Integration of two-dimensional complex functions

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a horizontal antenna can be removed if it is situated at least $(0.7/\sqrt{|\epsilon|})\lambda$ away from the ground plane. For the worst case $\phi(x, r) = 0$, it has been found that the reflection-coefficient method yields a solution within 10 percent of the exact Sommerfeld formulation both in the real and imaginary parts of impedance elements even when two horizontal currents are as much as 1000λ apart. All of the above restrictions are valid even for low values of the dielectric constant of the ground plane ($|\epsilon| \approx 2$). However, for parallel vertical wires over an imperfect ground plane the reflection coefficient method yields a result accurate to within 10 percent of the exact Sommerfeld formulation in both the real and imaginary parts of the impedance elements under all conditions of the ground as long as they are $(0.7/\sqrt{|\epsilon|})\lambda$ away from the surface of the ground plane.

Input data required for this program include the conductivity and the dielectric constant of the ground, the operating frequency, the total number of elements in the linear array, the radii and feed voltages of the wires, the lengths and spacings of individual elements, and the angular steps at which the specified field pattern is to be computed.

Computer output consists of all input data together with the current distribution of each wire, input impedances corresponding to feed points, and the normalized E -field pattern above earth with the appropriate normalization constant.

Storage required for execution of the analysis program which has been written in Fortran IV is 45.8 k-bytes. Time required by an IBM 370/150 computer for analyzing a linear array consisting of two half-wave wires using 7 expansion functions over each wire was 1.92 and 19.36 s for compilation and execution, respectively.

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Integration of Two-Dimensional Complex Functions

NAMES: QATR2, QATR3, SIMP2, SIMP3.

PURPOSE: To compute an approximation for the integral of complex function $F(x, y)$, summed over real values of x and y from x_l to x_u and y_l to y_u .

LANGUAGE:

Fortran IV.

AUTHORS:

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AVAILABILITY: ASIS-NAPS Document NAPS 02786.

DESCRIPTION: The package consists of four integration programs each requiring approximately 3.2 K bytes on an IBM 370/165.

Consider the integral J of a complex valued function F of two real variables (x, y)

$$J = \int_{y_l}^{y_u} \int_{x_l}^{x_u} F(x, y) dx dy \quad (1)$$

taken over the rectangular domain

$$x_l \leq x \leq x_u, \quad y_l \leq y \leq y_u. \quad (2)$$

An approximation to J may be obtained by recursive application of a one-dimensional rule:

$$J = \int_{y_l}^{y_u} f(y) dy \quad (3)$$

where

$$f(y) = \int_{x_l}^{x_u} F(x, y) dx \quad (4)$$

where J and $f(y)$ are the outer and inner integral, respectively.

Two one-dimensional rules have been implemented for evaluation of (3) and (4), namely the Simpson and Romberg methods [1]. Both make use of the trapezoidal approximation $T_r^{(0)}$ to the integral calculated by means of a recurrence algorithm [2]. The Romberg extrapolated approximations to the integral are given by

$$T_r^{(m)} = \frac{4^m T_r^{(m-1)} - T_{r-1}^{(m-1)}}{4^m - 1}, \quad m = 1, 2, \dots, r \quad (5)$$

and the successive values of T are stored in the T -table shown in Table I. The Simpson method falls out as the special case $m = 1$ corresponding to the second column in Table I.

Fig. 1(a) shows the hierarchical structure of the computer programs applying the two methods. QATR2 (Romberg) or SIMP2 (Simpson) calculates the outer integral and QATR1 or SIMP1 the inner integral. MAIN and FCT are user-supplied programs of which FCT evaluates the complex function $F(x, y)$.

The Romberg integration process on the inner and outer integral is stopped whenever

$$\left| \frac{T_r^{(m)} - T_{r-1}^{(m-1)}}{T_{r-1}^{(m-1)}} \right| < \epsilon, \quad \text{for } m = r, r > 2 \quad (6)$$

TABLE I
COMPLEX ROMBERG T -TABLE IN ONE-DIMENSIONAL INTEGRAL

r	Number of Integrand Samples $2^r + 1$	Trapezoidal and Extrapolated Approximations			
0	2	$T_0^{(0)}$			
1	3	$T_1^{(0)}$	$T_1^{(1)}$		
2	5	$T_2^{(0)}$	$T_2^{(1)}$	$T_2^{(2)}$	
3	9	$T_3^{(0)}$	$T_3^{(1)}$	$T_3^{(2)}$	$T_3^{(3)}$
4	17				...
5	33				

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Fig. 1. Hierarchical structure of complex two-dimensional integration programs. (a) Romberg (Simpson) integration of inner and outer integration consecutively. (b) Romberg (Simpson) integration by two-dimensional algorithm.

where ε is a user specified small positive number. The corresponding stop criterion in the Simpson integration is

$$\left| \frac{T_r^{(1)} - T_{r-1}^{(1)}}{T_{r-1}^{(1)}} \right| < \varepsilon, \quad r > 2. \quad (7)$$

The programs QATR3 and SIMP3 make use of a two-dimensional trapezoidal algorithm [3] to approximate (1) directly:

$$T_r^{(0)} = \frac{1}{4} T_{r-1}^{(0)} + \frac{\Delta x \Delta y}{2^{2r}} \left[\frac{1}{2} \sum_j F_j^{bo} + \sum_k F_k^{in} \right]$$

$$T_0^{(0)} = \Delta x \Delta y \frac{1}{4} \sum_{i=1}^4 F_i^{co}$$

$$\Delta x = x_u - x_l, \Delta y = y_u - y_l \quad (8)$$

where the i -summation includes the integrand sample points in the four corners of the rectangular integration interval. It is understood from (8) that the r th trapezoidal approximation $T_r^{(0)}$ to the integrand is defined by $T_{r-1}^{(0)}$ and two summations over new sample points not involved in the calculation of $T_{r-1}^{(0)}$. The new sample points are homogeneously distributed over the integration interval by halving the mesh size (see Fig. 2). In (8) the superscript *bo* indicates new sample points at the boundary of the integration interval whereas superscript *in* indicates interior points. Subsequently (5) is used for the Romberg extrapolations (QATR3) and with $m = 1$ for the Simpson method (SIMP3). The convergence criteria used are as defined in (6) and (7). The program structure is shown in Fig. 1(b).

The program listings contain extensive information on input and output parameters. An example is shown in Fig. 3 for QATR2.

The programs described above have been used for the calculation of two-dimensional phase integrals and numerical results are given in [4].

Fig. 2. Two-dimensional trapezoidal rule, $r = 2$.

PURPOSE

TO COMPUTE AN APPROXIMATION FOR INTEGRAL OF COMPLEX FUNCTION FCT(x_1, x_2), SUMMED OVER x_1 AND x_2 FROM x_{l1} TO x_{u1} AND x_{l2} TO x_{u2} .

USAGE

CALL QATR2($x_{l1}, x_{u1}, x_{l2}, x_{u2}, \text{EPS}, \text{NDIM}, \text{FCT}, Y, \text{IER}, \text{AU1}, \text{AU2}$)
PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT.

DESCRIPTION OF PARAMETERS

x_{l1}, x_{l2} - THE LOWER BOUNDS OF THE INTERVALS.
 x_{u1}, x_{u2} - THE UPPER BOUNDS OF THE INTERVALS.
EPS - THE UPPER BOUND OF THE RELATIVE ERROR.
NDIM - THE DIMENSION OF THE AUXILIARY STORAGE ARRAYS.
NDIM-1 IS THE MAXIMAL NUMBER OF BISECTIONS OF THE INTERVALS (x_{l1}, x_{u1}) AND (x_{l2}, x_{u2}).
FCT - THE NAME OF THE EXTERNAL FUNCTION SUBPROGRAM USED.
Y - THE RESULTING APPROXIMATION FOR THE INTEGRAL VALUE.
IER - A RESULTING ERROR PARAMETER.
 $\text{AU1}, \text{AU2}$ - AUXILIARY STORAGE ARRAYS WITH DIMENSIONS NDIM.

REMARKS

ERROR PARAMETER IER IS CODED IN THE FOLLOWING FORM
IER=0 - IT WAS POSSIBLE TO REACH THE REQUIRED ACCURACY.
NO ERROR.
IER=2 - IT WAS IMPOSSIBLE TO CHECK ACCURACY BECAUSE NDIM IS LESS THAN 3, OR THE REQUIRED ACCURACY COULD NOT BE REACHED WITHIN NDIM-1 STEPS. NDIM (OR EPS) SHOULD BE INCREASED.

Fig. 3. Input and output parameters for integration program QATR2.

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