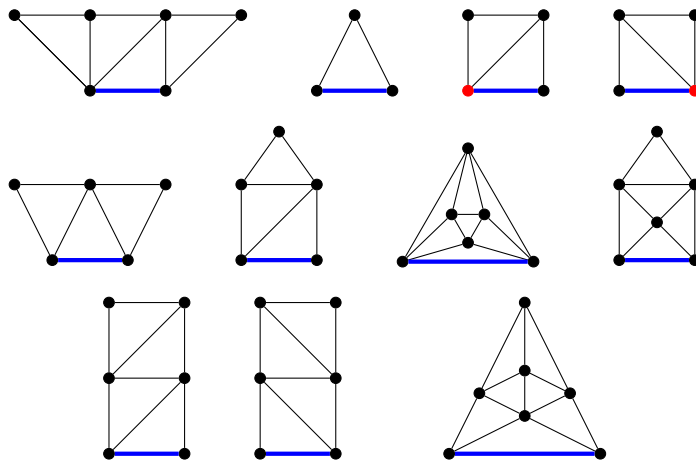


Figure 13: Small examples, with base edge in blue and bold. Each row to be read from left to right. Top row: A+B, OR, A, B, the later two with their red vertex. Middle row: AND, L+R, OCTA, L OR R. Bottom row: L, R, None. It is worth noting that the only non-outerplanar examples here are OCTA, L OR R and None.

Act as	s	ϕ	formula	lower bound
A+B	2	5	$\phi = 3.5s - 2$	$\phi \geq 3.5s - 2$
OR	1	1.5	$\phi = 3.5s - 2$	$\phi \geq 3.5s - 2$
A, B	1	2.5	$\phi = 3.5s - 1$	$\phi \geq 3.5s - 1$
AND	1	3.5	$\phi = 3.5s$	$\phi \geq 3.5s$
L+R	1	3.5	$\phi = 3.5s$	$\phi \geq 3.5s$
OCTA	1	4	$\phi = 3.5s + 0.5$	$\phi \geq 3.5s$
L OR R	1	4.5	$\phi = 3.5s + 1$	$\phi \geq 3.5s$
L, R	1	4.5	$\phi = 3.5s + 1$	$\phi \geq 3.5s + 0.5$
None	1	5	$\phi = 3.5s + 1.5$	$\phi \geq 3.5s + 1.5$

Table 3: The relation between ϕ and the size s of a minimum rooted dominating set. For each act-as type, we compare the small example in Figure 13 to the lower bound in Proposition 5.3.

AND, L+R, OCTA, L OR R then $\phi \geq 3.5s$.

L, R then $\phi \geq 3.5s + 0.5$.

None then $\phi \geq 3.5s + 1.5$.

Proof. Follows from Theorem 5.1, see Section 4. \square

Figure 13 depicts a small example for each case. The most important ones are small A, small B and small OR. Take special note of the *red vertex* in the small A, B. Note in the OCTA, L OR R, L, R cases, the bound in proposition 5.3 is not tight. This is illustrated in Table 3. The loose bounds are sufficient for our proof.

5.3 Toolbox: Replacing attachments

LEMMA 5.1. Let G_0 be a near triangulation with boundary edge u_0, v_0 . Let (G_1, u_1, v_1) and (G_2, u_2, v_2) be rooted near-triangulations that act as the same type. Let H_i be the result of attaching G_i to u_0, v_0 . Then

$$s(H_2) - s(H_1) = s(G_2, u_2, v_2) - s(G_1, u_1, v_1)$$

$$\Phi(H_2) - \Phi(H_1) = \phi(G_2, u_2, v_2) - \phi(G_1, u_1, v_1)$$

Proof. For the statement on s , note that a H_i -dominating set is the disjoint union of a set in G_0 and a rooted dominating set in G_i . Since G_1 and G_2 act as the same type, we can switch between rooted dominating sets of the two that both contain u and/or v or both do not contain u and/or v . The statement on Φ follows from the equality cases in Lemmas 4.3 and 4.4 (unrooting and detaching). \square

5.4 Deleting one problem creates at most one new one Sometimes, we want to delete a low degree vertex that is already dominated for one reason or another. It is crucial that this does not increase Φ by too much.

LEMMA 5.2. (PROBLEMS ARE NOT ADJACENT) *Let G be a skeletal triangulation. Suppose u, v are each involved in distinct low-degree problems that are not degree-2 cut vertices, i.e. each a degree-3 vertex in a 3-pair of a distinct bad 5-wheel or each a degree-2 vertex in a distinct isolated triangle / pivoting triangle / ear. Then u is not adjacent to v .*

Proof. A tedious but straight-forward case analysis. \square

COROLLARY 5.1. (DEGREE BOUND ON PROBLEMS) *Let G be a near-triangulation with boundary vertex u . Suppose $G - u$ is a skeletal triangulation. Then $\Phi(G - u) \leq \Phi(G) - 1 + \lfloor \frac{\deg(u)+1}{2} \rfloor$. In other words, deleting u creates at most $\lfloor \frac{\deg(u)+1}{2} \rfloor$ new low-degree problems.*

Proof. Every newly created low-degree problem contains at least one vertex adjacent to u . Pick one such vertex for each problem, then by the previous lemma, these vertices are not adjacent to each other, so they form an independent set in $N(u)$. $N(u)$ is a path on $\deg(u)$ vertices. \square

LEMMA 5.3. (DELETING PROBLEMS) *Let G be a skeletal triangulation with boundary vertex u . Suppose that $\Phi(G) = \phi(G, u) + 1.5$ and that u is not a cut vertex. Then $u \in G$ is an ear tip or in every 3-pair of a bad 5-wheel and $H := G - u$ is a skeletal triangulation with*

$$\Phi(H) \leq \phi(G, u) + 0.5 = \Phi(G) - 1.$$

Let G be a near-triangulation with boundary edge u, v . Suppose that $\Phi(G) = \phi(G, u, v) + 2.5$. Then at least one of u, v is an ear tip or in the (unique) 3-pair of a bad 5-wheel, suppose it is u . Then $H := G - u$ is a near-triangulation with

$$\Phi(H) \leq \phi(G, u, v) + 1 + 0.5 = \Phi(G) - 1.$$

Proof. Lemma 4.3 shows that u an ear tip or in a 3-pair. If $\deg(u) = 2$, then Corollary 5.1 gives the result. If u is in a 3-pair, then deleting u creates exactly one ear and no new bad 5-wheels. \square

5.5 Toolbox: Covering by Fusing a small LR In the main proof, we sometimes modify the graph and would like to “remember” that some vertex u is already dominated in the original graph, meaning it does not have to be dominated again. Fusing a small LR to u achieves exactly this.

LEMMA 5.4. *Let G be a skeletal triangulation with boundary vertex u . Let H be the graph resulting from fusing a small LR (Figure 12) to u . Then:*

- $\Phi(H) \leq \Phi(G) + 3.5$, with equality if $\deg_G(u) \geq 4$.
- If H has a dominating set of size s , then G has a set of size $\leq s - 1$ that dominates every vertex except maybe u .

Proof. Let $H' \subseteq H$ be the small LR that was fused, including u . Then,

$$\Phi(H) \leq \Phi(G) + \phi(H', u) = \Phi(G) + 3.5.$$

If $\deg(u) \geq 4$, then u is not involved in any low-degree problems, hence equality holds. For the second statement, let D be a H -dominating set of size s . Put $D' = D \setminus (H' - u)$, then clearly D' dominates $G - H'$ as u is a cut vertex in H . Moreover $|D'| < |D| = s$ as D contains at least one vertex of $H' - u$. \square

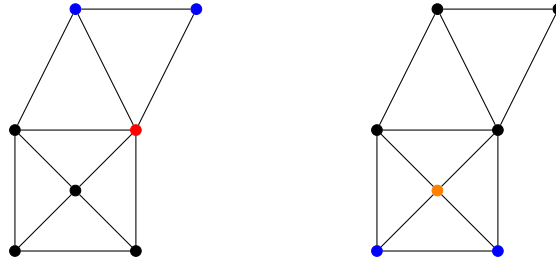


Figure 14: On the left: In the attached B, the red vertex is contained in every neat dominating set, whereas the blue vertices are not. On the right: The blue vertices, which form a 3-pair for the bad 5-wheel centered at the orange vertex, are not contained in any neat dominating set, as the orange vertex has strictly larger closed neighborhood. Note however that the orange vertex is not contained in every neat dominating set.

This lemma generalizes to many vertices u_i :

LEMMA 5.5. (COVERING WITH LRS) *Let G be a skeletal triangulation with boundary vertices u_1, \dots, u_k . Let H be the graph resulting from fusing a small LR to each u_i . Then:*

- $\Phi(H) \leq \Phi(G) + 3.5k$.
- If H has a dominating set of size s , then G has a set of size $s - k$ that dominates every vertex except maybe some of the u_i .

Proof. Similar to the previous proof. □

5.6 Toolbox: Neat dominating sets In some graphs, there are vertices that appear “weakly suboptimal” to include in a dominating set.

DEFINITION 5.4. (NEAT) *Let G be a skeletal triangulation. A dominating set $S \subseteq G$ is neat if for every $u \in S$, there is no $v \in V(G)$ with $N[u] \subsetneq N[v]$.*

LEMMA 5.6. *Every skeletal triangulation has a minimum dominating set that is neat.*

Proof. Take a minimum dominating set S that maximizes

$$\sum_{u \in S} |N[u]|,$$

then S is neat. □

In many cases, neat dominating sets allow us to assume that some vertices are not contained or have to be contained in a dominating set. Let $S \subseteq G$ be a near dominating set. Here are some examples, many of which occur in Figure 14.

- If u is the tip of an ear or in a 3-pair of a bad 5-wheel, then $u \notin S$.
- If $\deg(u) = 3$ and G is a 3-connected near-triangulation, then $u \notin S$.
- If G is the result of fusing a small AB to u , then $u \in S$.
- If G is the result of attaching a small A (or B) to u, v , then the red vertex (see Figure 13) of the A (or B) is in S .

5.7 Toolbox: Forcing a Vertex We have just seen that fusing a small AB allows us to force a vertex into every neat dominating set. This turns out to be extremely useful.

LEMMA 5.7. (FORCING A VERTEX) *Let G be a skeletal triangulation with boundary vertex u . Let H be the graph resulting from fusing a small AB to u . Then:*

- $\Phi(H) \leq \Phi(G) + 2.5$.
- *Any neat dominating set in H is a dominating set in G that contains u .*

Proof. Let $H' \subseteq H$ be the small AB that was fused, including u . Then,

$$\Phi(H) \leq \Phi(G) + \phi(H', u, v) = \Phi(G) + 2.5.$$

Any dominating set in H contains at least one vertex of the AB (including u), so a neat one has to contain u . \square

Due to the following reason, forcing a vertex is a lot more versatile than simply “picking” a vertex u and deleting u together with some of its neighbors: If we delete u , this by necessity turns all interior neighbors of u into boundary vertices, which might create many low-degree problems. If we instead force u and are somewhat picky with the other neighbors we delete, we can keep most interior neighbors in the interior, which avoids having to discuss them in detail. Intuitively, forcing is “efficient” in the sense that Φ increases by 2.5 and keeping u around instead of deleting it is another +1, so we get an “increase” of 3.5 while potentially having 1 extra vertex in a minimum dominating set.

5.8 Proof of Theorem 5.2 In this section, we show that Theorem 5.1 implies Theorem 5.2, in a way that can be used inside the induction step of the main proof.

PROPOSITION 5.2. *Let (G, u) be a rooted skeletal triangulation. Let $s = s(G, u)$ and let $\phi = \phi(G, u)$. If (G, u) acts as*

AB then $\phi \geq 3.5s - 1$ under the following assumption: Let H be the graph resulting from fusing a small LR to u . Assume Theorem 5.1 holds for H .

LR then $\phi \geq 3.5s$ under the following assumption: Let $\{u, v\}$ be a boundary edge incident to u . Let H be the graph resulting from attaching an A with red vertex u to u, v . Assume Theorem 5.1 holds for H .

Nope then $\phi \geq 3.5s + 1.5$ under the following assumptions: If G is a sporadic example, assume nothing. If $\deg(u) \geq 2$, assume Theorem 5.1 holds for G . If $\deg(u) = 1$, assume $H := G - u$ satisfies the attaching-an-A assumption of the LR case.

Proof. If G is a sporadic example, then G acts as Nope and we check $\phi \geq 3.5s + 1.5$ by hand. Otherwise, suppose G acts as

AB Fuse a small LR to u . By assumption, the resulting graph H satisfies Theorem 5.1, hence $\Phi(H) \geq 3.5s(H)$. As G acts as AB and the small LR acts as LR,

$$s(H) = s(G, u) + s(\text{small LR}, u) = s(G, u) + 1.$$

The small LR has $\phi = 3.5$, hence by Lemma 4.4

$$\Phi(H) = \phi(G, u) + 3.5 + 1 = \phi(G, u) + 4.5.$$

Chaining inequalities yields

$$\phi(G, u) = \Phi(H) - 4.5 \geq 3.5s(H) - 4.5 = 3.5s(G, u) - 1.$$

LR Let $\{u, v\}$ be a boundary edge incident to u . Attach a small A with red vertex u to u, v . By assumption, the resulting graph H satisfies Theorem 5.1, hence $\Phi(H) \geq 3.5s(H)$. As G acts as LR and the small A acts as A,

$$s(H) = s(G, u) + s(\text{small A}, u) = s(G) + 1.$$

The small A has $\phi = 2.5$, hence by a slight modification of Lemma 4.4

$$\Phi(H) \leq \phi(G, u) + \phi(\text{small A}, u, v) + 1 = \phi(G, u) + 3.5.$$

Chaining inequalities yields

$$\phi(G, u) \geq \phi(H) - 3.5 \geq 3.5s(H) - 3.5 = 3.5s(G, u).$$

Nope We distinguish whether $u \in G$ is a degree-1 root.

- If $\deg(u) \geq 2$, then G is a skeletal triangulation. By Lemma 4.3, $\Phi(G) \leq \phi(G, u) + 1.5$. As (G, u) acts as Nope, $s(G) = s(G, u) + 1$. By assumption, G satisfies Theorem 5.1, hence $\Phi(G) \geq 3.5s(G)$. Chaining inequalities yields

$$\phi(G, u) \geq \Phi(G) - 1.5 \geq 3.5s(G) - 1.5 = 3.5s(G, u) + 2.$$

- If $\deg(u) = 1$, then let w be the unique neighbor of u and let $H = G - w$. H is a rooted skeletal triangulation with root w and $\phi(H, w) = \phi(G, u) - 1.5$. As G acts as Nope, H acts as LR and $s(G, u) = s(H, w)$. By assumption, H satisfies the assumptions needed for the LR case of this proposition. Therefore, by the LR case, $\phi(H, w) \geq 3.5s(H, w)$. Chaining inequalities yields

$$\phi(G, u) = \phi(H, w) + 1.5 \geq 3.5s(H, w) + 1.5 = 3.5s(G, u) + 1.5.$$

□

5.9 Proof of Theorem 5.3 In this section, we show that Theorem 5.1 implies Theorem 5.3, in a way that can be used inside the induction step of the main proof.

PROPOSITION 5.3. *Let (G, u, v) be a rooted near-triangulation. Let $s = s(G, u, v)$ and let $\phi = \phi(G, u, v)$. Assume Theorem 5.1 holds for any near-triangulation with the same number of interior vertices as G . Then: If (G, u, v) acts as*

$$A+B, \text{ OR then } \phi \geq 3.5s - 2.$$

$$A, B \text{ then } \phi \geq 3.5s - 1.$$

AND, $L+R, \text{ OCTA}, L \text{ OR } R$ then $\phi \geq 3.5s$.

$$L, R \text{ then } \phi \geq 3.5s + 0.5.$$

$$\text{None then } \phi \geq 3.5s + 1.5.$$

Proof. If G is a sporadic example, then G acts as OCTA and $\phi \geq 3.5s + 0.5$. Otherwise, suppose G acts as

A+B, OR Let H be the graph resulting from attaching a small L+R to u, v . Then $s(H) = s(G, u, v) + 1$, and $\Phi(H) = \phi(G, u, v) + 2 + 3.5$ by Lemma 4.4. By assumption, Theorem 5.1 holds for H , therefore $\Phi(H) \geq 3.5s(H)$. Chaining inequalities yields $\phi(G, u, v) \geq 3.5s(G, u, v) - 2$.

A, B Let H be the graph resulting from attaching a small B / A to u, v , in a way that forces the other base vertex compared to G . Then $s(H) = s(G, u, v) + 1$ and $\Phi(H) = \phi(G, u, v) + 2 + 2.5$. By assumption, Theorem 5.1 holds for H , therefore $\Phi(H) \geq 3.5s(H)$. Chaining inequalities yields $\phi(G, u, v) \geq 3.5s(G, u, v) - 1$.

AND, L+R, L OR R, OCTA Let H be the graph resulting from attaching a small OR to u, v . Then $s(H) = s(G, u, v) + 1$ as G acts as AND, L+R, L OR R, OCTA, and $\Phi(H) = \phi(G, u, v) + 2 + 1.5$. By assumption, Theorem 5.1 holds for H , therefore $\Phi(H) \geq 3.5s(H)$. Chaining inequalities yields $\phi(G, u, v) \geq 3.5s(G, u, v)$

L, R Let H be the graph resulting from attaching a small L / R to u, v in a way that dominates the same base vertex compared to G . Then $\Phi(H) = \phi(G, u, v) + 2 + 4.5$ and $s(H) = s(G, u, v) + 2$ (one +1 from the small L / R and one +1 from a u or v that is not dominated by either side.) Chaining inequalities yields $\phi(G, u, v) \geq 3.5s(G, u, v) + 0.5$.

None Consider the two cases in the attachment part of Lemma 4.3. Suppose first that $\Phi(G) = \phi(G, u, v) + 2$. As G acts as None, $s(G) = s(G, u, v) + 1$. By assumption, Theorem 5.1 holds for G , therefore $\Phi(G) \geq 3.5s(G)$. Chaining inequalities yields

$$\phi(G, u, v) \geq \Phi(G) - 2 \geq 3.5s(G) - 2 \geq 3.5s(G, u, v) + 1.5.$$

In all remaining cases, $\Phi(G) = \phi(G, u, v) + 2.5$. Suppose that u (or v) is not involved in a low-degree problem. Then $\phi(G, v) = \phi(G, u, v) + 1$. As (G, u, v) acts as None, $s(G, v) = s(G, u, v) + 1$. By Proposition 5.2, $\phi(G, v) \geq 3.5s(G, v) - 1$, independent of what (G, v) acts as. Chaining inequalities yields

$$\phi(G, u, v) = \phi(G, v) - 1 \geq 3.5s(G, v) - 2 = 3.5s(G) + 1.5.$$

In the remaining case, u, v is a 3-pair in a bad 5-wheel. Let w be the interior vertex adjacent to u, v . Then $H = G / \{u, v\}$ is a skeletal triangulation. Let $x \in H$ be the vertex corresponding to $\{u, v\}$. Then (H, x) acts as Nope with $s(H, x) = s(G, u, v)$ and $\phi(H, x) = \phi(G, u, v)$. By Proposition 5.2, $\phi(H, x) \geq 3.5s(H, x) + 1.5$. Chaining inequalities yields

$$\phi(G, u, v) = \phi(H, x) \geq 3.5s(H, x) + 1.5 = 3.5s(G, u, v) + 1.5.$$

□

6 The Proof

In this section, we prove Proposition 5.1. The proof consists of many cases. In each case, we assume none of the previously discussed cases apply. In particular, for later cases, we get to make stronger and stronger assumptions on the given graph.

6.1 Bridge Suppose G has a bridge u, v . Let G_1, G_2 be the two components resulting from deleting $\{u, v\}$.

CLAIM 6.1. *Then*

$$\Phi(G) = \phi(G_1, u) + \phi(G_2, v) + 2.$$

Proof. If u (or v) is a degree-1 root in (G_1, u) (or (G_2, v)), then it is a degree-2 cut vertex in G and vice versa. Otherwise, u and v are not involved in any low-degree problems, as they are cut vertices in G and roots in G_1, G_2 . All other low-degree problems are the same in G and G_1, G_2 , but u, v are not counted in the later (as they are roots), so we get a +2. □

CLAIM 6.2. *Theorem 5.2 holds for (G_1, u) and (G_2, v) .*

Proof. We check that Proposition 5.2 applies: The graphs resulting from attaching an A / B, or fusing a small LR to G_1 or to G_2 all have fewer bridges than G^1 , hence these satisfy Theorem 5.1 by the induction hypothesis. □

¹And the same number of interior vertices and at most the same Φ .

With the claim, we conclude as follows: If one of G_1, G_2 acts as AB, then $s(G) = s(G_1, u_1) + s(G_2, u_2)$ and

$$\begin{aligned}\Phi(G) &= \phi(G_1, u_1) + \phi(G_2, u_2) + 2 \\ &\geq 3.5s(G_1, u_1) - 1 + 3.5s(G_2, u_2) - 1 + 2 = 3.5s(G).\end{aligned}$$

Otherwise, if one of G_1, G_2 acts as Nope, say G_1 , then $s(G) = s(G_1, u_1) + s(G_2, u_2) + 1$ and

$$\begin{aligned}\Phi(G) &= \phi(G_1, u_1) + \phi(G_2, u_2) + 2 \\ &\geq 3.5s(G_1, u_1) + 1.5 + 3.5s(G_2, u_2) + 2 = 3.5s(G).\end{aligned}$$

Otherwise, both G_1 and G_2 act as LR, then $s(G) = s(G_1, u_1) + s(G_2, u_2)$ and

$$\begin{aligned}\Phi(G) &= \phi(G_1, u_1) + \Phi(G_2, u_1) + 2 \\ &\geq 3.5s(G_1, u_1) + 3.5s(G_2, u_2) + 2 = 3.5s(G) + 2.\end{aligned}$$

Conclusion From now on, we assume that G does not contain any bridges.

6.2 Cut vertex Suppose G has a cut vertex u . Let us split G at u into two pieces in the obvious way: Let G_1, G_2 each be subgraphs induced by u together with one or more components in $G - u$ such that each component occurs in exactly one of G_1 and G_2 . Then, (G_1, u) and (G_2, u) are rooted skeletal triangulations and the result of fusing them is G . As G has no bridges, neither (G_1, u) nor (G_2, u) has a degree-1 root. Therefore

$$\Phi(G) = \phi(G_1, u) + \phi(G_2, u) + 1$$

by Lemma 4.4. Similar to the bridge case, we would like to conclude via Theorem 5.2, but some care has to be taken to avoid circular arguments.

6.2.1 One side acts as Nope Suppose that G_1 (or G_2) acts as Nope. The graphs in the Nope case of Proposition 5.2 have fewer blocks than G , hence G_1 satisfies Theorem 5.1, therefore $\phi(G_1, u) \geq 3.5s(G_1, u) + 1.5$. If G_2 is a sporadic example, then $\Phi(G_2) \geq 3.5s(G_2) - 1$. Otherwise, by the induction hypothesis, $\Phi(G_2) \geq 3.5s(G_2)$. By Lemma 4.4, $\Phi(G) \geq \phi(G_1, u) + \Phi(G_2) - 0.5$. By Lemma 4.2, $s(G) \leq s(G_2) + s(G_1, u)$. Combining everything yields

$$\begin{aligned}\Phi(G) &\geq \phi(G_1, u) + \Phi(G_2, u) - 0.5 \\ &\geq 3.5s(G_1, u) + 1.5 + 3.5s(G_2) - 1 - 0.5 \geq 3.5s(G).\end{aligned}$$

6.2.2 Both sides act as AB Suppose both G_1 and G_2 act as AB, then $s(G) = s(G_1, u) + s(G_2, u) - 1$. As G_1, G_2 are both smaller than G and not a sporadic example, by the induction hypothesis, $\Phi(G_i) \geq 3.5s(G_i)$. By Lemma 4.3, $\Phi(G_i) \leq \phi(G_i, u) + 1.5$. Chaining inequalities yields

$$\begin{aligned}\Phi(G) &= \phi(G_1, u) + \phi(G_2, u) + 1 \geq \Phi(G_1) + \Phi(G_2) - 2 \\ &\geq 3.5s(G_1) + 3.5s(G_2) - 2 = 3.5s(G) + 1.\end{aligned}$$

6.2.3 One side is a small LR Suppose G_1 is a small LR, then $s(G) = s(G_2, u) + 1$. By Lemma 4.4, $\Phi(G) = \phi(G_1, u) + \phi(G_2, u) + 1$. As G_2 does not act as Nope, $s(G_2) = s(G_2, u)$. As G_2 is smaller than G and not a sporadic example, by the induction hypothesis, $\Phi(G_2) \geq 3.5s(G_2)$. If $\Phi(G_2) = \phi(G_2, u) + 1$, then chaining inequalities yields

$$\begin{aligned}\Phi(G) &= \phi(G_1, u) + \phi(G_2, u) + 1 = 3.5 + \Phi(G_2) \\ &\geq 3.5 + 3.5s(G_2) = 3.5s(G).\end{aligned}$$

Otherwise, $\Phi(G_2) = \phi(G_2, u) + 1.5$, then by Lemma 5.3, $u \in G_2$ is an ear tip or part of a 3-pair in a bad 5-wheel. Let $H = G_2 - u$, then $s(H) = s(G_2)$ as a neat rooted dominating set in G_2 does not contain u and $s(G_2) = s(G_2, u)$ as G_2 does not act as Nope. By Lemma 5.3, $\Phi(H) \leq \phi(G_2, u) + 0.5$. Chaining inequalities yields

$$\begin{aligned}\Phi(G) &= \phi(G_1, u) + \phi(G_2, u) + 1 \geq 3.5 + \Phi(H) + 0.5 \\ &\geq 3.5 + 3.5s(H) + 0.5 = 3.5s(G) + 0.5.\end{aligned}$$

6.2.4 One side acts as LR In all remaining cases, G_1 (or G_2) acts as LR, then $s(G) = s(G_1, u) + s(G_2, u)$. By Lemma 4.4, $\Phi(G) = \phi(G_1, u) + \phi(G_2) + 1$. As G_1 is not a small LR, both G_1 and G_2 satisfy the assumptions of Proposition 5.2, as the involved graphs have fewer blocks and/or have smaller Φ than G . G_1 acts as LR, so this yields $\phi(G_1, u) \geq 3.5s(G_1, u)$. G_2 acts as LR or AB, so this yields $\phi(G_2, u) \geq 3.5s(G_2, u) - 1$. Chaining inequalities yields

$$\Phi(G) = \phi(G_1, u) + \phi(G_2, u) + 1 \geq 3.5s(G_1, u) + 3.5s(G_2, u) = 3.5s(G).$$

Conclusion From now on, we may assume that G does not contain any cut vertices. In particular, from now on, G is a near-triangulation.

6.3 Outerplanar Suppose G is an outerplanar near-triangulation with k vertices of degree 2. There are two known bounds:

1. $s(G) \leq n/3$, see [12].
2. $s(G) \leq (n+k)/4$, see [2].

The linear combination $\frac{3}{7} \cdot (1) + \frac{4}{7} \cdot (2)$ yields

$$s(G) \leq \frac{3}{7} \cdot \frac{n}{3} + \frac{4}{7} \cdot \frac{n+k}{4} = \frac{2n+k}{7} \leq \frac{2}{7} \cdot \Phi(G).$$

In other words, we ignore bad 3-pairs and just use known bounds.

REMARK 6.1. *There is also a direct proof based on finding 7 dominating sets such that every ear tip is contained in 3 of them and every other vertex in 2 of them.*

Conclusion From now on, G is not outerplanar. In particular, it has at least one interior vertex.

6.4 Non-trivial chord Suppose G has a chord u, v . Let G_1, G_2 be the two sides, with G_1 having at least one interior vertex².

CLAIM 6.3. *Then, G_2 satisfies Proposition 5.3.*

Proof. G_1 has at least one interior vertex, hence any skeletal triangulation with the same number of interior vertices as G_2 is smaller than G due to having fewer interior vertices. In particular, any such graph satisfies Theorem 5.1 by the induction hypothesis. \square

Intuitively speaking, the claim allows us to replace G_2 by a small OR, A, B that acts as the same type or delete G_2 and argue about the low-degree problems we create. Formally, suppose G_2 acts as

A+B, OR Let H be the result of attaching a small OR to $u, v \in G_2$. Then H has smaller Φ than G as the small OR is the unique smallest rooted near triangulation that acts as OR, so by the induction hypothesis, $\Phi(H) \geq 3.5s(H)$. By Claim 6.3 and Lemma 5.1³, also $\Phi(G) \geq 3.5s(G)$.

A, B Let H be the result of attaching a small A, B to u, v . Then H has smaller Φ than G , so by the induction hypothesis, $\Phi(H) \geq 3.5s(H)$. By Claim 6.3 and Lemma 5.1, also $\Phi(G) \geq 3.5s(G)$.

L, R, None Let $H = G_1$, then H has smaller Φ than G , so by the induction hypothesis, $\Phi(H) \geq 3.5s(H)$ unless H is a sporadic example. By Lemma 4.4, $\Phi(G) + 0.5 \geq \Phi(H) + \phi(G_2, u_2, v_2)$. By Lemma 4.1, $s(G) \leq s(H) + s(G_2, u, v)$. By Claim 6.3, $\phi(G_2, u_2, v_2) \geq 3.5s(G_2, u_2, v_2) + 0.5$. Chaining inequalities yields

$$\begin{aligned} \Phi(G) &\geq \Phi(H) + \phi(G_2, u_2, v_2) - 0.5 \\ &\geq 3.5s(H) + 3.5s(G_2, u_2, v_2) + 0.5 - 0.5 \geq 3.5s(G) \end{aligned}$$

²In particular, G_1 is not a small A, B, OR, L+R, L, R

³Technically speaking, the lemma as stated only applies to the OR case, but as A+B is “strictly stronger” than OR, a more careful look shows that the A+B case also works.

If H is a sporadic example, and G_2 acts as None, then $\Phi(H) \geq 3.5s(H) - 1$ while $\phi(G_2, u, v) \geq 3.5s + 1.5$, so a similar chain of inequalities works. If H is sporadic and G_2 acts as L, R, then G_2 can be used to cover one vertex of H : Then $\Phi(H) \geq 3.5s(H, u, v) + 2.5$ and, since H acts as OCTA and G_2 as L / R, $s(G) \leq s(H, u, v) + s(G_2, u, v)$. Chaining inequalities yields

$$\begin{aligned}\Phi(G) &\geq \Phi(H) + \phi(G_2, u_2, v_2) - 0.5 \\ &\geq 3.5s(H, u, v) + 2.5 + 3.5s(G_2, u_2, v_2) + 0.5 - 0.5 \geq 3.5s(G) + 2.5\end{aligned}$$

AND, L+R, L OR R, OCTA By Claim 6.3, $\phi(G_2, u_2, v_2) \geq 3.5s(G_2, u_2, v_2) + 0.5$. Similar to lemma 4.3, if there is no low-degree problem in G_1 at u, v , then $\Phi(G) = \Phi(G_1) + \phi(G_2, u_2, v_2)$ and we conclude $\phi(G) \geq 3.5s(G)$ as in the L, R case. This includes the case where G_2 is a sporadic example. Suppose now there is a low-degree problem in G_1 that involves u . Let $H = G_1 - u$, then by Lemma 5.3 $\Phi(H) \leq \Phi(G_1) - 0.5$. As G_2 acts as AND, L+R, L OR R, OCTA, there is a rooted dominating set in G_2 that dominates u . Therefore, $s(G) \leq s(H) + s(G_2, u, v)$. Chaining inequalities similar to the L, R case yields

$$\begin{aligned}\Phi(G) &\geq \Phi(G_1) + \phi(G_2, u_2, v_2) - 0.5 \geq \Phi(H) + \phi(G_2, u_2, v_2) \\ &\geq 3.5s(H) + 3.5s(G_2, u_2, v_2) \geq 3.5s(G)\end{aligned}$$

6.5 Conclusion From now on, for every chord, one side is a small OR, A, B attachment. More precisely, G has exactly one 3-connected component with interior vertices. All other 3-connected components are copies of a small OR / A / B.

DEFINITION 6.1. *The 3-connected component with interior vertices is called the polygon. All other ones are the A, B, OR-attachments of G .*

In fact, G is the result of attaching its A, B, OR-attachments to its polygon.

6.6 Notation for further cases

6.6.1 Polygon vertices As concluded in the previous case, G now consists of a *polygon* with *attachments*.

DEFINITION 6.2. *A polygon vertex is a boundary vertex of G that is part of the polygon.*

In the remaining cases, we (implicitly) use s, t, u, v, w, x, y, z to denote a range of consecutive polygon vertices, either in clockwise or counter-clockwise order. (These may not be distinct if the polygon is small.) For example, if we say ‘‘Suppose $\deg(v) = 3$ and $\deg(x) = 4$.’’, we really mean: Suppose there is a polygon vertex v with $\deg(v) = 3$ and a polygon vertex x with $\deg(x) = 4$ with exactly one polygon vertex w in between.

6.6.2 Simplified framework Explicitly arguing with inequalities and the induction hypothesis gets very tedious and distracts from more important parts of the proof. For the remaining cases, we use the following simplified framework: Let G be the graph we consider. By modifying G slightly, we construct a new skeletal triangulation H . We require H to be smaller than G and not one of the sporadic examples. Then, by the induction hypothesis, $\Phi(H) \geq 3.5s(H)$. Let $\Delta\Phi := \Phi(G) - \Phi(H)$ and $\Delta s := s(G) - s(H)$. We show that the decrease in Φ satisfies

$$\Delta\Phi = \Phi(G) - \Phi(H) \geq 3.5(s(G) - s(H)) = 3.5\Delta s.$$

Together with the previous equation, this implies $\Phi(G) \geq 3.5s(G)$, as desired. To show this, we usually state a bound $\Delta s \leq C$. We then prove the bound $-\Delta\Phi \leq -3.5C$ ⁴. Our argument will be phrased as ‘‘ Φ decreases by 2, decreases by 1.5 and increases 0.5’’, which really means $\Phi(H) = \Phi(G) - 2 - 1.5 + 0.5$, or equivalently, $-\Delta\Phi = -2 - 1.5 + 0.5$.

6.7 A, B attachments Suppose G has one or more A, B attachments.

⁴Note the minus here. In general, $\Delta\Phi \geq 0$ describes the decrease in Φ whereas $-\Delta\Phi \leq 0$ describes the net change in Φ . We prefer this minus here, as it avoid a bunch of minuses in the next sentence. Working with Δs and $-\Delta\Phi$ also has the advantage of having to prove an *upper* bound for both.

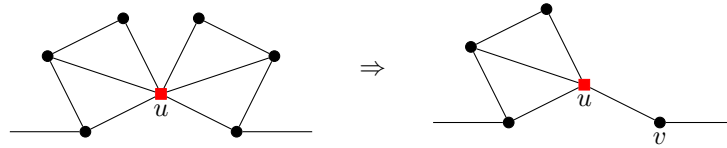


Figure 15: A and B attachment with same red vertex.

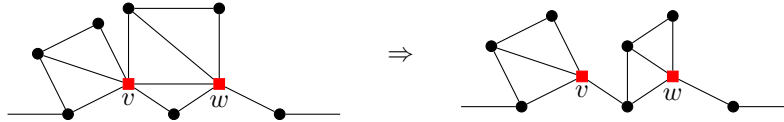


Figure 16: Two B attachments with consecutive red vertices.

6.7.1 Same red vertex Suppose G has an A and a B attachment with the same red vertex u . See Figure 15.

(Construction) Construct H by deleting the A attachment from G .

(Domination) A minimum neat dominating set $S \subseteq H$ contains u , due to the B attachment. In G , u dominates the deleted A, hence S dominates G . Therefore $\Delta s \leq 0$.

(Penalty) Deleting the A decreases Φ by 2.5. This may create up to one low-degree problem, namely one involving v , increasing Φ by ≤ 0.5 . Overall, $-\Delta\Phi \leq -2.0$

(Smaller) H has smaller Φ than G^5

6.7.2 Consecutive red vertex Suppose G has an A / B attachment with red vertex v and another one with red vertex w . Then, by the previous case, there are the only attachments on u, v, w, x . See Figure 16.

(Construction) Temporarily remove both A / B attachments. Delete the edge $\{v, w\}$. Add the A / B attachments back, with the same red vertex as before, but possibly different base.

(Domination) A minimum neat dominating set $S \subseteq H$ contains both v and w , hence S dominates G . Therefore $\Delta s \leq 0$.

(Penalty) Removing the A / B attachments and adding them back does not change Φ . Deleting the edge does not create any low-degree problems. Therefore $-\Delta\Phi \leq 0$.

(Smaller) H has one fewer interior vertex than G .

6.7.3 Red vertex with OR Suppose G has an A / B attachment with red vertex u and an OR attachment with base u, v . See Figure 17.

(Construction) Delete the OR.

(Domination) A minimum neat dominating set $S \subseteq H$ contains u due to the A / B. In G , u dominates the deleted OR, hence S dominates G . Therefore $\Delta s \leq 0$.

(Penalty) Deleting the OR decreases Φ by 1.5. This may create up to one low-degree problem at v , increasing Φ by ≤ 0.5 . Overall, $-\Delta\Phi \leq -1.0$.

(Smaller) H has smaller Φ than G .

⁵And the same number of interior vertices. We will only write down the highest “priority” difference between H and G .

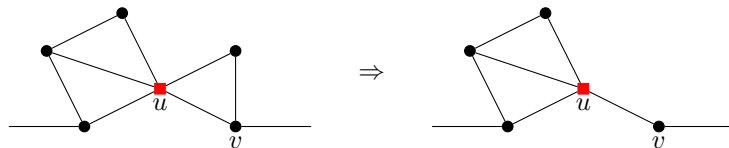


Figure 17: B attachment with OR.

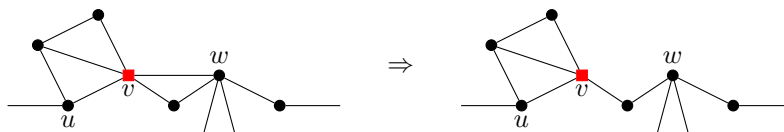


Figure 18: Red vertex next to 5^+ vertex.

6.7.4 Red vertex next to 5^+ vertex or next to vertex with attachment Suppose G has an A / B attachment with red vertex v and that $\deg(w) \geq 5$ or that there is an attachment with base w, x . See Figure 18.

(Construction) Delete the edge v, w .

(Domination) H is a spanning subgraph of G , hence $\Delta s \leq 0$.

(Penalty) After the deletion, $\deg(w) \geq 4$ or there is an attachment with base w, x . In either case, w is not involved in a low-degree problem. Thus $\Delta\Phi = 0$.

(Smaller) H has one fewer interior vertex than G .

6.7.5 Red vertex next to 3, 4 vertex Suppose G has an A / B attachment with base u, v and red vertex v and that $\deg(w) \in \{3, 4\}$. By the previous cases, there is no attachment with base containing w . See Figure 19.

(Construction) Delete w .

(Domination) A minimum neat dominating set $S \subseteq H$ contains v , which dominates w . Therefore $\Delta s \leq 0$.

(Penalty) Deleting w decreases Φ by ≥ 1.0 . As $\deg(w) \leq 4$, this creates at most two new low-degree problems, increasing Φ by ≤ 1.0 . Overall, $-\Delta\Phi \leq 0.0$.

(Smaller) H has at least one fewer interior vertex than G .

6.7.6 Conclusion From now on, G has no A / B attachments. Thus, G has only OR attachments. In particular, every boundary vertex of degree ≥ 3 is a polygon vertex and is hence adjacent to an interior vertex.

6.8 Consecutive low-degree vertices.

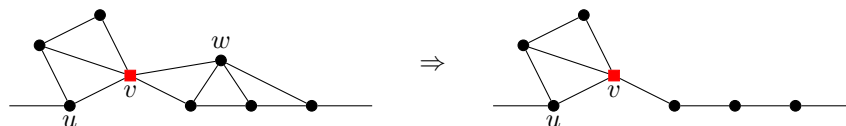


Figure 19: Red vertex next to 3, 4 vertex.

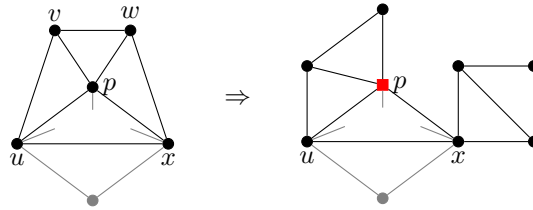


Figure 20: Bad 5-wheel. The gray things may or may not exist.

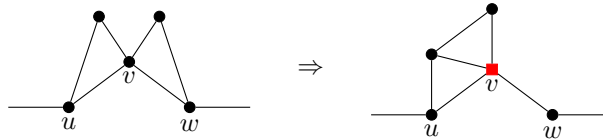


Figure 21: Consecutive OR attachments.

6.8.1 Bad 5-wheel Suppose v, w is a 3-pair in a bad 5-wheel. Then, as u, x is an edge between boundary vertices and p is an interior vertex, the polygon is just u, v, w, x , i.e. $y = u$. There may or may not be at OR attachment at x, u and there may or may not be vertices inside the triangle u, p, x . See Figure 20.

(Construction) Let p be the interior vertex adjacent to v and w . Delete v, w . Force p by attaching an B to u, p . Cover x by fusing a small LR to x .

(Domination) A minimum neat dominating set $S \subseteq H$ contains p and contains exactly one vertex, say $s \neq x$ in the small LR. Then s dominates only the small LR (including x). Then $S - s$ dominates G , as x is dominated by p . $\Delta s \leq -1$.

(Penalty) Deleting v, w decreases Φ by 2.5. Attaching the A increases Φ by 2.5. Fusing the LR increases Φ by 3.5. Overall, $-\Delta\Phi \leq 3.5$ (no minus here as Δs is negative).

(Smaller) H has one fewer interior vertex than G , so H is smaller despite $\Phi(H) > \Phi(G)$.

6.8.2 Consecutive OR attachments Suppose there is an OR attachment at u, v and another one at v, w . See Figure 21.

(Construction) Delete both ORs. Force v by attaching a B to u, v .

(Domination) A minimum neat dominating set $S \subseteq H$ contains v . In G , v dominates both ORs, hence S dominates G . $\Delta s \leq 0$.

(Penalty) Deleting both ORs decreases Φ by 3. Attaching the A increases Φ by 2.5. The deletion might create a low-degree problem involving w , increasing Φ by 0.5. Overall, $-\Delta\Phi \leq 0$.

(Smaller) H has fewer degree-2 vertices than G .

6.8.3 Degree-3 triple Suppose that $\deg(v) = \deg(w) = \deg(x) = 3$. We allow $u = x$, i.e. $G = K_4$.) Let p be the interior vertex adjacent to v, w, x . See Figure 22.

(Construction) Delete v, w, x . Force p by attaching an A to p, y . Fuse a small LR to u .

(Domination) A minimum neat dominating set $S \subseteq H$ contains p and contains exactly one vertex $s \neq u$ in the small LR. Then $S - s$ dominates G , as p dominates u, v, w, x, y . $\Delta s \leq -1$.

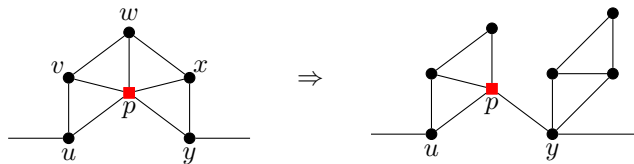


Figure 22: Degree-3 triple.

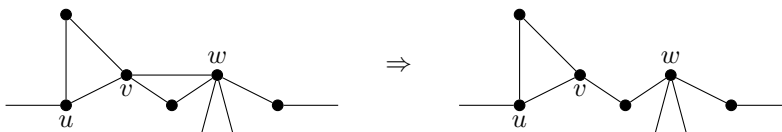


Figure 23: OR next to 5^+ vertex.

(Penalty) Deleting v, w, x decreases Φ by 3. Attaching the A increases Φ by 2.5. Fusing an LR increases Φ by 3.5. The deletions do not create any low-degree problems, as u, p, y each get something fused / attached to them. Overall, $-\Delta\Phi \leq 3$.

(Smaller) H has fewer interior vertices.

6.8.4 Conclusion Now G has no bad 5-wheels and all OR attachments have disjoint base vertices. Moreover, no three consecutive boundary vertices all have degree 3.

6.9 Unproblematic ORs

6.9.1 OR next to 5^+ vertex Suppose there is an OR attachment at u, v and that $\deg(w) \geq 5$. See Figure 23.

(Construction) Delete the edge v, w .

(Domination) H is a spanning subgraph of G , hence $\Delta s \leq 0$.

(Penalty) After the deletion, $\deg(w) \geq 4$, so this does not create low-degree problems. $-\Delta\Phi = 0$.

(Smaller) H has one fewer interior vertex.

6.9.2 Two ORs one edge apart Suppose there is an OR attachment at u, v and another OR attachment at w, x . See Figure 24.

(Construction) Delete the edge v, w .

(Domination) H is a spanning subgraph of G , hence $\Delta s \leq 0$.

(Penalty) Only the degrees of v, w are affected. Both are incident to a chord, so this does not create low-degree problems by Remark 4.1.

(Smaller) H has one fewer interior vertex.

6.9.3 OR on triangle boundary Suppose there is an OR attachment at v, w and that $x = u$, i.e. the polygon is a triangle. See Figure 25.

(Construction) Delete w .

(Domination) A neat dominating set in H contains either u or v . That vertex then dominates w . $\Delta s = 0$.

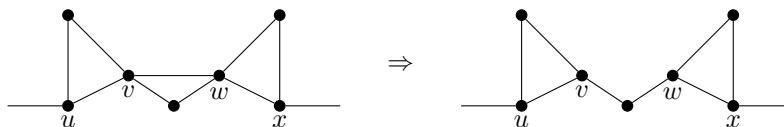


Figure 24: Two ORs one edge apart.

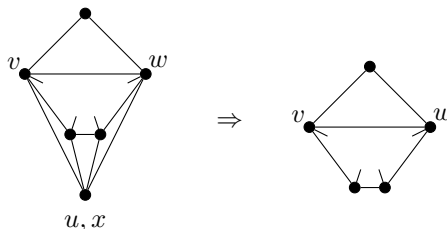


Figure 25: OR on triangle polygon.

(Penalty) Deleting w decreases Φ by 1. By the previous case, $\deg(w) \leq 4$, hence the deletion creates at most one low-degree problem, increasing Φ by ≤ 0.5 . Overall, $-\Delta\Phi \leq -0.5$.

(Smaller) H has at least one fewer interior vertex.

6.9.4 Conclusion Now every OR has a base that lies between two distinct polygon vertices, both of degree $\{3, 4\}$, both not part of any attachment.

6.10 Deleting ORs Suppose there is an OR attachment at x, y . Then $\deg(w), \deg(z) \in \{3, 4\}$ and $w \neq z$. (It could happen that $v = z$.) Let p be the interior vertex adjacent to x, y s.t. $\{p, x, y\}$ is a facial triangle.

6.10.1 Interior degree-3 neighbor Suppose there is an interior vertex r with $\deg(r) = 3$ adjacent to x (or y). See Figure 26.

(Construction) Delete r . Replace the OR by an A with red vertex x . Fuse a small LR to w .

(Domination) If S is a minim neat dominating set in H and $L \subseteq S$ is the vertex in the fused LR, then $S \setminus L$ is a dominating set in G containing x , which dominates w, r . $\Delta s = -1$.

(Penalty) Deleting r and the OR decreases Φ by 2.5. Attaching the A increases Φ by 2.5. Fusing the LR increases Φ by 3.5. Deleting r does not create any low-degree problem, as the only boundary vertices possibly adjacent to r are w, x, y . Overall, $-\Delta\Phi \leq 3.5$.

(Smaller) H has fewer interior vertices.

6.10.2 Antipodal 5-wheels Suppose there is an interior vertex $q \neq w$ adjacent to both x and y with $\deg(q) = 4$. Then $N[q]$ is a 5-wheel with x, y on antipodal sides of the wheel. Let r_1, r_2 be the other neighbors of q , i.e. $N(q) = \{x, y, r_1, r_2\}$. See Figure 27.

(Construction) Delete w . Add the edge r_1, r_2 .

(Domination) A neat dominating set in H contains either x or y . That vertex then dominates w, r_1, r_2 . In particular, the added edge has no effect on s and $\Delta s \leq 0$.

(Penalty) Deleting w decreases Φ by 1. The deletion only decreases the degrees of x, y so it does not create low degree problems. $-\Delta\Phi \leq -1$.

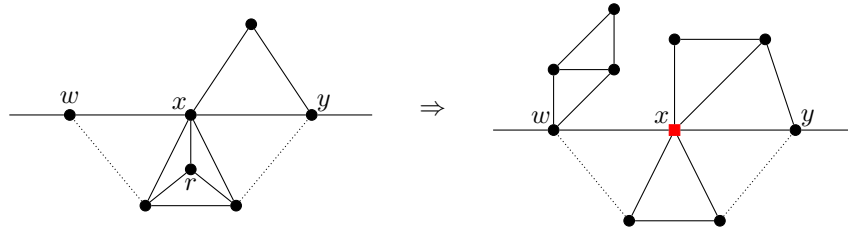


Figure 26: OR with interior degree-3 neighbor.

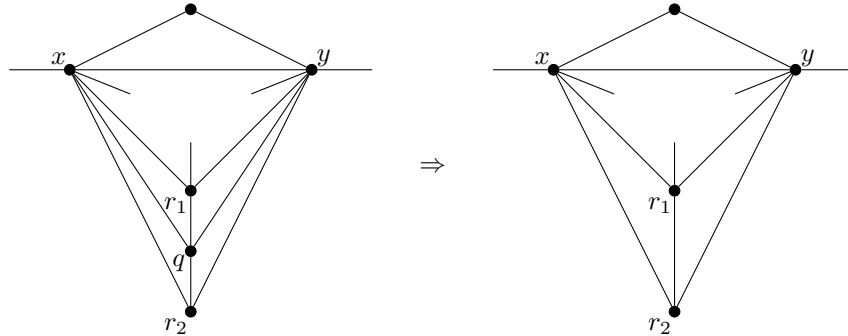


Figure 27: OR with Antipodal 5-wheel.

(Smaller) H has one fewer interior vertex.

6.10.3 Octahedral interior 4-pairs Let $c(x)$ denote the number of octahedral interior 4-pairs adjacent to x . Suppose $c(x) \geq 1$. See Figure 28.

(Construction) Delete the OR and x , but keep y . Delete both vertices of every interior 4-pair adjacent to x . Fuse a small LR to every remaining neighbor of x . Suppose we fuse k LRs this way.

(Domination) If S is a minimum dominating set in H and $L \subseteq S$ are the k vertices in the fused LRs, then $S \cup \{x\} \setminus L$ is a dominating set in G . $\Delta s \leq k - 1$.

(Penalty) Deleting the OR and x decreases Φ by 2.5. Deleting interior 4-pairs decreases Φ by $2 \cdot c(x)$. Any low-degree problem created by deleting the OR and x get deleted or covered by an LR. Deleting interior 4-pairs creates exactly one ear per pair, increasing Φ by $0.5 \cdot c(x)$. Fusing the LRs increases Φ by $3.5k$. Overall,

$$-\Delta\Phi \leq -2.5 - 1.5c(x) + 3.5k \leq -4 + 3.5k.$$

(Smaller) H has fewer interior vertices.

6.10.4 No interior problems The previous cases now allow us to delete both x and y , without creating too many low-degree problems. If $v = z$, then $\deg(w) = \deg(z) = 3$ makes w, z a 3-pair in the bad 5-wheel w, x, y, z, p , which is covered by Case 6.8.1. Therefore, suppose that $v \neq z$ or that WLOG $\deg(z) \geq 4$. See Figure 29.

(Construction) Delete the OR, w, x, y . Fuse a small LR to every remaining neighbor of x . Suppose we fuse k LRs this way.

(Domination) If S is a minimum dominating set in H and $L \subseteq S$ are the k vertices in the LRs, then $S \cup \{x\} \setminus L$ is a dominating set in G . $\Delta s \leq k - 1$.

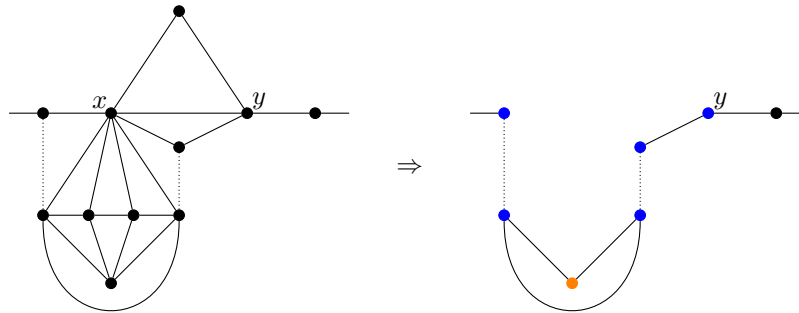


Figure 28: Octahedral interior 4-pairs. Here, $c(x)$. The dashed lines represent a path of vertices all adjacent to x . The blue vertices each have an LR fused to them (not drawn here, to keep the picture clean). The orange vertices mark locations where a low-degree problem might be created.

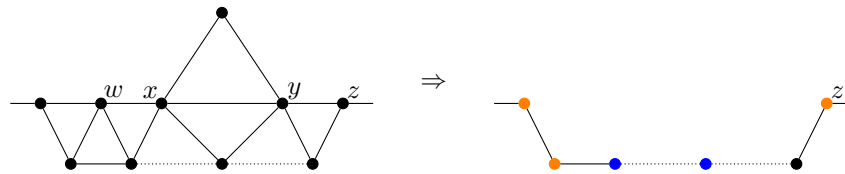


Figure 29: OR with no interior 4-pairs. The dashed lines represent a path of vertices all adjacent to x (or y). The blue vertices each have an LR fused to them (not drawn here). The orange vertices mark locations where a low-degree problem might arise.

(Penalty) The deletions decrease Φ by 4.5. Fusing LRs increases Φ by $3.5k$. The deletions may create one low-degree problem involving z and, as $\deg(w) \leq 4$, at most one involving a former neighbor of w . Overall, these increase Φ by ≤ 1 . If $v = z$, then $\deg(z)$ decreases by two, so we need $\deg(z) \geq 4$ to avoid creating a leaf. There are no other low-degree problems: All former neighbors of x get covered by an LR. Former neighbors of y cannot be involved in a low degree problem in which w, z are not involved, as that would require an interior vertex of degree 3 (for ears or pivoting triangles), an interior octahedral 4-pair (for 3-pairs in a bad 5-wheel) or a chord from y to a degree-3 boundary vertex (for a degree-2 cut vertex)⁶. All of these are covered by previous cases. Overall,

$$-\Delta\Phi \leq -4.5 + 1 + 3.5k \leq -3.5(k - 1).$$

(Smaller) H has fewer interior vertices.

6.10.5 Conclusion Now G has no OR attachments. If G is a single triangle, then $\Phi = 3.5$ and $s = 1$. Otherwise, and G is a 3-connected near-triangulation. This is great for deleting boundary vertices:

DEFINITION 6.3. (INTERIOR GRAPH) Let G be a skeletal triangulation. The interior graph $\text{Int}(G)$ is the graph induced by all interior vertices in G . An interior leaf is a vertex $u \in \text{Int}(G)$ with $\deg_{\text{Int}(G)}(u) = 1$.

LEMMA 6.1. If G is a 3-connected near-triangulation G , then $\text{Int}(G)$ is connected.

Proof. Let $s, t \in \text{Int}(G)$ be arbitrary. By Menger's theorem, there are 3 vertex-disjoint s - t -paths. If all of these contain a boundary vertex, then adding a vertex connected to all boundary vertices to the unbounded face creates a planar $K_{3,3}$ -subdivision, contradiction. Hence, at least one of the s - t -paths lies fully in $\text{Int}(G)$. \square

⁶There is a bit of subtlety if w and y share interior neighbors of degree 4.

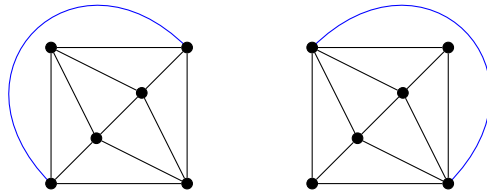


Figure 30: The two ways of adding one edge to the 3-bifan. On the left: the octahedron. On the right: a graph with $s = 1$.

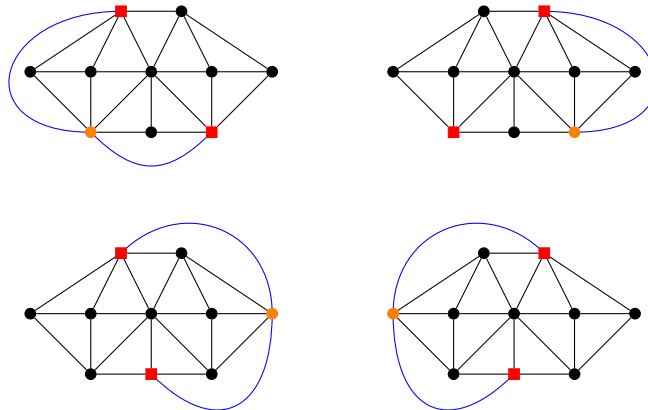


Figure 31: The special 4343434-heptagon with four sets of size 2 (red) that each dominate all vertices except one (orange). In each case, adding any one of the blue edges makes the red set dominating.

COROLLARY 6.1. *Let G be a 3-connected near-triangulation and let $B \subseteq G$ be any set of boundary vertices. Then $G - B$ is connected.*

Proof. By the Lemma, $\text{Int}(G)$ is connected. Every boundary vertex of G has a neighbor in $\text{Int}(G)$. □

6.11 Deletable boundary edges

6.11.1 Sporadic examples Suppose deleting the (boundary) edge $\{v, w\}$ yields a sporadic example H . In other words, G is the result of adding an edge to a sporadic example in a way that creates a facial triangle.

H contains at least four boundary vertices, so H is not the octahedron. If H is the 3-bifan, then $n = 6$ and either G is the octahedron, or G has two vertices of degree 5 and hence $s = 1$. See Figure 30. Finally, if H is the special 4343434-heptagon, then G has six boundary vertices and $n = 10$. There are seven ways of adding an edge to the special 4343434-heptagon and one can check that $s = 2$ in each case. See Figure 31.

6.11.2 No problems Suppose deleting the (boundary) edge $\{v, w\}$ does not create any low-degree problems.

(Construction) Delete the edge $\{v, w\}$.

(Domination) $H \subseteq G$ is a spanning subgraph. $\Delta s \leq 0$.

(Penalty) By assumption, no low-degree problems are created. $-\Delta\Phi \leq 0$. By the previous case, H is not a sporadic example.

(Smaller) H has fewer interior vertices.

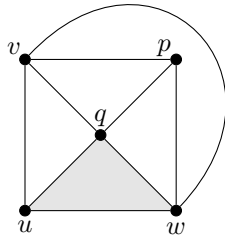


Figure 32: Boundary edge with interior degree-3 vertex. Any additional vertices lie in the shaded area.

6.11.3 Interior degree-3 vertex. Suppose there is an interior degree-3 vertex p such that v, p, w is a facial triangle. By Case 6.11.2, deleting $\{v, w\}$ creates at least one low-degree problem.

If the problem involves p , then v, p (or p, w) is a 3-pair into some bad 5-wheel, centered around some vertex q . Then, $\{u, w\}$ is an edge, hence the polygon of G is a triangle. See Figure 32. If $\deg(u) = 3$, then G is a 5-wheel with one extra edge, with $\Phi = 5$ and $s = 1$. Otherwise, deleting the boundary edge $\{u, v\}$ does not create any low-degree problems and Case 6.11.2 applies.

Otherwise, if the problem is an ear, say at v , then $\deg_G(v) = 3$ so $G = K_4$. If the problem is a 3-pair, say u, v , in some bad 5-wheel. Then G is the 5-wheel with one extra edge.

Observation A direct consequence this case is the following: If p is an interior degree-3 vertex adjacent to some boundary vertex v , then p is not adjacent to u (nor w). Moreover, as G is 3-connected, v is the only boundary vertex adjacent to p . In addition, this implies $\deg(v) \geq 5$.

6.11.4 Conclusion Now, deleting any boundary edge $\{v, w\}$ creates at least one low-degree problem, which involves only boundary vertices in G . Let H be resulting graph. As G is 3-connected, H is 2-connected, so any newly created low-degree problem is an ear tip or a bad 5-wheel, and in both cases, a 2-cut is created. More precisely, at least one of the following is true.

- $\deg_G(v) = 3$ and $v \in H$ is an ear tip.
- $\deg_G(w) = 3$ and $w \in H$ is an ear tip.
- $\deg_G(u) = 3$, $\deg_G(v) = 4$ and $u, v \in H$ is a 3-pair in a bad 5-wheel.
- $\deg_G(w) = 4$, $\deg_G(x) = 3$ and $w, x \in H$ is a 3-pair in a bad 5-wheel.

Therefore, in G , there is at least one boundary vertex of degree 3, and, in between two boundary vertices of degree 3, the degrees of boundary vertices form one of the following patterns:

- 345^+43 , i.e. $\deg(u) = 3$, $\deg(v) = 4$, $\deg(w) \geq 5$, $\deg(x) = 4$ and $\deg(y) = 3$.
- 345^+3 or 35^+43 .
- 35^+3 .
- 34443 , 3443 , or 343 .
- 33 .

6.12 Triangular boundary Suppose the polygon is a triangle, i.e. $x = u$. If at least two of u, v, w have degree 3, then $G = K_4$ with $\Phi = 4$ and $s = 1$. Otherwise, WLOG let $\deg(u) = 3$ and $\deg(v), \deg(w) \geq 4$. By Case 6.11, deleting the edge $\{v, w\}$ creates a 3-pair u, v (or w, u) in a bad 5-wheel, hence $\deg_G(v) = 4$. See Figure 33.

(Construction) Delete v .

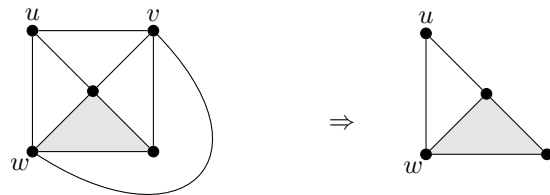


Figure 33: Triangular boundary. All additional vertices lie in the shaded area.

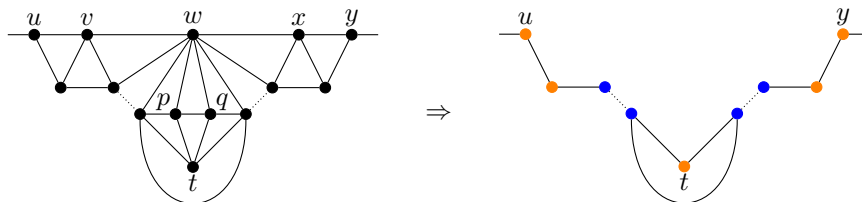


Figure 34: Boundary vertex with interior octahedron.

(Domination) A dominating set in H contains a vertex that dominates u . As $N[u] \subseteq N[v]$, that vertex also dominates v . $\Delta s \leq 0$.

(Penalty) Deleting v decreases Φ by 1. As $\deg(v) = 4$, this creates at most two new low-degree problems, increasing Φ by ≤ 1 . $-\Delta\Phi \leq 0$.

Conclusion The polygon is not a triangle, hence there are at least four boundary vertices.

6.13 Boundary vertex with interior octahedron Suppose deleting w creates a 3-pair p, q in a bad 5-wheel with $p, q \notin \{v, x\}$. Then, p, q are interior vertices and $N[p, q]$ is the octahedron. Let $t \in N[p, q]$ be the vertex antipodal to w . See Figure 34. If $v \in N[p, q]$ (or $x \in N[p, q]$), then deleting $\{v, w\}$ (or $\{w, x\}$) does not create a low-degree problem, contradiction. In particular, $\deg(w) \geq 5$ and $\deg(v), \deg(x) \leq 4$.

(Construction) Delete v, w, x, p, q . Fuse a small LR to every remaining neighbor of w . Suppose we fuse k LR's this way.

(Domination) If S is a minimum dominating set in H and $L \subseteq S$ are the k vertices in the fused LR's, then $S \cup \{w\} \setminus L$ is a dominating set in G . $\Delta s \leq 1 - k$.

(Penalty) The deletions decrease Φ by 5. Fusing the LR's increases Φ by $3.5k$. The deletions may create up to three low-degree problems, involving u, y and t respectively⁷ This increases Φ by ≤ 1.5 . Overall, $-\Delta\Phi \leq -5 + 3.5k + 1.5 = -3.5(1 - k)$.

(No Leaves) The only vertices that could end up as leaves are u and y . If $u \neq y$, then u and y each lose only one neighbor and hence cannot end up as leaves. Therefore, $u = y$ and $\deg(u) = 3$. If $\deg(v) = 4$ and $\deg(x) = 3$, then deleting $\{v, w\}$ does not create a bad 5-wheel, contradicting Case 6.11.4. Finally, if $\deg(v) = \deg(x) = 4$, then deleting $\{v, w\}$ or $\{w, x\}$ cannot both create a bad 5-wheel, as that would force $\deg(w) = 3$.

⁷One can show that it is actually u and y , and not their formerly-interior neighbor, but it is only important that low-degree problems are never adjacent.

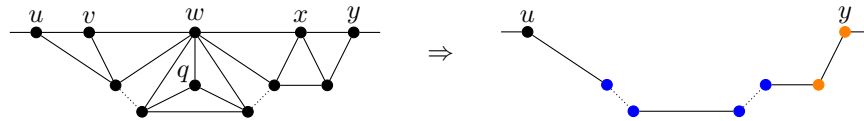


Figure 35: Boundary vertex w with interior K_4 .

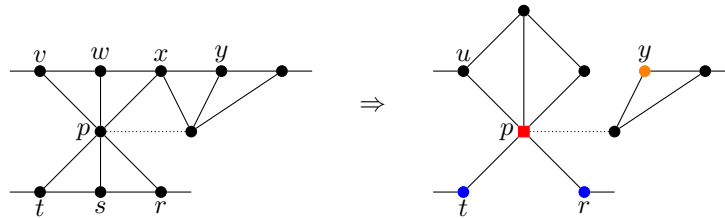


Figure 36: Degree-3 boundary vertices on two sides.

6.13.1 Conclusion Now, deleting a boundary vertex never creates a 3-pair of former interior vertices in a bad 5-wheel.

6.14 Boundary vertex with deletable K_4 Suppose deleting w creates an ear tip q with $q \notin \{v, x\}$. Then $\deg(w) \geq 5$, hence $\deg(v), \deg(x) \leq 4$. Suppose moreover that $\deg(v) = 3$ and that deleting v does not create a low-degree problem involving u . See Figure 35.

(Construction) Delete v, w, x, q . Fuse a small LR to every remaining neighbor of w . Suppose we fuse k LR's this way.

(Domination) If S is a minimum dominating set in H and $L \subseteq S$ are the k vertices in the fused LR's, then $S \cup \{w\} \setminus L$ is a dominating set in G . $\Delta s = 1 - k$.

(Penalty) The deletions decrease Φ by 4. Fusing the LR's increases Φ by $3.5k$. The deletions may create a single low-degree problem, involving y . This increases Φ by 0.5. Overall, $-\Delta\Phi \leq -4 + 3.5k + 0.5 = -3.5(1 - k)$.

6.14.1 Conclusion If w is a boundary vertex with an interior degree-3 neighbor, then deleting either v (or x) creates a low-degree problem involving u (or y). In particular, $\deg_G(u), \deg_G(y) \leq 4$, hence u and y have no interior degree-3 neighbor.

6.15 Interior vertex adjacent to two non-consecutive degree-3 boundary vertices Suppose $\deg(v) \geq 4$, $\deg(w) = 3$ and $\deg(x) \geq 4$. Let p be the interior vertex adjacent to v, w, x . Suppose p is adjacent to another boundary vertex $s \neq w$ with $\deg(s) = 3$. Let r, t be the boundary neighbors of s . See Figure 36. If both v and x have an interior degree-3 neighbor, then Case 6.14 applies, as deleting w does not create any low-degree problems. Hence, WLOG assume that x does not have an interior degree-3 neighbor.

(Construction) Delete x . Delete w and s . Force p by attaching a B to v, p . Fuse a small LR's to r and t .

(Domination) If $S \subseteq H$ is a minimum neat dominating set and $L \subseteq S$ are the two vertices in the LR's, then $S \setminus L$ contains p , which dominates H . $\Delta s \leq -2$.

(Penalty) Deleting three vertices decreases Φ by 3. Attaching the B increases Φ by 2.5. Fusing two LR's increases Φ by 7. The deletions may create a low-degree problem involving y but no further ones, increasing Φ by ≤ 0.5 . Overall, $-\Delta\Phi \leq -3 + 2.5 + 7 + 0.5 = 7$.

6.15.1 Conclusion Now any two non-consecutive degree-3 boundary vertices are adjacent to distinct interior vertices.

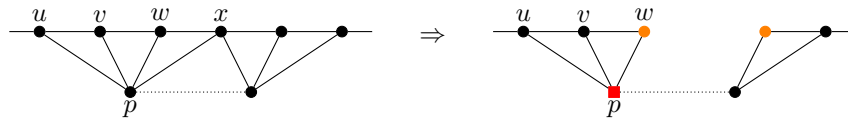


Figure 37: Consecutive degree-3 boundary vertices with no interior K_4 at x .

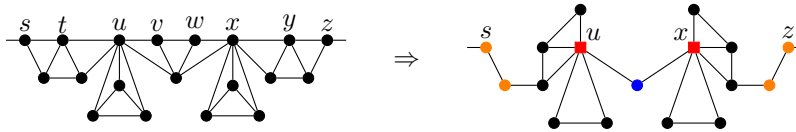


Figure 38: Consecutive degree-3 boundary vertices with interior K_4 s on both sides.

6.16 Consecutive degree-3 boundary vertices Suppose $\deg(u) \geq 4$, $\deg(v) = \deg(w) = 3$ and $\deg(x) \geq 4$. Let p be the interior vertex adjacent to u, v, w, x .

6.16.1 No interior degree-3 neighbor Suppose that x (or u) has no interior degree-3 neighbor. See Figure 37.

(Construction) Delete x . This turns v, w into a B with red vertex p .

(Domination) A minimum neat dominating set in H contains p , which dominates x . $\Delta s \leq 0$.

(Penalty) Deleting one vertex decreases Φ by 1. Deleting x creates at most two low-degree problems. This increases Φ by ≤ 1 . Overall, $-\Delta\Phi \leq -1 + 1 = 0$.

6.16.2 Both sides have an interior K_4 . Suppose that both u and x have an interior degree-3 neighbor, then $\deg(u), \deg(x) \geq 5$, and $t \neq y$ due to Case 6.14. See Figure 38.

(Construction) For each of u, x , delete one interior degree-3 neighbor. Delete t, v, w, y . Force u and x by attaching a B with red vertex u and an A with red vertex x . Fuse a small LR to p .

(Domination) If S is a minimum neat dominating set in H and L are the vertices in the small LR, then $S \setminus L$ is a dominating set in G , as it contains both u and x . $\Delta s \leq -1$.

(Penalty) Deleting six vertices decreases Φ by 6. Attaching the A and B increases Φ by 5. Fusing a small LR increases Φ by 3.5. If $s \neq y$, then the deletion may create two low degree problems, involving s and z respectively, increasing Φ by 1. If $s = z$, then Case 6.14 forces $\deg(y) = \deg(t) = 3$ and the deletions create at most one low-degree problem. No other low-degree problems are created, not even degree-2 cut vertices. Overall, $-\Delta\Phi \leq -6 + 5 + 3.5 + 1 = 3.5$.

(No leaves) We might create a leaf, but only if $s = z$, $\deg(s) = 3$ and $\deg(y) = \deg(t) = 4$. We treat this special case up next.

6.16.3 The remaining special case. Suppose that both u and x have an interior degree-3 neighbor, then $\deg(u), \deg(x) \geq 5$. Suppose that there are exactly 7 polygon vertices, i.e. $s = z$, and that $\deg(z) = 3$. Then $\deg(t) = \deg(y) = 4$. Let q be the interior vertex adjacent to z , then $N[q]$ is a 5-wheel. See Figure 39.

(Construction) Delete y, z, t, q .

(Domination) If S is a minimum dominating set in H , then $S \cup q$ dominates G . $\Delta s = 1$.

(Penalty) Deleting four vertices decreases Φ by 4. No low-degree problems are created. $-\Delta\Phi = -4$.



Figure 39: Consecutive degree-3 boundary vertices, the remaining special case.

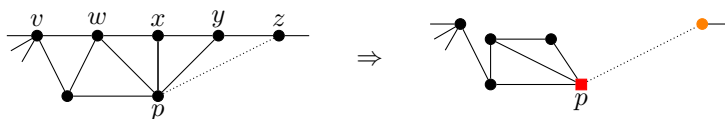


Figure 40: Degree Pattern $5+4$.

6.16.4 Conclusion Now every degree-3 boundary vertex is adjacent to a distinct interior vertex.

6.17 Degree Patterns $5+4$ and 444

6.17.1 Degree Pattern $5+4$ Suppose $\deg(v) \geq 5$ and $\deg(w) = 4$, then due to previous cases, $\deg(x) = 3$ and $\deg(y) \geq 4$. Let p be the interior vertex adjacent to w, x, y . See Figure 40. Deleting x does not create a low-degree problem at w , hence y has no interior degree-3 neighbor due to Case 6.14.

(Construction) Delete y . Delete x and w . Force p by attaching a B to v, p .

(Domination) Any neat dominating set $S \subseteq H$ contains p , which dominates w, x, y . $\Delta s \leq 0$.

(Penalty) Deleting three vertices decreases Φ by 3. Attaching a B increases Φ by 2.5. Deleting y creates exactly two low-degree problems, namely at x and z . Deleting x, w removes the former and does not create a new one, as $\deg(v) \geq 5$. Overall, there remains a single low-degree problem, increasing Φ by 0.5. In total, $-\Delta\Phi \leq -3 + 2.5 + 0.5 = 0$.

(Smaller) H has fewer interior vertices.

6.17.2 Degree Pattern 444 Suppose $\deg(u) = \deg(v) = \deg(w) = 4$, then $\deg(x) = 3$ and $\deg(y) \geq 4$. Let p be the interior vertex adjacent to w, x, y . If $v \neq z$, then argue as in the previous case. It might happen that $\deg_H(v) = 3$, but this cannot create a second low-degree problem: Either $u = z$, or $\deg_H(u) = 4$ and u, v is not a 3-pair. See Figure 41. If $v = z$, i.e. the polygon is a square, then the graph looks as in Figure 42, but then, deleting the edge $\{u, v\}$ does not create a low degree problem, contradiction.

6.17.3 Conclusion We can eliminate some degree patterns from Section 6.11.4. The remaining possibilities are:

- $35+3$.

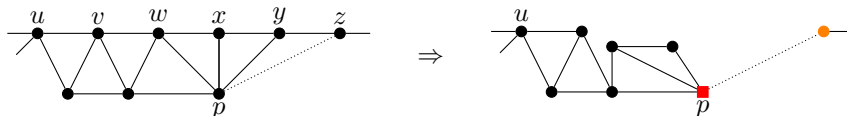


Figure 41: Degree Pattern 444.

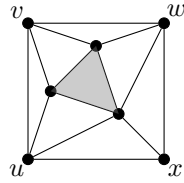


Figure 42: Degrees 4443 on a square. There might be additional vertices, but they all lie in the shaded triangle.

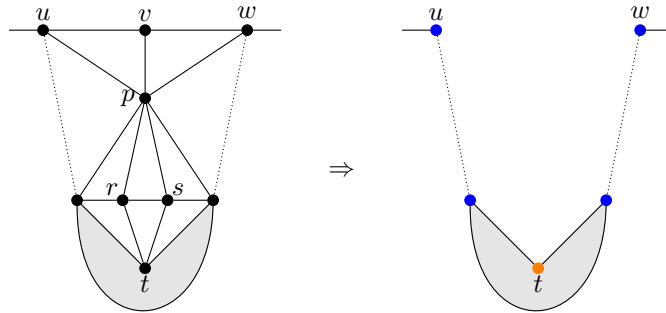


Figure 43: Degree 3 vertex with interior Octahedron. There may or may not be additional vertices in the shaded area.

- 3443.
- 343.

6.18 Degree 3 boundary vertex with interior problem Suppose $\deg(u) \geq 4$, $\deg(v) = 3$ and $\deg(w) \geq 4$. Let p be the interior vertex adjacent to v . Suppose deleting v and p creates a low-degree problem not involving u or w . By Case 6.15, it is not a degree-2 cut vertex.

6.18.1 Interior octahedron Suppose it is a 3-pair r, s in a bad 5-wheel with central vertex t . This 5-wheel together with p forms an octahedron. See Figure 43.

(Construction) Delete v, p . Delete r, s . Fuse a small LR to every remaining neighbor of p . Suppose k small LRs are fused this way.

(Domination) If $S \subseteq H$ is a minimum dominating set and $L \subseteq S$ are the k vertices in the small LRs, then $S \cup \{p\} \setminus L$ is a dominating set in G . $\Delta s \leq 1 - k$.

(Penalty) Deleting four vertices decreases Φ by 4. Fusing the small LRs increases Φ by $3.5k$. The deletions may create an ear tip at t , but no other low degree problem: every other vertex that lost some neighbors got fused to a small LR. Overall, $-\Delta\Phi \leq -4 + 3.5k + 0.5 = -3.5(1 - k)$.

6.18.2 Interior K_4 Suppose there is an interior degree-3 vertex r adjacent to p . By Case 6.14, WLOG assume that w has no interior degree-3 neighbor. In particular, r is not adjacent to w . See Figures 44 and 45.

(Construction) Delete w, v . Force p by attaching an A to u, p . Delete r . Fuse a small LR to every former neighbor of r that ends up as a boundary vertex. Suppose k LRs are fused this way.

(Domination) If S is a minimum neat dominating set in H and L are the k vertices in the LRs, then $S \setminus L$ contains p , which dominates r, v, w and any former neighbor of r . $\Delta s = -k$.

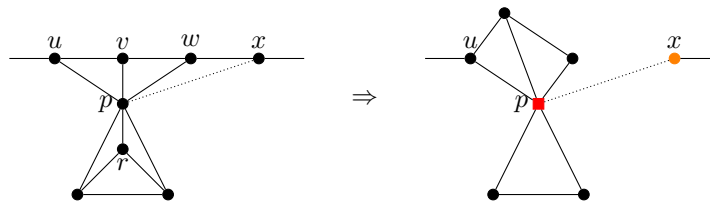


Figure 44: Interior K_4 , deleting r does not affect new boundary vertices.

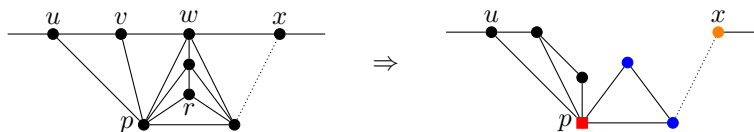


Figure 45: Interior K_4 , deleting r after v, w creates an ear. (There is a similar situation with a 3-pair in a bad 5-wheel instead of an ear.)

(Penalty) Deleting three vertices decreases Φ by 3. Attaching an A increases Φ by 2.5. Fusing the LR increases Φ by $3.5k$. Deleting w, v may create a low-degree problem involving x , but nowhere else. Deleting r does not create low-degree problems due to the fused LR. Overall, $-\Delta\Phi \leq -3 + 2.5 + 3.5k + 0.5 = -3.5(-k)$.

6.18.3 Conclusion For v, p as defined above, deleting v and p creates at most two low-degree problems and these involve u and w respectively.

6.19 Deleting a degree-3 boundary vertex and its interior neighbor The following technical case turns out to be useful in multiple later cases: Suppose there are at least five polygon vertices. Suppose $\deg(u) \geq 4$, $\deg(v) = 3$, $\deg(w) \geq 4$, $\deg(x) = 3$, then $\deg(y) \geq 4$. Let p be the interior vertex adjacent to v and let q be the interior vertex adjacent to x . Suppose that at least one of the following is true (see Figure 46):

- $\deg(y) \geq 5$.
- y is not adjacent to p and deleting (only) the edge $\{x, y\}$ does not turn y, z into a 3-pair in a bad 5-wheel.

(Construction) Delete the edge $\{x, y\}$. Delete v, p . Delete x . Force w by attaching an A to w, q . Fuse a small LR to the other H -boundary neighbor of w , which might be q .

(Domination) If $S \subseteq H$ is a minimum neat dominating set in H and $L \subseteq S$ is the vertex in the small LR, then $S \setminus L$ is a G -dominating set, as it contains w , which dominates v, p, x and the vertex to which the LR got fused. $\Delta s = -1$.

(Penalty) Deleting three vertices decreases Φ by 3. Attaching the A increases Φ by 2.5. Fusing the LR increases Φ by 3.5. Deleting the edge $\{x, y\}$ decreases the degree of y by one, but does not create a low-degree problem involving y . By Case 6.18, deleting v, p may create low degree problem at u , but nowhere else, as w gets covered by the A. In total, $-\Delta\Phi \leq -3 + 2.5 + 3.5 + 0.5 = -3.5 \cdot (-1)$.

6.20 Degree pattern 3443, big polygon Suppose there are at least eight boundary vertices. Suppose $\deg(v) = 3$, $\deg(w) = \deg(x) = 4$ and $\deg(y) = 3$. Then $\deg(u) \geq 4$ and $\deg(z) \geq 4$. Let p and q be the interior vertices adjacent to v and y respectively. Let r be the interior vertex adjacent to p, w, x, q .

6.20.1 No degree-2 cut vertices Suppose that deleting u and z does not create any degree-2 cut vertices. See Figure 47. Note that deleting v (or y) does not create a low-degree problem at w (or x), hence u and z do not have interior degree-3 neighbors due to Case 6.14.

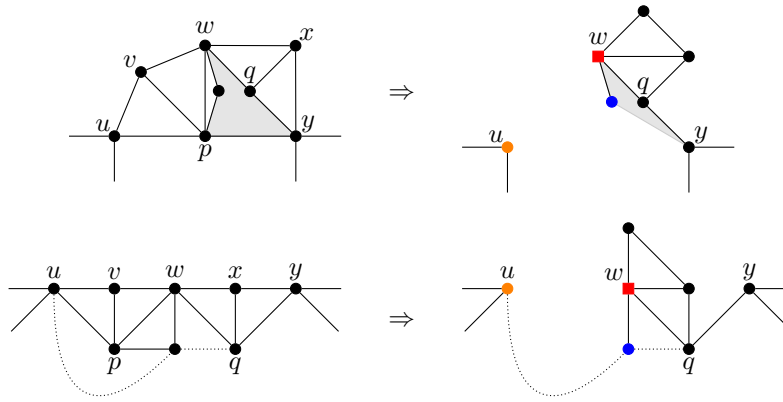


Figure 46: Degree 3 boundary vertex with deletable interior neighbor. Note: The blue vertex is equal to q if $\deg(w) = 4$. Above: p adjacent to y , but $\deg(y) \geq 5$. Below: p not adjacent to y , then we assume that deleting $\{x, y\}$ does not create a 3-pair.

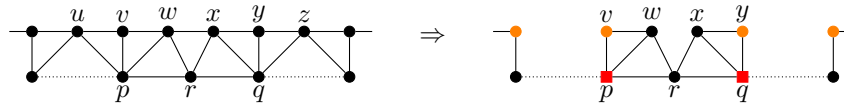


Figure 47: Degree pattern 3443, big polygon, no degree-2 cut vertices.

- (Construction) Delete u and z . Delete the edge $\{w, x\}$, turning v, w and x, y into an A with red vertex p and a B with red vertex q , respectively.
- (Domination) A minimum neat dominating set in H contains p and q , which dominate u and z , respectively. $\Delta s \leq 0$.
- (Penalty) Deleting two vertices decreases Φ by 2. Deleting u creates two low-degree problems, involving t and v , due to Cases 6.14 and 6.13. Similar for z . Deleting both u and z does not create any additional low-degree problems, by assumption. Deleting $\{w, x\}$ does not create any low-degree problems. Overall, $-\Delta\Phi \leq -2 + 4 \cdot 0.5 = 0$.
- (No Leaves) By assumption, there are at least eight boundary vertices, hence u and z do not share any boundary neighbors, that could end up as leaves.

6.20.2 Interior 5-wheel Suppose that deleting u and z creates a degree-2 cut vertex ℓ . Then $N[\ell] \subseteq G$ is a 5-wheel with u, z being antipodal vertices in $N(\ell)$. The other two vertices in $N(\ell)$, say, j, k , are both interior vertices, as there are at least eight boundary vertices. In particular, $\deg(u), \deg(z) \geq 5$ and $\deg(t) = 3, \deg(s) \geq 4$. See Figure 48.

- (Construction) Delete ℓ and add the edge $\{j, k\}$. Delete v and t . Force u by attaching a small A to u, p .
- (Domination) A minimum neat dominating set $S \subseteq H$ contains u and hence dominates G , as u dominates v, t, j, k, ℓ . In particular, the added edge $\{j, k\}$ is irrelevant. $\Delta s \leq 0$.
- (Penalty) Deleting three vertices decreases Φ by 3. Attaching a small A increases Φ by 2.5. Replacing ℓ by an edge only affects u and z . Since $\deg_G(u), \deg_G(z) \geq 5$, this does not create low-degree problems. Deleting v and t may create a low-degree problem involving s , but nothing else, increasing Φ by 0.5. Overall, $-\Delta\Phi \leq -3 + 2.5 + 0.5 = 0$.

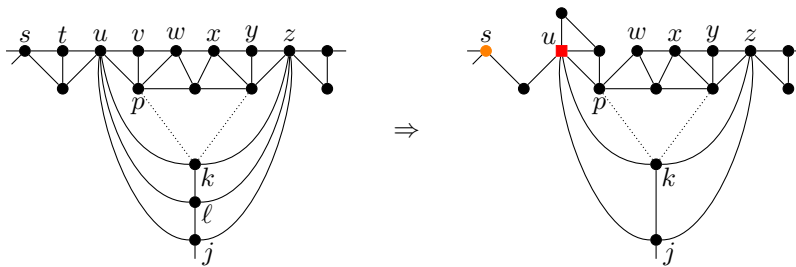


Figure 48: Degree pattern 3443, big polygon, with interior 5-wheel. The red vertex is forced and the orange vertex marks a potential low-degree problem. There is a small LR fused to the blue vertex (not drawn).

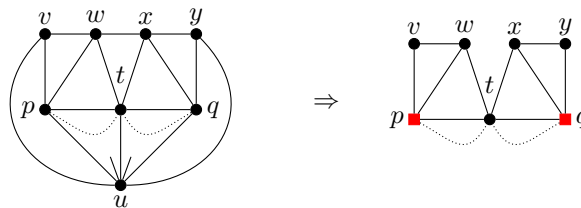


Figure 49: Degree pattern 3443, pentagon.

(Smaller) H has fewer interior vertices.

6.20.3 Conclusion Now, if there are consecutive boundary vertices with degrees 3, 4, 4, 3, then G has at most seven boundary vertices. Recall that due to previous cases, the polygon has at least four boundary vertices.

6.21 Degree pattern 3443, small polygon Suppose $\deg(v) = 3$, $\deg(w) = \deg(x) = 4$ and $\deg(y) = 3$. Let p be the interior vertex adjacent to v and let q be the interior vertex adjacent to y , then $p \neq q$ due to Case 6.15. Let t be the shared interior neighbor of p, w, x, q .

6.21.1 Square Suppose the polygon is a square. Then v, y are adjacent and both of degree 3, hence $p = q$, which is covered by Case 6.15.

6.21.2 Pentagon Suppose the polygon is a pentagon. Then $u = z$ and $\deg(u) \geq 4$. See Figure 49. Note that u has no interior degree-3 neighbor due to Case 6.14.

(Construction) Delete u . Delete the edge $\{w, x\}$, turning v, w into an A with red vertex p and x, y into a B with red vertex q .

(Domination) There is a minimum neat dominating set in H containing p and q , which dominate y . This works even if $\deg_H(p) = 3$ or $\deg_H(q) = 3$.

(Penalty) Deleting u decreases Φ by 1. This creates two low-degree problems, at v and y , increasing Φ by 1, but nothing else. Deleting the edge $\{w, x\}$ does not create any low-degree problems. $-\Delta\Phi \leq 0$.

6.21.3 Hexagon Suppose the polygon is a hexagon. Then $\deg(u) = 4$, $\deg(z) = 4$ and u, z are adjacent. Let $s \notin \{p, q\}$ be the interior vertex adjacent to u and z . If $s = r$, then G is the graph depicted in Figure 50, with $\Phi = 9$ and $s = 2$. Otherwise, u and z are not adjacent to r . Then, deleting the edge $\{u, z\}$ does not create any low-degree problems, contradiction. See Figure 51.

6.21.4 Conclusion Now, if there are consecutive boundary vertices with degrees 3, 4, 4, 3, then G has at exactly seven boundary vertices.

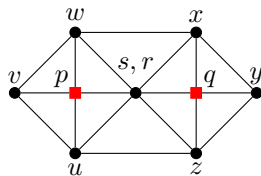


Figure 50: Degree pattern 3443, hexagon with few interior vertices. The depicted graph has $\Phi = 9$ and a dominating set of size 2, drawn in red.

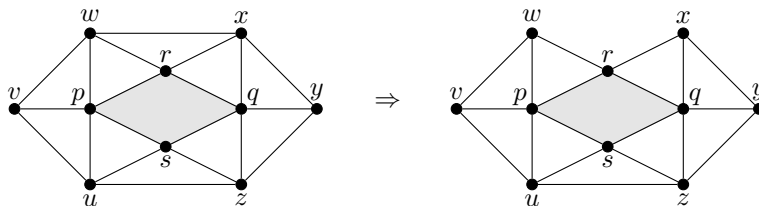


Figure 51: Degree pattern 3443, hexagon with many interior vertices.

6.22 Degree pattern 3443, Heptagon Suppose G has exactly seven boundary vertices, namely t, u, v, w, x, y, z , with t adjacent to z . Suppose that $\deg(v) = 3, \deg(w) = \deg(x) = 4$ and $\deg(y) = 3$, then always $\deg(t) = 3, \deg(u) \geq 4, \deg(z) = 3$ and $\deg(y) \geq 4$. Let p, q, s be the interior vertices adjacent to v, y, t respectively and let r be the interior vertex adjacent to p, w, x, q . See Figure 52. Then, p, q, s are interior vertices adjacent to a degree-3 boundary vertex and hence distinct, and $p \neq r \neq q$ as $\deg(u), \deg(z) \geq 4$. To summarize, out of all vertices we defined so far, the only two that can be equal are s and r .

Note that deleting $\{x, y\}$ does not create a 3-pair involving x . Similar for $\{v, w\}$ and w . If $s \neq r$, then x is not adjacent to s and Case 6.19 applies (which involves deleting s, t, y and forcing z). Therefore, assume $s = r$. If $\deg(z) \geq 5$, then Case 6.19 applies once again. Similar if $\deg(u) \geq 5$. The only remaining case is $\deg(u) = \deg(z) = 4$, then G is the special 4343434-Heptagon.

6.22.1 Conclusion Now G has no consecutive boundary vertices with degrees 3, 4, 4, 3. The only remaining degree patterns are 3, 5⁺, 3 and 3, 4, 3. In particular, every other boundary vertex has degree 3.

6.23 Degree 5⁺ boundary vertices Suppose there is a boundary vertex of degree ≥ 5 .

6.23.1 Big polygon Suppose there are at least five boundary vertices. Let $\deg(y) \geq 5$. Then Case 6.19 applies.

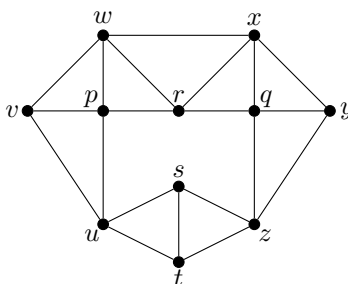


Figure 52: Degree pattern 3443, Heptagon.

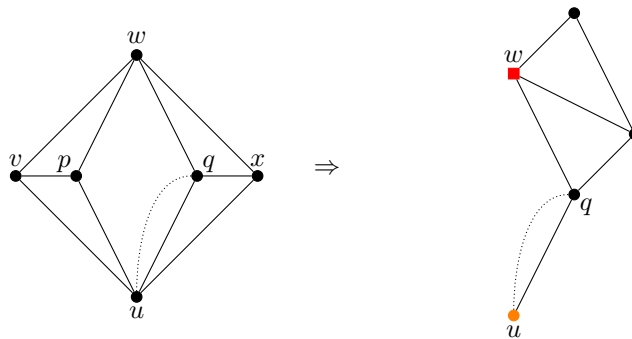


Figure 53: Degree 5^+ boundary vertices, square polygon.

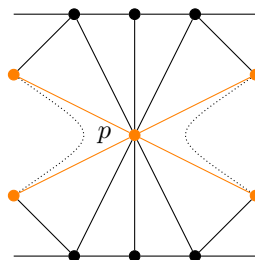


Figure 54: $\text{Int}(G)$ drawn in orange. The interior cut vertex p is adjacent to multiple boundary vertices of degree 3.

6.23.2 Square polygon Suppose there are exactly four boundary vertices. Let $\deg(u) \geq 5$, $\deg(v) = 3$, $\deg(w) \geq 4$ and $\deg(x) = 3$. Let p and q be the interior vertices adjacent to v and x , respectively. See Figure 53. The following construction closely mimics Case 6.19.

(Construction) Delete v, p . Delete x . Force w by attaching an A to w, q .

(Domination) There is a minimum neat dominating set in H that contains w , which dominates v, p, x . $\Delta s \leq 0$.

(Penalty) Deleting three vertices decreases Φ by 3. Attaching an A increases Φ by 2.5. The deletions may create a low-degree problem at w , but nowhere else. $-\Delta\Phi \leq -3 + 2.5 + 0.5 = 0$.

(No leaves) u does not end up as a leaf, as $\deg_G(u) \geq 5$.

6.23.3 Conclusion Now, all boundary vertices have degrees 3 and 4, in alternating fashion. If $\deg(u) = 3$, $\deg(v) = 4$ and $\deg(w) = 3$, then any interior vertex adjacent to v is adjacent to u or w . Therefore, $\text{Int}(G)$ is 2-connected unless it is a path on two vertices: any interior cut vertex would be adjacent to two distinct degree-3 boundary vertices, see Figure 54.

6.24 Degree pattern 34343 Let $\deg(t) = 3$, $\deg(u) = 4$, $\deg(v) = 3$, $\deg(w) = 4$ and $\deg(x) = 3$. Suppose G has at least five boundary vertices, then $t \neq x$. Let p be the interior vertex adjacent to v and let q be the interior vertex adjacent to x .

As G has at least five boundary vertices, $\text{Int}(G)$ is not a path on two vertices. Suppose we delete $\{w, x\}$. If this results in a 3-pair u, v in some bad 5-wheel, then $q \in \text{Int}(G)$ is a cut vertex, contradiction. See Figure 55.

Therefore, deleting any one boundary edge never creates a 3-pair in a bad 5-wheel. Then, the second case of Case 6.19 applies: Deleting $\{x, y\}$ does not create a 3-pair in a bad 5-wheel and y is not adjacent to p : otherwise, deleting $\{v, w\}$ would create the 3-pair w, x in the bad 5-wheel $N[q]$, see Figure 56.

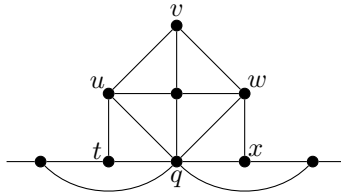


Figure 55: Degree pattern 34343: If deleting $\{w, x\}$ creates a 3-pair v, w in a bad 5-wheel, then u, q is the base of the 5-wheel and q is an interior cut vertex.

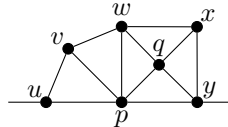


Figure 56: Degree pattern 34343: If y is adjacent to p , then deleting $\{v, w\}$ creates the 3-pair w, x in a bad 5-wheel $N[q]$.

6.24.1 Conclusion Now, G has exactly four boundary vertices, with degrees 3, 4, 3, 4. Therefore, G is the 3-bifan.

7 A quadratic time algorithm

Finally, let us discuss how to turn our proof into an algorithm.

THEOREM 7.1. *Let G be a skeletal triangulation on n vertices that is not a sporadic example. Then, there is an algorithm that finds a dominating set of size $\lfloor \frac{\Phi(G)}{3.5} \rfloor$ in $\mathcal{O}(n^2)$ time.*

Roughly speaking, our proof of Theorem 5.1 involves two different types of steps:

- (a) Find a small configuration in G , delete some vertices and edges to obtain H and inductively find a dominating set in H . Turn that into a neat dominating set in H and then into a dominating set in G .
- (b) Split the graph into two or more parts, determine the act as type of each part and, for each part, recursively find a rooted dominating set that conforms to this Act as type.

Note that sections 6.1, 6.2 and 6.4 are the only ones that involve steps of type (b). Steps of type (a) can easily be implemented with a single recursive call to Theorem 7.1 and $\mathcal{O}(n)$ additional time. Steps of type (b) are problematic, as determining the Act as type is hard. Trying multiple possible acts-as types is also not feasible, as doing multiple recursive calls into the same part results in exponential running time.

To get around this, for each part, we instead compute ϕ and then guess the acts type to be the “worst” possible one according to Theorems 5.2 and 5.3. This guess turns out to have all the properties needed for our proof, even if it might not match the actual acts as type. We also show that a “conforming” dominating set can be found with a single recursive call to Theorem 7.1 on each part. This ensures an $\mathcal{O}(n^2)$ running time.

LEMMA 7.1. *Let (G, u) be a rooted skeletal triangulation. Let s be an arbitrary integer and let $\phi = \phi(G, u)$.*

1. *If $\phi < 3.5s$, then G has a rooted dominating set of size s that contains u .*
2. *If $\phi < 3.5s + 1.5$, then G has a dominating set of size s .*
3. *If $\phi < 3.5(s + 1) - 1$, then G has a rooted dominating set of size s .*

Moreover, these dominating sets can be found algorithmically via a single call to Theorem 7.1.

Proof. We mimic the proof of Proposition 5.2.

1. Suppose $\phi < 3.5s$. Let $\{u, v\}$ be a boundary edge incident to u . Attach a small A with red vertex u to u, v . Then $\phi(H) = \phi + 3.5 < 3.5(s + 1)$. By Theorem 7.1, H has a dominating set of size $\lfloor \frac{\Phi(H)}{3.5} \rfloor = s$. In linear time, we turn this into a neat dominating set of size s , which contains u due to the attached A .
2. Suppose $\phi < 3.5s + 1.5$. If G is a sporadic example, check by hand. If $\deg(u) = 1$, then apply case (1) to $G - u$. Otherwise, G is a skeletal triangulation, with $\Phi(G) < 3.5s + 3 + 0.5$. Then, Theorem 7.1 yields a dominating set of size $\lfloor \frac{\Phi(G)}{3.5} \rfloor = s$.
3. Suppose $\phi < 3.5(s + 1) - 1$. Fuse a small LR to u , then $\Phi(H) = \phi + 4.5 < 3.5(s + 2)$. By Theorem 7.1, H has a dominating set S of size $\lfloor \frac{\Phi(H)}{3.5} \rfloor = s + 1$, then $S \cap H$ is a rooted G -dominating set of size s .

□

With this lemma, we guess as follows:

1. If $3.5s - 1 \leq \phi < 3.5s$, guess that G acts as AB.
2. If $3.5s \leq \phi < 3.5s + 1.5$, guess that G acts as LR.
3. If $3.5s + 1.5 \leq \phi < 3.5(s + 1) - 1$, guess that G acts as Nope.

This guessing strategy ensures that G both has the requisite dominating sets (Lemma 7.1) and that ϕ satisfies the bounds in Theorem 5.2. This ensures that the steps in Cases 6.1 and 6.2 work.

Chords For Step 6.4, we use similar ideas, but some care has to be taken to distinguish A from B and L from R.

LEMMA 7.2. *Let (G, u, v) be a rooted near-triangulation. Let s be an arbitrary integer and let $\phi = \phi(G, u, v)$.*

1. *If $\phi < 3.5s - 1$, then G has two dominating set of size s , one containing u and one containing v .*
2. *If $\phi < 3.5s$, then G has a dominating set of size s that contains u or v .*
3. *If $\phi < 3.5s + 0.5$, then G has two rooted dominating set of size s , one dominating u and one dominating v .*
4. *If $\phi < 3.5s + 1.5$, then G has a rooted dominating set of size s that dominates u or v .*
5. *If $\phi < 3.5(s + 1) - 2$, then G has a rooted dominating set of size s .*

Moreover, these dominating sets can be found algorithmically via Theorem 7.1.

Proof. The proof follows along the same lines as the proof of Proposition 5.3. We only prove (2) and (4), which are the most interesting parts.

2. Suppose $\phi < 3.5s$. Attach a small OR to u, v . Then, $\Phi(H) = \phi(G) + 3.5 < 3.5(s + 1)$. By Theorem 7.1, H has a dominating set of size $\lfloor \frac{\Phi(H)}{3.5} \rfloor = s$, which, due to the OR, contains u or v .
4. Suppose $\phi < 3.5s + 1.5$. If $\phi(G, u) = \phi(G, u, v) + 1$, then $\phi(G, u) < 3.5(s + 1) - 1$ and Lemma 7.1 yields a rooted dominating set of size s that dominates v . Similarly, if $\phi(G, v) = \phi(G, u, v) + 1$, then there is a rooted dominating set of size s that dominates u . In all other cases, u, v is a 3-pair in a bad 5-wheel. Let $H = G/\{u, v\}$, then $\phi(H, uv) = \phi(G, u, v) < 3.5s + 1.5$. By Lemma 7.1, there is a H -dominating set of size s . A neat such set does not contain uv , and hence is a rooted G -dominating set that dominates at least one of u and v .

□

With this lemma, we guess as follows:

1. If $3.5s - 2 \leq \phi < 3.5s - 1$, guess that G acts as OR.

2. If $3.5s - 1 \leq \phi < 3.5s$, guess that G acts as one of A, B.
3. If $3.5s \leq \phi < 3.5s + 0.5$, guess that G acts as L OR R.
4. If $3.5s + 0.5 \leq \phi < 3.5s + 1.5$, guess that G acts as one of L, R.
5. If $3.5s + 1.5 \leq \phi < 3.5(s + 1) - 2$, guess that G acts as None. (This never happens.)

Proceed as in Step 6.4 of the proof. Note that in cases (2) and (4), the algorithmic nature of Lemma 7.2 yields the actual (rooted) dominating sets, which then allows us to distinguish A from B and L from R. Also note that, instead of guessing that G acts as None, we guess that G acts as OR. Intuitively, this makes sense, as we can just add one of u, v to a rooted dominating set.

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