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[1] We use the signature $(-, +, +, +)$:

$$\Gamma^{\sigma}_{\mu\nu} \equiv \tfrac{1}{2} g^{\sigma\tau}(g_{\mu\tau,\nu} + g_{\nu\tau,\mu} - g_{\mu\nu,\tau}),$$
$$R_{\mu\nu\sigma}{}^{\tau} \equiv \Gamma_{\nu\sigma,}{}^{\tau}{}_{\mu} - \Gamma_{\mu\sigma,}{}^{\tau}{}_{\nu} + \Gamma_{\nu\sigma}{}^{\rho}\Gamma_{\mu\rho}{}^{\tau} - \Gamma_{\mu\sigma}{}^{\rho}\Gamma_{\nu\rho}{}^{\tau},$$
$$R_{\mu\nu} \equiv R_{\mu\tau\nu}{}^{\tau}.$$

A comma denotes ordinary differentiation; a semicolon denotes covariant differentiation; $16\pi G = 1$, $c = 1$, G is the gravitational constant, and c is the velocity of light. All other notation is defined in the text or is standard.

[2] For a review of the geometric theory, see *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), Chap. 9.

[3] J. L. Synge, *Relativity: The General Theory* (Interscience, New York, 1960), Chap. VIII; K. S. Thorne, Ph.D. thesis, University Microfilms, Ann Arbor, 1965.

[4] The general solutions to Cases I and II of the possible field configurations and a particular solution of Case III is given in Ref. 2 and also in a paper by L. Witten, Colloq. Theor. Relativity, Centre Belge Rech. Math., Univ. (Louvain, Belgium), p. 59, 1960.

[5] This differs slightly from the line element given in Ref. 4. $\mu \equiv 0$ requires that we take the limit $l \to \infty$ in the line elements of Ref. 4. The necessity that l approach ∞ can be argued for on a variety of grounds, which we shall not present here.

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Hypergeometric Functions with Integral Parameter Differences

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For a generalized hypergeometric function ${}_pF_q(z)$ with positive integral differences between certain numerator and denominator parameters, a formula expressing the ${}_pF_q(z)$ as a finite sum of lower-order functions is proved. From this formula, Minton's two summation theorems for $p = q + 1$, $z = 1$ are deduced, one of these under less restrictive conditions than assumed by Minton.

This paper deals with generalized hypergeometric functions ${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z)$ having the special property that, with suitable enumeration of parameters, $a_i = b_i + m_i$, $i = 1, 2, \cdots, n$, where $m_1, \cdots, m_n$ are positive integers and $n \le \min(p, q)$. It is assumed that $p \le q + 1$ and that no denominator parameter b is a negative integer or zero. A function of this type may be expressed as a finite sum of ${}_{p-n}F_{q-n}$ functions in the following way:

$$ {}_pF_q\begin{bmatrix} b_1 + m_1, \cdots, b_n + m_n, a_{n+1}, \cdots, a_p; z \\ b_1, \cdots, b_n, b_{n+1}, \cdots, b_q \end{bmatrix} = \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} A(j_1, \cdots, j_n) z^{J_n} \, {}_{p-n}F_{q-n}\begin{bmatrix} a_{n+1} + J_n, \cdots, a_p + J_n; z \\ b_{n+1} + J_n, \cdots, b_q + J_n \end{bmatrix}, \quad (1)$$

where

$$J_n = j_1 + \cdots + j_n, \quad (2)$$

$$A(j_1, \cdots, j_n) = \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} \frac{(b_2 + m_2)_{J_1}(b_3 + m_3)_{J_2} \cdots (b_n + m_n)_{J_{n-1}}(a_{n+1})_{J_n} \cdots (a_p)_{J_n}}{(b_1)_{J_1}(b_2)_{J_2} \cdots (b_n)_{J_n}(b_{n+1})_{J_n} \cdots (b_q)_{J_n}}, \quad (3)$$

and

$$(c)_r = \Gamma(c + r)/\Gamma(c). \quad (4)$$

By the principle of analytical continuation, Eq. (1) is valid whenever the functions involved are all analytic; restrictions upon the parameters imposed in the proof may thus be removed.

The proof is based upon an Eulerian integral representation given by Erdélyi,[1] viz.,

$$ {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \frac{\Gamma(b_1)\Gamma(1 - b_1 + a_1)}{\Gamma(a_1)\exp(i\pi(b_1 - a_1))} \frac{i}{2\pi} \times \int_0^{(1+)} (1 - t)^{b_1 - a_1 - 1} f(t)\, dt, \quad (5)$$

where

$$f(t) = t^{a_1 - 1} \, {}_{p-1}F_{q-1}(a_2, \cdots, a_p; b_2, \cdots, b_q; zt), \quad (6)$$

valid when $\mathrm{Re}\, a_1 > 0$, b_1 is not a negative integer or zero, and $|\arg(1-z)| < \pi$ if $p = q+1$. Now, as $a_1 = b_1 + m_1$, the branch point of the integrand at $t = 1$ disappears, and the integral takes the form $\int_C f(t)\,dt/(t-1)^{m_1+1}$, where C is a closed contour encircling the point $t = 1$ counterclockwise and f is analytic within and on C. From Cauchy's integral formula, we then find that Eq. (5) becomes

$$ {}_pF_q\begin{bmatrix} b_1+m_1, a_2, \cdots, a_p; z \\ b_1, b_2, \cdots, b_q \end{bmatrix} = \frac{D^{m_1}f(1)}{(b_1)_{m_1}}, \quad (7) $$

D denoting differentiation with respect to t. Application of Leibniz's differentiation formula and the well-known expression for the derivative of a ${}_pF_q$ then yields

$$ {}_pF_q\begin{bmatrix} b_1+m_1, a_2, \cdots, a_p; z \\ b_1, b_2, \cdots, b_q \end{bmatrix} = \sum_{j=0}^{m_1} z^j \binom{m_1}{j} \frac{(a_2)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \times {}_{p-1}F_{q-1}\begin{bmatrix} a_2+j, \cdots, a_p+j; z \\ b_2+j, \cdots, b_q+j \end{bmatrix}. \quad (8) $$

This result can itself be applied to each member of its rhs if $a_2 = b_2 + m_2$, etc. It is easily seen that the general result (1) is obtained in this way.

A special case ($p = 3$, $q = 2$) of Eq. (8) has been derived by Rösler[2] from the series representation.

From formula (1) we now derive two summation theorems for $p = q+1$, $z = 1$. These have been given recently by Minton,[3] the first one, however, under more restrictive conditions than those given below.

To deduce the first theorem, we take $q = p - 1 = n + 1$, $a_{n+1} = b = b_{n+1} - 1$, $z = 1$, and for brevity $a_{n+2} = a$. The hypergeometric functions in the multiple sum of Eq. (1) then become ${}_2F_1(1)$'s, which all exist provided that

$$ \mathrm{Re}\,(-a) > m_1 + \cdots + m_n - 1. \quad (9) $$

By Gauss' summation theorem we then get, after some rearrangements,

$$ {}_{n+2}F_{n+1}(b_1+m_1, \cdots, b_n+m_n, b, a; b_1, \cdots, b_n, b+1; 1) = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(b+1-a)} \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} B_n(j_1, \cdots, j_n), $$

where

$$ B_n(j_1, \cdots, j_n) = \frac{(b+1)_{J_n}}{(a)_{J_n}} (-1)^{J_n} A(j_1, \cdots, j_n). $$

Next, the definition (3) is applied, the binomial coefficients being written in the form $(-1)^j(-m)_j/j!$; this leads to

$$ B_n(j_1, \cdots, j_n) = B_{n-1}(j_1, \cdots, j_{n-1}) \times \frac{(b_n+m_n)_{J_{n-1}}}{(b_n)_{J_{n-1}}} \frac{(-m_n)_{j_n}(b+J_{n-1})_{j_n}}{j_n!\,(b_n+J_{n-1})_{j_n}}. $$

The terms containing j_n obviously constitute a terminating ${}_2F_1(1)$, which is summed by Gauss' theorem. After some rearrangements we obtain (summation limits understood)

$$ \sum_{j_1, \cdots, j_n} B_n(j_1, \cdots, j_n) = \frac{(b_n+m_n)_{-b}}{(b_n)_{-b}} \sum_{j_1, \cdots, j_{n-1}} B_{n-1}(j_1, \cdots, j_{n-1}). $$

Repeating this procedure, we finally arrive at Minton's first theorem,

$$ {}_{n+2}F_{n+1}(b_1+m_1, \cdots, b_n+m_n, b, a; b_1, \cdots, b_n, b+1; 1) = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(b+1-a)} \prod_{k=1}^{n} \frac{(b_k+m_k)_{-b}}{(b_k)_{-b}}, \quad (10) $$

valid under the condition (9), i.e., if the lhs of (10) exists at all. In Minton's proof,[3] a was required to be negative integral.

The particular case $n = 1$ of Eq. (10) was obtained by Mitra[4] by series manipulations.

The second summation theorem may be deduced from the first[3] by letting $b \to \infty$. It may, however, also be deduced directly from Eq. (1) by taking $q = p - 1 = n$, $a_{n+1} = -(m_1 + \cdots + m_n)$, and $z \to 1$. The hypergeometric functions in the multiple sum of Eq. (1) then reduce to power functions $(1-z)^h$, where $h = -a_{n+1} - J_n$ and $h \geq 0$ for all terms. When $z \to 1$, all terms of the multiple sum will thus tend to zero, except the one for which $-a_{n+1} = J_n$, i.e., $j_i = m_i$, $i = 1, 2, \cdots, n$; the limit of this term is $A(m_1, \cdots, m_n)$. After some reductions we find the summation formula

$$ {}_{n+1}F_n(b_1+m_1, \cdots, b_n+m_n, -(m_1+\cdots+m_n); b_1, \cdots, b_n; 1) = \frac{(-1)^{m_1+\cdots+m_n}(m_1+\cdots+m_n)!}{(b_1)_{m_1} \cdots (b_n)_{m_n}}, \quad (11) $$

which is easily transformed to the form given by Minton.[3]

It may be of interest to compare Eq. (11) with the special case of (10) obtained by taking $b_n = b$, $m_n = 1$ and then replacing $n - 1$ by n, viz.,

$$ {}_{n+1}F_n(b_1+m_1, \cdots, b_n+m_n, a; b_1, \cdots, b_n; 1) = 0, \quad \mathrm{Re}\,(-a) > m_1 + \cdots + m_n. \quad (12) $$

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