DTU Library

Analytic weak-stable manifolds in unfoldings of saddle-nodes

Kristiansen, Kristian Uldall; Szmolyan, Peter

Published in:
Nonlinearity

Link to article, DOI:
10.1088/1361-6544/ada67a

Publication date:
2025

Document Version
Publisher's PDF, also known as Version of record

OPEN ACCESS
IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity **38** (2025) 025019 (70pp) https://doi.org/10.1088/1361-6544/ada67a

Analytic weak-stable manifolds in unfoldings of saddle-nodes*

Kristian Uldall Kristiansen[1,**] and **Peter Szmolyan**[2]

[1] Department of Applied Mathematics and Computer Science, Technical University of Denmark, 2800 Kgs. Lyngby, Denmark
[2] Institute of Analysis and Scientific Computing, TU Wien, Vienna, Austria

E-mail: krkri@dtu.dk and peter.szmolyan@tuwien.ac.at

Received 26 June 2024; revised 25 November 2024
Accepted for publication 6 January 2025
Published 24 January 2025

Recommended by Dr Rafael de la Llave

Abstract

Any attracting, hyperbolic and proper node of a two-dimensional analytic vector-field has a unique strong-stable manifold. This manifold is analytic. The corresponding weak-stable manifolds are, on the other hand, not unique, but in the nonresonant case there is a unique weak-stable manifold that is analytic. As the system approaches a saddle-node (under parameter variation), a sequence of resonances (of increasing order) occur. In this paper, we give a detailed description of the analytic weak-stable manifolds during this process. In particular, we relate a 'flapping-mechanism', corresponding to a dramatic change of the position of the analytic weak-stable manifold as the parameter passes through the infinitely many resonances, to the lack of analyticity of the centre manifold at the saddle-node. Our work is motivated and inspired by the work of Merle, Raphaël, Rodnianski, and Szeftel, where this flapping mechanism is the crucial ingredient in the construction of C^∞-smooth self-similar solutions of the compressible Euler equations.

* Dedicated to the memory of Claudia Wulff. A dear friend and a respected colleague.
** Author to whom any correspondence should be addressed.

Keywords: analytic weak-stable manifolds, centre manifolds, Gevrey properties, saddle-nodes

Mathematics Subject Classification numbers: 34C23, 34C45, 37G05, 37G10

Contents

1. Introduction

In this paper, we consider the following analytic normal form (based upon [19, theorem 2.2]) for the unfolding of a planar saddle-node bifurcation:

$$\begin{aligned}\dot{x} &= (x-\epsilon)x,\\ \dot{y} &= -y(1+a^\epsilon x)+g^\epsilon(x,y),\end{aligned} \tag{1}$$

with $g^\epsilon(x,y)=\mathcal{O}(x^2,x^2y,xy^2)$, see theorem 3.1 below. Here it is important to emphasise that this formulation of the unfolding of the saddle-node is only valid on the singularity side of the saddle-node; this means that the unfolding parameter $\epsilon\geqslant 0$ is treated nonnegative only. In particular, the functions a^ϵ and g^ϵ depend analytically on the unfolding parameter $\epsilon\in[0,\epsilon_0)$, $\epsilon_0>0$. For $\epsilon\in(0,\epsilon_0)$, there is a saddle at $(\epsilon,\mathcal{O}(\epsilon^2))$ and a node at the origin, see figure 1. It is well-known that the saddle's stable and unstable manifolds, W^s and W^u, are analytic. The linearisation of the node has eigenvalues $-\epsilon$ and -1. It is therefore resonant for $\epsilon^{-1}\in\mathbb{N}$. When

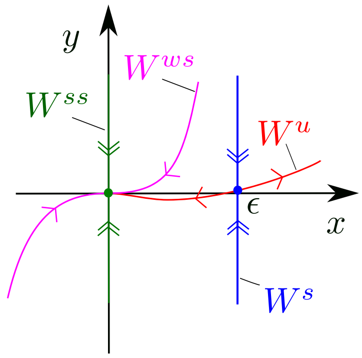

Figure 1. Phaseportrait of (1) for $0<\epsilon\ll 1$, $\epsilon^{-1}\notin\mathbb{N}$, with a hyperbolic saddle at $(\epsilon,\mathcal{O}(\epsilon^2))$ with stable and unstable manifolds (W^s and W^u in blue and red, respectively) and a hyperbolic proper node at the origin. The node always has a unique strong-stable manifold (W^{ss} in green) and in the nonresonant case ($\epsilon^{-1}\notin\mathbb{N}$) a unique analytic weak-stable manifold (W^{ws} in magenta).

the node is nonresonant ($\epsilon^{-1}\notin\mathbb{N}$) it is known [7, theorem 2.15] that the node can be linearised locally by an analytic change of coordinates to the form

$$\begin{aligned}\dot{x} &= -\epsilon x,\\ \dot{y} &= -y.\end{aligned} \tag{2}$$

Here $x=0$ is the strong-stable manifold W^{ss}, which is analytic. The invariant curves $y=c|x|^{\epsilon^{-1}}$, $c\neq 0$, tangent to the weak eigendirection at $x=0$, are all weak-stable invariant manifolds with finite smoothness at $x=0$. The set $\{y=0\}$ is therefore the unique analytic weak-stable manifold W^{ws} of (2). At a resonance $\epsilon^{-1}=N\in\mathbb{N}$, the node can be brought into the analytic normal form

$$\begin{aligned}\dot{x} &= -N^{-1}x,\\ \dot{y} &= -y+bx^N,\end{aligned} \tag{3}$$

see [7, theorem 2.15]. All weak-stable manifolds of (3) take the graph form

$$y = x^N c - bNx^N\log|x|,\quad c\in\mathbb{R},$$

and therefore have finite smoothness at the origin in the generic case $b\neq 0$. Specifically, there is no analytic weak-stable manifold in this case. Note that this classification in the context of the normal form (1) is (clearly) nonuniform with respect to $\epsilon>0$.

For further background on normal forms, including formal and analytic linearisations, centre manifolds and stable and unstable manifolds, we refer to the excellent book [7].

In the present paper, we provide a detailed description of the analytic weak-stable manifold W^{ws} of (1) for all $0<\epsilon\ll 1$ (see our hypotheses 1 and 2 below). Our overall strategy follows [16]. Here the authors constructed C^∞-smooth invariant manifolds (for a specific rational system) by matching a global unstable manifold with an analytic weak-stable manifold close to a saddle-node $\epsilon\to 0$. These invariant manifolds correspond to C^∞-smooth self-similar solutions of the isentropic ideal compressible Euler equations that were used in [17] to determine finite time energy blowup solutions of Navier–Stokes equations (isentropic ideal compressible), see also [15] for applications to the defocusing nonlinear Schrödinger equation.

In order to control the analytic weak-stable manifolds, the authors of [16] first apply a new approach for the centre manifold W^c at $\epsilon = 0$. In particular, they define a number S^0_∞, which depends on the nonlinearity (in our case, it will depend on the full jet of g^0), and show that if this quantity is nonzero $S^0_\infty \neq 0$, then a 'leading order term' of the analytic weak-stable manifold can be determined. The proof of the main result of [16] is not based upon dynamical systems theory but rather on careful estimation and boot-strapping arguments in order to bound the growth of the coefficients of certain series expansions.

In the context of (1), the centre manifold W^c is defined for $\epsilon = 0$:

$$\begin{aligned}\dot{x} &= x^2,\\ \dot{y} &= -y\left(1+a^0x\right)+g^0\left(x,y\right),\end{aligned} \tag{4}$$

as an invariant manifold of the graph form $y = m^0(x)$ tangent to $y = 0$. This means that $y = m^0(x)$ solves

$$x^2y' = -y\left(1+a^0x\right)+g^0\left(x,y\right),$$

obtained from (4) by eliminating time. It is well-known that although m^0 has a well-defined formal series expansion, which we denote by

$$\widehat{m}^0\left(x\right) = \sum_{k=2}^{\infty} m^0_k x^k, \tag{5}$$

it is in general only C^∞-smooth, see e.g. [7, theorem 2.19]. As an example of a nonanalytic centre manifold, consider $a^0 = 0$, $g^0(x,y) = x^2$:

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = -y + x^2.$$

This y-linear case corresponds to Euler's famous example. Here one can easily show that $y = \sum_{k=2}^{\infty} m^0_k x^k$ (by term-wise differentiation of the series) leads to

$$m^0_k = \left(-1\right)^k\left(k-1\right)! \quad \forall k \geqslant 2.$$

Consequently, we have $m^0_k x^k \not\to 0$ as $k \to \infty$ for any $x \neq 0$ and the centre manifold is therefore nonanalytic. The nonanalyticity of centre manifolds is also intrinsically related to their nonuniqueness (see e.g. figure 2 below for $x < 0$).

In general, it is also known, see e.g. [4], that the formal series expansion $\widehat{m}^0(x) = \sum_{k=2}^{\infty} m^0_k x^k$ is Gevrey-1:

$$|m^0_k| \leqslant CD^{-k}k! \quad \forall k \geqslant 2, \tag{6}$$

for some $C, D > 0$. (In fact, this formal series is 1-summable along any sector that is not centred along the negative real axis, see [2, chapter 3] and [4] for further details.) The Gevrey-1 property of the formal series in (6) also holds true for one-dimensional centre manifolds of higher dimensional saddle-nodes (with one single zero eigenvalue of the linearisation). We refer to [4] for further details. In contrast, higher dimensional centre manifolds may only have finite C^k-smoothness, see [22].

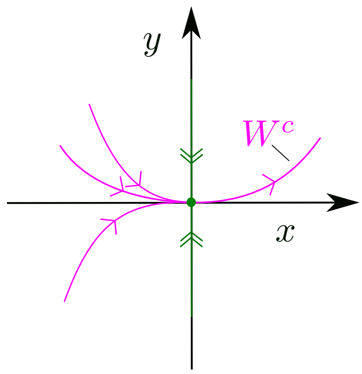

Figure 2. The saddle node for $\epsilon = 0$. The centre manifold (W^c in magenta) is only unique on the positive side of $x = 0$.

1.1. Informal statement of the main results

In this paper, we consider the general case (1) (as opposed to a specific g^ϵ as in [16]), while adding the following technical hypotheses:

$$a^0 > -2, \quad g^\epsilon(x,y) - g^\epsilon(x,0) = \mathcal{O}(\mu), \tag{7}$$

with $0 \leqslant \mu < \mu_0$ small enough, see further details below. Here μ_0 is independent of $\epsilon \geqslant 0$. We conjecture that our results are true without these assumptions (i.e. for any analytic and generic unfolding of a saddle-node), but leave this to future work (see section 6). We feel that (7) gives a suitable forum to present the phenomenon in an accessible way. We then summarise our results as follows:

Theorem 1.1. *Suppose (7) and let $y = \sum_{k=2}^\infty m_k^0 x^k$ denote the formal series expansion of the centre manifold W^c for $\epsilon = 0$. Then*

$$S_\infty^0 := \lim_{k\to\infty} \frac{(-1)^k m_k^0}{\Gamma(k+a^0)}, \tag{8}$$

where Γ is the gamma function (see appendix A below), is well-defined for all $\mu \in [0,\mu_0)$, with $\mu_0 > 0$ small enough. Moreover, if $S_\infty^0 \neq 0$ then the following holds true:

1. *The centre manifold W^{cs} for $\epsilon = 0$ is nonanalytic (see theorem 3.2).*
2. *$W^{ws} \cap W^u = \emptyset$ for all $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$ (see theorem 3.5).*
3. *The sign of $S_\infty^0 \neq 0$ determines on what side of W^u the analytic weak-stable manifold W^{ws} lands (within the strip defined by $x \in (0,\epsilon)$, see theorem 3.5 and figure 1.)*
4. *On the far side of the saddle (i.e. for $x < 0$), the weak-stable analytic manifold W^{ws} transitions from intersecting $y = c > 0$ to intersecting $y = -c < 0$ ($c > 0$ small but $\mathcal{O}(1)$) as ϵ^{-1} transverses (sufficiently large) integers (the resonances) (see corollary 3.6 and figure 6). The position of W^{ws} close to each resonance $\epsilon^{-1} \in \mathbb{N}$, $\epsilon^{-1} \gg 1$, is determined by the sign of S_∞^0 and on whether ϵ^{-1} is even or odd.*

The results of theorem 1.1 are (as indicated) stated precisely in our main result section, see section 3. The (dramatic) movement of W^{ws} as ϵ^{-1} transverses (sufficiently large) integers (described informally in item 4 of theorem 1.1) was central in [16] for the construction of their special C^∞ self-similar solutions. We will refer to this phenomenon as 'flapping', see also section 2.

While the discovery of the phenomenon in a specific problem is due to [16], our treatment of the underlying general mechanism is novel and more in the spirit of dynamical systems theory. We also feel that our proof streamlines the approach of [16], used for their specific nonlinearity (rational with numerator cubic, denominator quadratic, see [16, equations (1.9) and (1.10)]). Moreover, we will perform the important estimates not by brute force calculations but by using appropriate arguments (including fixed-point theorems) in suitable normed spaces.

Analyticity will throughout refer to real analyticity and we will work with $x \in \mathbb{R}$. This is in contrast to related studies of exponentially small phenomena, see e.g. [1, 10], where extensions into the complex plane $x \in \mathbb{C}$ play a crucial role.

1.2. Further background

We emphasise that item 2 of theorem 1.1 is in line with the statement in [20, example 3.1, p 13], which says that

$$W^c \notin C^\omega \Longrightarrow W^{ws} \cap W^u = \emptyset \quad \forall 0 < \epsilon \ll 1, \epsilon^{-1} \notin \mathbb{N}.$$

This is of course the generic situation. (Keep in mind that, as the branches (away from the equilibria) of W^{ws} and W^u are orbits of the planar problem (1), W^{ws} and W^u either coincide or do not intersect). However, a novel aspect of our results is that we show that the sign of $S^0_\infty \neq 0$ determines (along with the parity of $\lfloor \epsilon^{-1} \rfloor \gg 1$) on what side of W^u the analytic weak-stable manifold W^{ws} lands.

In [14], Martinet and Ramis presented their analytic characterisation of saddle-nodes through *analytic invariants*. The quantity a^0 is here known as *the formal analytic invariant*. The name is motivated by the fact that there exists a *formal* transformation (a formal series with respect to x and y) that brings the saddle-node for $\epsilon = 0$ into

$$\begin{aligned}
\dot{x} &= x^2, \\
\dot{y} &= -y\left(1 + a^0 x\right).
\end{aligned}$$

In general (also for Poincaré ranks $r \in \mathbb{N} \setminus \{1\}$ where $\dot{x} = x^{r+1}$), the formal transformation can be summed (in the sense of Borel-Laplace) along complex sectors (whose union cover the origin) and in essence the analytic invariants encode the relationship between these on overlapping domains. Here there is a deep connection to the Ecalle-Voronin classification of tangent-to-the-identity maps, see [8, 13, 24].

The quantity S^0_∞ (somehow) relates to the so-called *translational part of the analytic invariants of Martinet and Ramis*. Indeed, the centre manifold is analytic if and only if this translational part is trivial, see also [13, section 1]. (Alternatively, the centre manifold is analytic if and only if the Borel transform of $\widehat{m}^0$ is entire). However, the details of this connection is still not clear to the authors. For example, we do not know whether $S^0_\infty = 0$ implies that the centre manifold is analytic (we will show below that it holds in the y-linear case). At the same time, S^0_∞ also acts like a Stokes constant that determines the properties of the unfolding. This is reminiscent of the Stokes constant in [1] for the unfolding of the zero-Hopf.

On the other hand, (8) also provides an estimate of the growth of the coefficients m^0_k as $k \to \infty$, insofar that there is a constant $F > 0$ such that

$$|m^0_k| \leqslant F\Gamma\left(k + a^0\right) \quad \forall k \geqslant 2. \tag{9}$$

By Stirling's formula (see (173) in appendix A):

$$\Gamma\left(k+a^0\right) = (1+o(1))\,k!k^{a^0-1}, \tag{10}$$

for $k \gg 1$, it follows that m_k^0 is Gevrey-1 (see (6)) for any $D<1$, also $D=1$ if $a^0 \leqslant 1$.

1.3. Outline

The paper is structured as follows: in section 2, we provide a first glimpse of the phenomena that we study through simple examples. We also use this section to introduce our terminology and (parts of our) notation. Subsequently, in section 3 we present our hypotheses and our main results in full technical details. In section 4, we then prove the statements relating to the centre manifold. Statements relating to $\epsilon > 0$ are proven in section 5. (Section 3.1 has a more detailed overview of the proofs.) We conclude the paper with a discussion section, see section 6. Appendix A contains some relevant properties of the gamma function that will be used throughout.

2. Motivating examples

Consider first the following simple example:

$$\begin{aligned}\dot{x} &= -\epsilon x,\\ \dot{y} &= -y + u(x),\end{aligned} \tag{11}$$

with

$$u(x) = \sum_{k=2}^{\infty} u_k x^k, \quad |u_k| \leqslant B\rho^{-k},$$

being analytic on the open disc $|x| < \rho$. Here $x=0$ is the strong stable manifold W^{ss} and for $\epsilon^{-1} \notin \mathbb{N}$ the analytic weak-stable manifold W^{ws} exists and takes the graph form

$$y = m^\epsilon(x), \quad m^\epsilon(x) = \sum_{k=2}^{\infty} \frac{u_k}{1-\epsilon k} x^k \quad \forall 0 \leqslant |x| < \rho. \tag{12}$$

This follows from a simple calculation. Notice that there are small divisors in the expression for m^ϵ for $\epsilon \approx \frac{1}{N}$, $N \in \mathbb{N}$ (the resonances). Let

$$\epsilon^{-1} = N^\epsilon + \alpha^\epsilon \notin \mathbb{N}, \quad N^\epsilon := \lfloor \epsilon^{-1} \rfloor, \quad \alpha^\epsilon \in (0,1),$$

and define

$$V^\epsilon(x) = N^\epsilon \frac{u_{N^\epsilon}}{\alpha^\epsilon} x^{N^\epsilon} - (N^\epsilon+1)\frac{u_{N^\epsilon+1}}{1-\alpha^\epsilon} x^{N^\epsilon+1}. \tag{13}$$

Then the sum of the terms in (12) with $k = N^\epsilon$ and $k = N^\epsilon + 1$ takes the following form

$$\begin{aligned}\sum_{k=N^\epsilon}^{N^\epsilon+1} \frac{u_k}{1-\epsilon k} x^k &= \epsilon^{-1}\frac{u_{N^\epsilon}}{\alpha^\epsilon} x^{N^\epsilon} - \epsilon^{-1}\frac{u_{N^\epsilon+1}}{1-\alpha^\epsilon} x^{N^\epsilon+1}\\ &= \left(N^\epsilon \frac{u_{N^\epsilon}}{\alpha^\epsilon} + u_{N^\epsilon}\right) x^{N^\epsilon} - \left((N^\epsilon+1)\frac{u_{N^\epsilon+1}}{1-\alpha^\epsilon} - u_{N^\epsilon+1}\right) x^{N^\epsilon+1}\\ &= V^\epsilon(x) + u_{N^\epsilon} x^{N^\epsilon} + u_{N^\epsilon+1} x^{N^\epsilon+1}.\end{aligned}$$

It follows that $B^\epsilon := m^\epsilon - V^\epsilon$ is uniformly bounded with respect to $\alpha^\epsilon \in [0,1)$, and for any $\upsilon > 0$ there is a $\delta > 0$ such that

$$|B^\epsilon(x)| \leqslant \upsilon \quad \forall 0 \leqslant |x| \leqslant \delta, \alpha^\epsilon \in (0,1). \tag{14}$$

The function V^ϵ, on the other hand, is not uniformly bounded if $u_{N^\epsilon} \neq 0$ or $u_{N^\epsilon+1} \neq 0$. Specifically, if $u_{N^\epsilon} u_{N^\epsilon+1} \neq 0$ then it follows that *we can track $W^{ws}: y = m^\epsilon(x)$ through $y = V^\epsilon(x)$ for $x \neq 0$, $\alpha^\epsilon \to 0^+$ and $\alpha^\epsilon \to 1^-$* (since $V^\epsilon(x)$, $x \neq 0$, goes unbounded in these limits). Here by 'track' we will mean that the position of W^{ws} can qualitatively be determined as follows:

Lemma 2.1. *Fix $N^\epsilon \in \mathbb{N}$, $K > 0$, suppose that $u_{N^\epsilon} \neq 0$ and define $s = \operatorname{sign}(u_{N^\epsilon})$. Let $W^{ws}: y = m^\epsilon(x), 0 \leqslant |x| < \rho$, denote the analytic weak-stable manifold of (11), $\epsilon^{-1} \notin \mathbb{N}$. Then the following holds true for all $0 < \delta \ll 1$:*

The position of W^{ws} for all $0 < \alpha^\epsilon < N^\epsilon \frac{|u_{N^\epsilon}|}{K} \delta^{N^\epsilon}$ can be determined as follows:

1. *Suppose that N^ϵ is even. Then W^{ws} intersects $\{y = \frac{sK}{2}\}$ for both $-\delta < x < 0$ and $0 < x < \delta$.*
2. *Suppose that N^ϵ is odd. Then W^{ws} intersects $\{y = -\frac{sK}{2}\}$ for $-\delta < x < 0$ and $\{y = \frac{sK}{2}\}$ for $0 < x < \delta$.*

Proof. For $u_{N^\epsilon} \neq 0$, we have

$$\left| N^\epsilon \frac{u_{N^\epsilon}}{\alpha^\epsilon} \delta^{N^\epsilon} \right| > K \quad \forall 0 < \alpha^\epsilon < N^\epsilon \frac{|u_{N^\epsilon}|}{K} \delta^{N^\epsilon}.$$

Then from (13), we obtain that the following holds true for $x = -\delta$ and $x = \delta$:

$$|V^\epsilon(x)| \geqslant \frac{3K}{4} \quad \forall 0 < \alpha^\epsilon < N^\epsilon \frac{|u_{N^\epsilon}|}{K} \delta^N, 0 < \delta \ll 1.$$

Then by using $m^\epsilon = B^\epsilon + V^\epsilon$ and (14) with $0 < \upsilon \ll K$ for $0 < \delta \ll 1$ the result follows. □

A similar result holds for $0 < 1 - \alpha^\epsilon \ll 1$ if $u_{N^\epsilon+1} \neq 0$, which we state without proof.

Lemma 2.2. *Fix $N^\epsilon \in \mathbb{N}$, $K > 0$, suppose that $u_{N^\epsilon+1} \neq 0$ and define $s = \operatorname{sign}(u_{N^\epsilon+1})$. Let $W^{ws}: y = m^\epsilon(x), 0 \leqslant |x| < \rho$, denote the analytic weak-stable manifold of (11), $\epsilon^{-1} \notin \mathbb{N}$. Then the following holds for all $0 < \delta \ll 1$:*

The position of W^{ws} for all $0 < 1 - \alpha^\epsilon < (N^\epsilon + 1) \frac{|u_{N^\epsilon+1}|}{K} \delta^{N^\epsilon+1}$ can be determined as follows:

1. *Suppose that N^ϵ is even. Then W^{ws} intersects $\{y = \frac{sK}{2}\}$ for $-\delta < x < 0$ and $\{y = -\frac{sK}{2}\}$ for $0 < x < \delta$.*
2. *Suppose that N^ϵ is odd. Then W^{ws} intersects $\{y = -\frac{sK}{2}\}$ for both $-\delta < x < 0$ and $0 < x < \delta$.*

By lemmas 2.1 and 2.2, we obtain a 'flapping phenomenon' when $u_{N^\epsilon} u_{N^\epsilon+1} \neq 0$, *whereby the position of W^{ws} (at least on one side of the node) changes dramatically as α^ϵ transverses the interval (0, 1)*. We illustrate this flapping phenomena in figure 3 for $u_{N^\epsilon} > 0, u_{N^\epsilon+1} < 0$ (which is relevant for (20) with $0 < \epsilon \ll 1$; the reader should compare the figure with figure 6).

Remark 1. It is essentially the flapping mechanism that (together with a basic continuity argument) allows the authors of [16] to connect their analytic weak-stable manifolds with a global analytic manifold (that does not 'flap') and construct C^∞-smooth self-similar solutions close to resonances (and close to a saddle-node where the resonances accumulate).

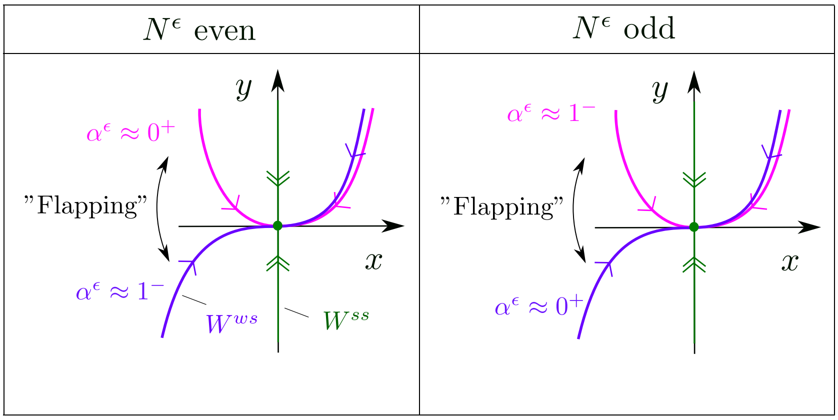

Figure 3. The 'flapping phenomenon' of the analytic weak-stable manifolds (W^{ws} in magenta and purple) of (11) for $u_{N^\epsilon} > 0, u_{N^\epsilon+1} < 0$.

For a general (fully nonlinear) analytic system, quantities corresponding to u_{N^ϵ} and $u_{N^\epsilon+1}$ for a hyperbolic node can in principle be computed for any fixed N^ϵ in terms of the jet of the nonlinearity (through normal form computations [7, chapter 2]). But in the context of (1), our results show (see section 3) that the condition $u_{N^\epsilon}u_{N^\epsilon+1} \neq 0$ can be related to the lack of analyticity (through S^0_∞) of the centre manifold $y = m^0(x)$ of the origin for $\epsilon = 0$.

Remark 2. Notice that in the context of the y-linear system (11), the flapping phenomena for $\epsilon \to 0$ does not appear if u is a polynomial. Indeed, the flapping is caused by an accumulation of resonances for $\epsilon \to 0$ and if u is a polynomial then there are only finitely many (possible) resonances for (11).

Next, to illustrate how (8) and the bound (9) occur, consider the case $g^0(x,y) = f^0(x)$ (so that g^0 is independent of y) in (4). As we are interested in invariant manifolds, we eliminate time to obtain the following differential equation for $y = y(x)$:

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y\left(1 + a^0x\right) = f^0(x), \quad f^0(x) = \sum_{k=2}^{\infty} f^0_k x^k. \tag{15}$$

This equation is linear in y and one can solve explicitly for the m^0_k's of the formal series. Indeed, inserting the formal series (5) into (15) leads to

$$\sum_{k=2}^{\infty}\left(\left(k + a^0\right)m^0_k x^{k+1} + m^0_k x^k\right) = \sum_{k=2}^{\infty} f^0_k x^k,$$

and therefore to the recursion relation:

$$m^0_k + \left(k - 1 + a^0\right)m^0_{k-1} = f^0_k \quad \forall k \geqslant 2, \tag{16}$$

with $m^0_1 = 0$.

Lemma 2.3. *Suppose that $a^0 > -2$ and define*

$$S^0_k := \sum_{j=2}^{k} \frac{(-1)^j f^0_j}{\Gamma\left(j + a^0\right)}. \tag{17}$$

Then the solution of the recursion relation (16) with $m_1^0 = 0$ is

$$m_k^0 = (-1)^k \Gamma\left(k + a^0\right) S_k^0. \tag{18}$$

Proof. The result can easily be proven by induction using the base case $m_1^0 = 0$ and the basic property of the gamma function: $\Gamma(z+1) = z\Gamma(z)$, see (170), in the induction step. □

Seeing that f^0 is analytic, we have

$$|f_k^0| \leqslant B\rho^{-k},$$

for some $B > 0$, $\rho > 0$, and the sum

$$S_\infty^0 := \lim_{k\to\infty} S_k^0 = \lim_{k\to\infty} \frac{(-1)^k m_k^0}{\Gamma(k+a^0)} = \sum_{j=2}^{\infty} \frac{(-1)^j f_j^0}{\Gamma(j+a^0)},$$

see (17), is therefore absolutely convergent for any $a^0 > -2$:

$$|S_\infty^0| \leqslant \sum_{j=2}^{\infty} \frac{|f_j^0|}{\Gamma(j+a^0)} \leqslant F := B\sum_{j=2}^{\infty} \frac{\rho^{-j}}{\Gamma(j+a^0)} < \infty.$$

The property (9) therefore follows from (18) in the context of (15).

Notice that if $S_\infty^0 \neq 0$, then by (10) we have

$$m_k^0 = (-1)^k \Gamma\left(k+a^0\right) S_k^0 = (-1)^k (1+o(1)) S_\infty^0 \Gamma\left(k+a^0\right) = (-1)^k (1+o(1)) S_\infty^0 k^{a^0-1} k! \tag{19}$$

for all $k \gg 1$. This implies that the centre manifold is nonanalytic. In the linear case (15), it is also possible to go the other way. We collect this in the following lemma:

Lemma 2.4. *Suppose that $a^0 > -2$. Then the centre manifold of the linear system (15) is analytic if and only if $S_\infty^0 = 0$.*

Proof. $\Rightarrow$: From (19), we have that for any $x \neq 0$, $m_k^0 x^k \not\to 0$ for $k \to 0$. Consequently, if $S_\infty^0 \neq 0$ then the centre manifold is nonanalytic.

$\Leftarrow$: If $S_\infty^0 = 0$ then

$$|S_k^0| = |S_\infty^0 - S_k^0| \leqslant B \sum_{j=k+1}^{\infty} \frac{\rho^{-j}}{\Gamma(j+a^0)},$$

and for any $k \geqslant k_0(a^0), j \in \mathbb{N}_0$:

$$\frac{\Gamma\left(k+a^0\right)}{\Gamma(k+1+j+a^0)} = \frac{(k-1)!}{(k+j)!}\left(1+o_{k_0\to\infty}(1)\right)\left(\frac{k}{k+1+j}\right)^{a^0} \leqslant \frac{2}{k^j}\left(\frac{k}{k+1+j}\right)^{-2} \leqslant 8(1+j)^2 k^{-j},$$

with $k_0 \gg 1$, using Stirling's formula (see (173) below) and

$$\left(\frac{k+1+j}{k}\right)^2 \leqslant \left(\frac{2k(1+j)}{k}\right)^2 = 4(1+j)^2.$$

Then upon using $\sum_{j=0}^{\infty} 2^{-j}(1+j)^2 = 12$ it follows that

$$|m_k^0| \leqslant 8B\rho^{-k-1}\sum_{j=0}^{\infty}(\rho k)^{-j}(1+j)^2 \leqslant 96B\rho^{-k-1} \quad \forall k \geqslant k_0 \geqslant 2\rho^{-1}.$$

We conclude that $\sum_{k=2}^{\infty} m_k^0 x^k$ converges absolutely for all $0 \leqslant |x| < \rho$ if $S_\infty^0 = 0$. □

A first important step of our approach is to carry the classification of the analyticity of the centre manifold for $\epsilon = 0$ over to the nonlinear case. For this, we will use a fixed-point argument in an appropriate Banach space of formal series. This leads to the definition of S_∞^0 for a nonlinearity g^0, satisfying the hypotheses 1 and 2 below.

Subsequently, for $\epsilon > 0$ and $S_\infty^0 \neq 0$, we (essentially) expand the analytic weak-stable invariant manifold $y = m^\epsilon(x)$ into the form

$$m^\epsilon = B^\epsilon + (-1)^{N^\epsilon} S_\infty^0 V^\epsilon, \quad N^\epsilon = \lfloor \epsilon^{-1} \rfloor,$$

on a subset $x \in I^\epsilon$, where (in essence, see theorem 3.5 for details) *only B^ϵ is uniformly bounded with respect to $\alpha^\epsilon = \epsilon^{-1} - \lfloor \epsilon^{-1} \rfloor \in (0,1)$. We will therefore track $y = m^\epsilon(x)$ for $S_\infty^0 \neq 0$ using $y = (-1)^{N^\epsilon} S_\infty^0 V^\epsilon(x)$ for $\alpha^\epsilon \to 0^+$ and $\alpha^\epsilon \to 1^-$* as in the example (11) above. (It would be more accurate to say that the tracking will first be done in scaled coordinates, see (30), and that $V^\epsilon(x) = \epsilon \overline{V}^\epsilon(\epsilon^{-1}x)$, see (33). Moreover, $x > 0$ and $x < 0$ will be treated slightly different, but we refer the reader to further details and the precise statements below.) In this context, it is (again) worth pointing out that $S_\infty^0 \neq 0$ essentially ensures that a condition like $u_{N^\epsilon} u_{N^\epsilon+1} \neq 0$ holds true near all resonances $\epsilon^{-1} \in \mathbb{N}$ for $0 < \epsilon \ll 1$, see theorem 3.5 and corollary 3.6.

3. Main results

We first state a general result (based upon [19, theorem 2.2]) on saddle-nodes.

Theorem 3.1. *For any analytic and generic family of two-dimensional vector-fields unfolding a saddle-node, there exists a locally defined analytic change of coordinates, parameters and time, such that on the singularity-side ($\epsilon \geqslant 0$) of the bifurcation, the system takes the following normal form:*

$$\begin{aligned} \dot{x} &= (x-\epsilon)x, \\ \dot{y} &= -y(1+a^\epsilon x) + g^\epsilon(x,y), \end{aligned} \tag{20}$$

where

$$\begin{gathered} g^\epsilon(x,y) = f^\epsilon(x) + u^\epsilon(x,y) \\ f^\epsilon(x) = \sum_{k=2}^{\infty} f_k^\epsilon x^k, \quad u^\epsilon(x,y) = \sum_{k=2}^{\infty} u_{k,1}^\epsilon x^k y + \sum_{k=1}^{\infty}\sum_{l=2}^{\infty} u_{k,l}^\epsilon x^k y^l. \end{gathered} \tag{21}$$

In particular, the following holds regarding the absolutely convergent power series expansions of f^ϵ and u^ϵ for all $\rho > 0$ small enough: Let

$$D_1 := [0,\epsilon_0) \times \{0 \leqslant |x| < \rho\}, \quad D_2 := [0,\epsilon_0) \times \{0 \leqslant |x| < \rho\} \times \{0 \leqslant |y| < \rho\},$$

and define

$$B := \sup_{(\epsilon,x)\in D_1} |f^\epsilon(x)|, \quad \mu := \sup_{(\epsilon,x,y)\in D_2} |u^\epsilon(x,y)|. \tag{22}$$

Then

$$|f_k^\epsilon| \leqslant B\rho^{-k}, \quad |u_{k,l}^\epsilon| \leqslant \mu\rho^{-k-l} \quad \textit{and} \quad u_{k,1}^0 = 0 \quad \forall k,l\in\mathbb{N}, \epsilon\in[0,\epsilon_0) \tag{23}$$

The proof of theorem 3.1 (available in appendix B) is obtained by applying elementary transformations to the normal form in [19, theorem 2.2].

In the remainder of the paper, we will assume the following conditions on a^ϵ and g^ϵ:

Hypothesis 1. The following inequality holds true:

$$a^0 := \lim_{\epsilon\to 0} a^\epsilon > -2.$$

Hypothesis 2. B and $\rho > 0$ are fixed and $\mu \geqslant 0$ in (22) is a parameter that is small enough (see details below).

Following hypothesis 2, we will henceforth write

$$u^\epsilon = \mu h^\epsilon \quad \text{and} \quad u_{k,l}^\epsilon = \mu h_{k,l}^\epsilon,$$

so that g^ϵ in (20) becomes

$$g^\epsilon(x,y) =: f^\epsilon(x) + \mu h^\epsilon(x,y), \tag{24}$$

where

$$f^\epsilon(x) = \sum_{k=2}^\infty f_k^\epsilon x^k, \quad h^\epsilon(x,y) = \sum_{k=2}^\infty h_{k,1}^\epsilon x^k y + \sum_{k=1}^\infty\sum_{l=2}^\infty h_{k,l}^\epsilon x^k y^l, \tag{25}$$

with

$$|f_k^\epsilon| \leqslant B\rho^{-k}, \quad |h_{k,l}^\epsilon| \leqslant \rho^{-k-l} \quad \text{and} \quad h_{k,1}^0 = 0 \quad \forall k,l\in\mathbb{N}, \epsilon\in[0,\epsilon_0). \tag{26}$$

The results below will be stated for (20) with g^ϵ given by (24) for $0<\mu\ll 1$ (in accordance with hypothesis 2).

In theorem 3.2, when we treat f_2^0 as a parameter, we will fix a compact interval I so that (23) holds (with $B>0$ large enough) for all $f_2^0\in I$.

The reference [16] also assumes a condition like hypothesis 1 (see [16, equation (5.3)]) in the context of their specific rational example of an analytic unfolding, see [16, equations (1.9) and (1.10)]. On the other hand, a condition like hypothesis 2, which can also be viewed as (7), does not appear in [16]. We conjecture that our results are true without hypotheses 1 and 2 (and therefore hold true for any analytic and generic unfolding of a saddle-node), but leave this extension to future work. Whereas hypothesis 1 seems relatively easy to relax, hypothesis 2 requires extra work. We will discuss the matter further in section 6.

Remark 3. Hypothesis 2 is only an assumption on the nonlinearity in y. This follows from the last equality in (26) and continuity with respect to ϵ (i.e. $h_{k,1}^\epsilon = o(1)$).

Remark 4. Obviously, from (22) we have $\mu = \mathcal{O}(\rho^3)$ as $\rho \to 0$ in general (since u^ϵ starts with cubic terms) and in this sense one can achieve μ small by taking $\rho > 0$ small. But this will not be helpful to us (and we do not expect it to be useful in general). This is in contrast to arguments based upon Nagumo norms (see e.g. [5]), where the size of the domain can be used as a small parameter to obtain the appropriate contraction of a fixed-point formulation of Gevrey-properties of formal series. At this stage, our approach in the present paper requires B and $\rho > 0$ fixed and $\mu > 0$ small enough, as stated in hypothesis 2.

Our first main result relates to the centre manifold.

Theorem 3.2. *Consider (20) with g^ϵ given by (24) for $\epsilon = 0$:*

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y\left(1 + a^0 x\right) + g^0\left(x, y\right), \end{aligned} \tag{27}$$

and suppose that hypotheses 1 and 2 hold true. Let $W^c: y = m^0(x)$, $m^0(0) = \frac{dm^0}{dx}(0) = 0$, with m^0 defined in a neighborhood of $x = 0$, denote the centre manifold of $(x, y) = (0, 0)$. Then there is a $\mu_0 > 0$ such that for all $0 \leqslant \mu < \mu_0$ the following statements hold true:

1. *There exists a number S_∞^0, which depends upon the full jet of g^0, such that:*
   (a)
$$\frac{(-1)^k}{\Gamma\left(k + a^0\right)} \frac{1}{k!} \frac{\mathrm{d}^k m^0}{\mathrm{d}x^k}(0) \to S_\infty^0 \quad \textit{for} \quad k \to \infty.$$
   (b) *The centre manifold W^c is nonanalytic if $S_\infty^0 \neq 0$.*
2. *$S_\infty^0 = S_\infty^0(f_2^0)$ is a C^1-function with respect to $f_2^0 \in I$ (as well as all other parameters of the system), recall (25), satisfying*
$$\frac{\partial S_\infty^0}{\partial f_2^0}(f_2^0) = \frac{1}{\Gamma(2 + a^0)} + \mathcal{O}(\mu) \neq 0 \quad \forall \mu \in [0, \mu_0), \quad \mu_0 = \mu_0(I) > 0.$$

The second statement shows that the centre manifold being nonanalytic for (20), under the hypotheses (1) and (2), is a generic condition. We exemplify this as follows:

Corollary 3.3. *Suppose that the conditions of theorem 3.2 hold true, in particular $0 \leqslant \mu \ll 1$ so that $\frac{\partial S_\infty^0}{\partial f_2^0}(f_2^0) \neq 0$, $f_2^0 \in I$, and suppose that the centre manifold of (27) is analytic ($\Rightarrow S_\infty^0(f_2^0) = 0$). Then the centre manifold of the perturbed system*

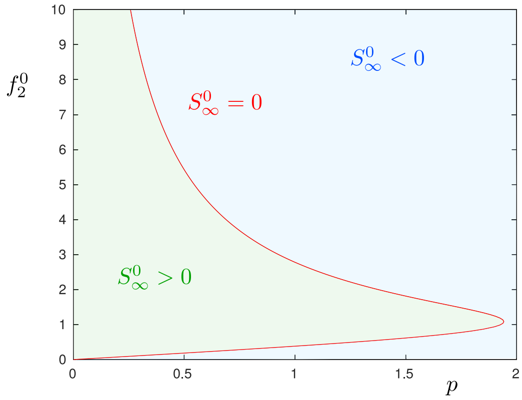

Figure 4. The locus $S_\infty^0(f_2^0,p) = 0$ (in red) for the family (28). The curve, which is computed numerically, see text for further details, has a fold point at $(p,f_2^0) \approx (1.94, 1.09)$, where $\frac{\partial S_\infty^0}{\partial f_2^0}$ changes sign, from $\frac{\partial S_\infty^0}{\partial f_2^0} > 0$ on the lower branch (in agreement with the statement of corollary 3.3) to $\frac{\partial S_\infty^0}{\partial f_2^0} < 0$ on the upper one.

$$\dot{x} = x^2,$$
$$\dot{y} = -y\left(1 + a^0 x\right) + g^0\left(x,y\right) + qx^2,$$

is nonanalytic for all $q \neq 0$ *small enough.*

Remark 5. The property that $\frac{\partial S_\infty^0}{\partial f_2^0}(f_2^0) \neq 0$ in corollary 3.3 is a perturbative result (obtained by perturbing away from $\mu = 0$) and it is (obviously) not expected to hold true in general ($\mu = 1$). In fact, in figure 4 we illustrate the locus

$$S_\infty^0\left(f_2^0,p\right) = 0,$$

computed numerically, see further details below, on the domain $(f_2^0,p) \in [0,10] \times [0,2]$ for the following (f_2^0,p)-family

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = -y + f_2^0 x^2 + p\left(\frac{x^3}{1-x} + 3xy^2 + xy^3\right). \tag{28}$$

Here p plays the role of μ, but as it includes parts of f^0 (the y-independent part) in the formulation of (27) we give it a different name. We clearly see that the curve has a fold ('far away' from $p = 0$) where necessarily $\frac{\partial S_\infty^0}{\partial f_2^0} = 0$ (we find that $\frac{\partial S_\infty^0}{\partial p} < 0$ at this point). At the same time, $\frac{\partial S_\infty^0}{\partial f_2^0} > 0$ (on the lower branch) for all $0 \leqslant p \lesssim 1.94$, in agreement with the statement of corollary 3.3.

As an approximation of $S_\infty^0 = S_\infty^0(f_2^0,p)$ we used S_{100}^0 (as a finite sum). In fact, we observed that $|S_{101}^0 - S_{100}^0| \sim 10^{-16}$ (i.e at the order of machine precision). To determine $m_2^0,\ldots,m_{97}^0$ (that are necessary to determine S_{100}^0 as a finite sum, see (17) and (66) below), we used the recursion relation offered by (16), starting from $m_1^0 = 0$.

Our next result relates to the analytic weak-stable invariant manifold W^{ws} of (20). To present this, we first write (20) in the form (upon eliminating time)

$$x(x-\epsilon)\frac{\mathrm{d}y}{\mathrm{d}x}+y(1+a^\epsilon x)=g^\epsilon(x,y). \tag{29}$$

For all $\epsilon^{-1}\notin\mathbb{N}$, W^{ws} takes the graph form

$$y=m^\epsilon(x)=\sum_{k=2}^\infty m_k^\epsilon x^k;$$

with the last equality valid locally $x\in(-\delta^\epsilon,\delta^\epsilon)$, $\lim_{\epsilon\to 0}\delta^\epsilon=0$. In particular, $y=m^\epsilon(x)$ is a (locally defined) solution of (29).

Now, the blowup transformation defined by

$$x=\epsilon\bar{x},\quad y=\epsilon\bar{y}, \tag{30}$$

for all $\epsilon>0$, separates the node and the saddle, so that the latter is at $\bar{x}=1$. By applying the change of variables defined by (30), (29) becomes

$$\epsilon\bar{x}(\bar{x}-1)\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}+\bar{y}(1+\epsilon a^\epsilon\bar{x})=\epsilon^{-1}g^\epsilon(\epsilon\bar{x},\epsilon\bar{y}), \tag{31}$$

where

$$\epsilon^{-1}g^\epsilon(\epsilon\bar{x},\epsilon\bar{y})=:\epsilon\bar{f}^\epsilon(\bar{x})+\epsilon\mu\bar{h}^\epsilon(\bar{x},\bar{y}),$$
$$\bar{f}^\epsilon(\bar{x})=\sum_{k=2}^\infty f_k^\epsilon\epsilon^{k-2}\bar{x}^k,\quad \bar{h}^\epsilon(\bar{x},\bar{y})=\sum_{k=2}^\infty h_{k,1}^\epsilon\epsilon^{k-1}\bar{x}^k\bar{y}+\sum_{k=1}^\infty\sum_{l=2}^\infty h_{k,l}^\epsilon\epsilon^{k+l-2}\bar{x}^k\bar{y}^l.$$

In these coordinates, (31) is a singularly perturbed system with respect to $0<\epsilon\ll 1$ and W^{ws} takes the following form

$$\bar{y}=\overline{m}^\epsilon(\bar{x}):=\epsilon^{-1}m^\epsilon(\epsilon\bar{x})=\sum_{k=2}^\infty\epsilon^{k-1}m_k^\epsilon\bar{x}^k,$$

where the last equality again holds true locally ($\bar{x}\in(-\epsilon^{-1}\delta^\epsilon,\epsilon^{-1}\delta^\epsilon)$). In the language of geometric singular perturbation theory (GSPT) [9, 11], the set $\{\bar{y}=0\}$ is a normally hyperbolic and attracting critical manifold of (31) for $\epsilon=0$. Therefore there is a (nonunique) slow manifold as a graph $\bar{y}=\mathcal{O}(\epsilon)$ over a compact subset $\bar{x}\in I$. This slow manifold only has finite smoothness (with respect to $\bar{x}$) in general, see [9]. However, the unstable manifold W^u of the saddle $(\bar{x},\bar{y})=(1,\mathcal{O}(\epsilon))$ is an example of an analytic slow manifold of the following graph form:

$$W^u:\quad \bar{y}=\epsilon\overline{H}^\epsilon(\bar{x}),\quad \bar{x}\in(0,2],\quad \overline{H}^\epsilon(0^+)=0; \tag{32}$$

here $\overline{H}^\epsilon$ extends C^k-smoothly ($1\leqslant k<\infty$, specifically not analytically, see corollary 3.6 item 1) to $\bar{x}=0$ for all $0<\epsilon\ll 1$. We will also need the following lemma (which we prove in section 5.5).

Lemma 3.4. *Suppose that* $\epsilon^{-1}\notin\mathbb{N}$, $0<\epsilon\ll 1$, *and write*

$$\epsilon^{-1} =: N^\epsilon+\alpha^\epsilon,\quad N^\epsilon := \lfloor \epsilon^{-1}\rfloor \quad \text{and} \quad \alpha^\epsilon\in(0,1).$$

Then the following holds true.

1. *The series*

$$\overline{V}^\epsilon(\overline{x}) := \frac{\Gamma(\alpha^\epsilon)\Gamma(1-\alpha^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})}\sum_{k=N^\epsilon}^{\infty}\frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\overline{x}^k, \tag{33}$$

is absolutely convergent for all $0\leqslant|\overline{x}|<1$*; in particular* $\overline{V}^\epsilon(0)=0$ *and*

$$\overline{V}^\epsilon(\overline{x})>0,\quad \frac{d}{d\overline{x}}\overline{V}^\epsilon(\overline{x})>0\quad \forall \overline{x}\in(0,1), \tag{34}$$

2. *Lower bound:*

$$\overline{V}^\epsilon(\overline{x})\geqslant \epsilon\left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^\epsilon+1}\qquad \forall 0\leqslant\overline{x}\leqslant\frac{3}{4}. \tag{35}$$

3. *At the same time, for any* $0<|\overline{x}|<1$,

$$|\overline{V}^\epsilon(\overline{x})|\to\infty\quad \text{for}\quad \alpha^\epsilon\to 0^+ \text{ and } 1^-.$$

4. *Asymptotics for* $\overline{x}=\mathcal{O}(\epsilon)$*: Let* $\overline{x}=\epsilon\overline{x}_2\in[-\epsilon\delta_2,\epsilon\delta_2]$, $\delta_2>0$ *fixed. Then for all* $0<\epsilon\ll 1$, $\epsilon^{-1}\notin\mathbb{N}$*:*

$$\begin{aligned}\overline{V}^\epsilon(\epsilon\overline{x}_2) &= (1+o(1))\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}(\epsilon\overline{x}_2)^{N^\epsilon}\\ &\quad\times\left(1+\frac{\overline{x}_2}{1-\alpha^\epsilon}\left[1+\overline{x}_2\int_0^1 e^{(1-v)\overline{x}_2}v^{1-\alpha^\epsilon}\mathrm{d}v+o(1)\right]\right),\end{aligned} \tag{36}$$

with each o(1) being uniform with respect to $\alpha^\epsilon\in(0,1)$.

Our main result on the analytic weak-stable manifold then takes the following form (see figure 5).

Theorem 3.5. *Fix* $K>0$, $\delta_2>0$, $0<\upsilon\ll K$ *and consider (31) with* g^ϵ *given by (24), satisfying hypotheses 1 and 2. Then the quantity* S_∞^0 *from theorem 3.2 is well-defined. We suppose that*

$$S_\infty^0\neq 0, \tag{37}$$

so that the centre manifold is nonanalytic.

Now, consider the convergent series $\overline{V}^\epsilon$ *defined in (33). Then the following holds for all* $0<\epsilon\ll 1$, $\epsilon^{-1}\notin\mathbb{N}$*: Let* $W^{ws}:\overline{y}=\overline{m}^\epsilon(\overline{x})$*, with* $\overline{m}^\epsilon$ *defined in a neighborhood of the origin, denote the analytic weak-stable manifold in the* $(\overline{x},\overline{y})$*-coordinates, see (30), and let* $I\subset\left[-\delta_2\epsilon,\frac{3}{4}\right]$ *be an interval so that*

$$|\overline{V}^\epsilon(\overline{x})|\leqslant K\quad\forall\overline{x}\in I. \tag{38}$$

Then $I\subset\mathrm{domain}(\overline{m}^\epsilon)$ *and*

$$|\overline{m}^\epsilon(\overline{x})-(-1)^{N^\epsilon}S_\infty^0\overline{V}^\epsilon(\overline{x})|\leqslant\upsilon\quad\forall\overline{x}\in I. \tag{39}$$

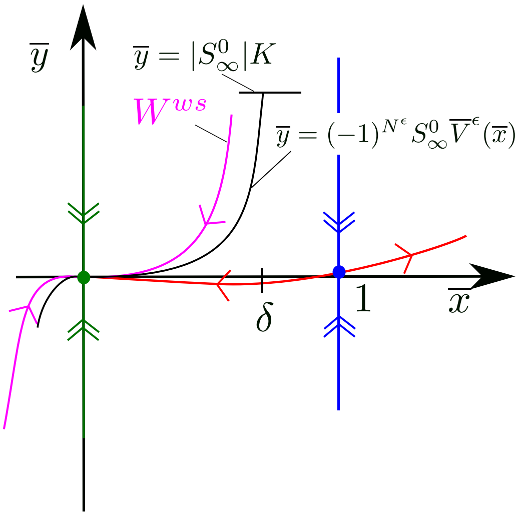

Figure 5. Phaseportrait of (31) for $\epsilon > 0$, $\epsilon^{-1} \notin \mathbb{N}$ (and $(-1)^{N^\epsilon} S^0_\infty > 0$); please compare with figure 1. Theorem 3.5 says that if $S^0_\infty \neq 0$ then we can track W^{ws} (in magenta) by the graph $\overline{y} = (-1)^{N^\epsilon} S^0_\infty \overline{V}^\epsilon(\overline{x})$, see also corollary 3.6 and further details in theorem 3.5.

In other words, when (37) holds true, then by taking $0 < \epsilon \ll 1$, *we can track* $W^{ws} : \overline{y} = \overline{m}^\epsilon(\overline{x})$ *through* $\overline{y} = (-1)^{N^\epsilon} S^0_\infty \overline{V}^\epsilon(\overline{x})$. Moreover, we have the following result, which we illustrate for $S^0_\infty > 0$ in figure 6; the weak manifold has to be reflected about the x-axis for $S^0_\infty < 0$.

Corollary 3.6. *Fix $c > 0$ small enough, suppose that hypotheses 1 and 2 hold true and that $S^0_\infty \neq 0$. Put $s = \operatorname{sign}(S^0_\infty)$ and let W^{ws} denote the analytic weak-stable manifold. Then the following holds true regarding the position of W^{ws} for all $N^\epsilon = \lfloor \epsilon^{-1} \rfloor \gg 1$:*

Intersections of W^{ws} with $\{y = \pm c\}$ for $x > 0$:

1. *W^{ws} does not intersect W^u. More precisely, we have the following:*
   (a) *Suppose that N^ϵ is even. Then W^{ws} intersects $\{y = sc\}$ for $x > 0$.*
   (b) *Suppose that N^ϵ is odd. Then W^{ws} intersects $\{y = -sc\}$ for $x > 0$.*

Intersections of W^{ws} with $\{y = \pm c\}$ for $x < 0$:
Define

$$\underline{\alpha}(N^\epsilon) := (N^\epsilon)^{a^0 - N^\epsilon}, \quad 1 - \overline{\alpha}(N^\epsilon) := (N^\epsilon)^{a^0 - 1 - N^\epsilon}. \tag{40}$$

2. *Suppose that N^ϵ is even. Then the following holds:*
   (a) *W^{ws} intersects $\{y = sc\}$ for $x < 0$ for all $0 < \alpha^\epsilon \leqslant \underline{\alpha}(N^\epsilon)$.*
   (b) *W^{ws} intersects $\{y = -sc\}$ for $x < 0$ for all $0 < 1 - \alpha^\epsilon \leqslant 1 - \overline{\alpha}(N^\epsilon)$.*
3. *Suppose that N^ϵ is odd. Then the following holds:*
   (a) *W^{ws} intersects $\{y = sc\}$ for $x < 0$ for all $0 < \alpha^\epsilon \leqslant \underline{\alpha}(N^\epsilon)$.*
   (b) *W^{ws} intersects $\{y = -sc\}$ for $x < 0$ for all $0 < 1 - \alpha^\epsilon \leqslant 1 - \overline{\alpha}(N^\epsilon)$.*

Proof. We first consider the statements in item 1 regarding the intersections of W^{ws} with $\{y = \pm c\}$ for $x > 0$ (proving items 1a and 1b). We let $K > 0$ be large enough and take $0 < \upsilon \ll$

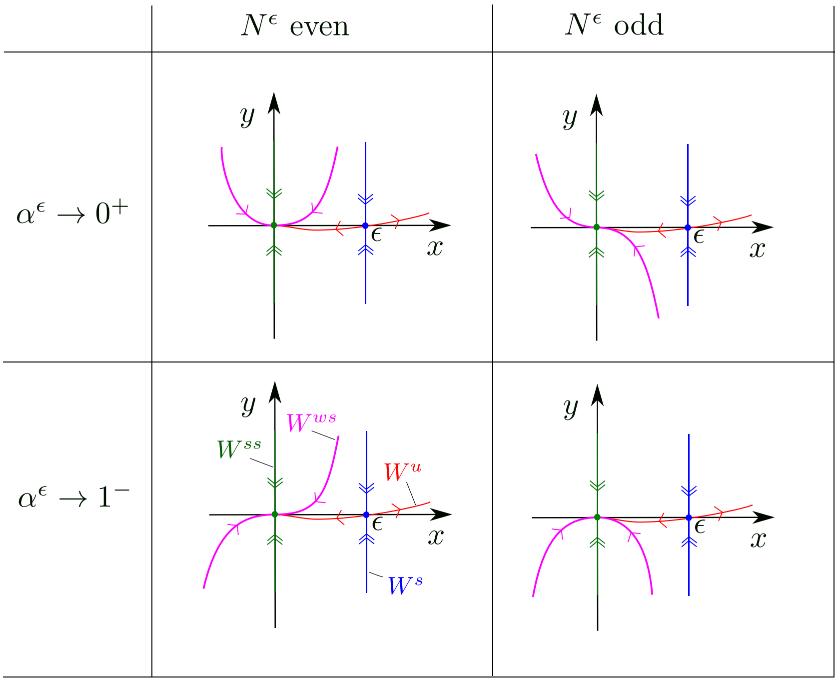

Figure 6. Illustration of the results of theorem 3.5, see also corollary 3.6. The strong stable manifold W^{ss} in green, the analytic weak-stable manifold W^{ws} in magenta, the stable manifold of the saddle W^s in blue, and finally the unstable manifold of the saddle W^u in red. The diagram assumes $S_\infty^0 > 0$; if $S_\infty^0 < 0$ then the diagram should be reflected about the x-axis. The analytic weak-stable manifold W^{ws} 'flaps' on the $x < 0$-side of the node as α^ϵ transverses $(0,1)$, aligning on $y > 0$ or $y < 0$ with W^{ss} as either $\alpha^\epsilon \to 0^+$ or $\alpha^\epsilon \to 1^-$. On the $x > 0$-side, W^{ws} remains on one side of W^u for all $\alpha^\epsilon \in (0,1)$ and only 'flaps' (discontinuously) when $N^\epsilon \gg 1$ changes parity. In particular, W^{ws} and W^u do not intersect.

K small enough. We let $0 < \epsilon \ll 1$ be so that $\overline{V}^\epsilon(\frac{3}{4}) > K$, see (35). Then since $\overline{V}^\epsilon(\overline{x})$ is an increasing function of $\overline{x}$, see (34), $\delta \in (0, \frac{3}{4})$ defined by the equation

$$\overline{V}^\epsilon(\delta) = K,$$

is uniquely determined. We then apply theorem 3.5 with $I = [0, \delta]$. In particular, from (39) we conclude that W^{ws} intersects $\{\overline{y} = \pm\frac{1}{2}S_\infty^0 K\}$ for $\overline{x} \in (0, \delta)$ when N^ϵ is even/odd, respectively. From $\{\overline{y} = \pm\frac{1}{2}S_\infty^0 K\}$, we undo the scaling (30) and return to (20) and apply the backward flow, see figure 5. This completes the proof of items 1a and 1b. To complete the proof of item 1, we recall that W^u and W^{ws} (away from the singularities) are orbits of planar systems and therefore if branches of these manifold intersect, then they coincide. We have that $\overline{y} = \mathcal{O}(\epsilon)$ along W^u for $\overline{x} \in [0,1]$, recall (32). Therefore W^u does not intersect $\{y = \pm c\}$ within $\overline{x} \in [0,1]$ for all $0 < \epsilon \ll 1$ and consequently W^u and W^{ws} do not coincide. Hence $W^s \cap W^{ws} = \emptyset$ as desired.

We then turn to the intersection of W^{ws} with $\{y = \pm c\}$ for $x < 0$. For this purpose, we again let $K > 0$ be large enough, put $\delta_2 = 1$ (for concreteness), take $0 < \upsilon \ll K$ small enough, $I = [-\delta, 0]$ with $0 < \delta \leqslant \epsilon$ and use the expansion (36) for $-\delta_2\epsilon \leqslant \overline{x} \leqslant 0$ to obtain the following for all $N^\epsilon \gg 1$: Consider $\underline{\alpha}(N^\epsilon)$ and $\overline{\alpha}(N^\epsilon)$ defined in (40). Then for any $0 < \alpha^\epsilon \leqslant \underline{\alpha}(N^\epsilon) \ll 1$,

$\overline{V}^{\epsilon}(\epsilon\bar{x}_2)$ is given by

$$\frac{1}{\alpha^{\epsilon}}(N^{\epsilon})^{a^{\epsilon}+1-N^{\epsilon}}\bar{x}_2^{N^{\epsilon}}e^{\bar{x}_2}, \tag{41}$$

to leading order, whereas for any $0<1-\alpha^{\epsilon}\leqslant 1-\overline{\alpha}(N^{\epsilon})\ll 1$, $\overline{V}^{\epsilon}(\epsilon\bar{x}_2)$ is given by

$$\frac{1}{1-\alpha^{\epsilon}}(N^{\epsilon})^{a^{\epsilon}-N^{\epsilon}}\bar{x}_2^{N^{\epsilon}+1}e^{\bar{x}_2}, \tag{42}$$

to leading order. We have here used (171),

$$\left[1+\bar{x}_2\int_0^1 e^{(1-v)\bar{x}_2}v\mathrm{d}v\right]=\frac{e^{\bar{x}_2}-1}{\bar{x}_2}\quad\text{and}\quad\left[1+\bar{x}_2\int_0^1 e^{(1-v)\bar{x}_2}\mathrm{d}v\right]=e^{\bar{x}_2}.$$

In both cases ((41) and (42)), there are remainder terms that we can assume are bounded by $\upsilon>0$, uniformly with respect to α^{ϵ} (whenever (41) and (42) do not exceed K in absolute value); this characterisation will be adequate for our purposes.

We now further claim that for any $0<\alpha^{\epsilon}\leqslant\underline{\alpha}(N^{\epsilon})$, then (41) with $\bar{x}_2=-1$ exceeds $K>0$ in absolute value. To show this, we just estimate

$$\left|\frac{1}{\alpha^{\epsilon}}(N^{\epsilon})^{a^{\epsilon}+1-N^{\epsilon}}(-1)^{N^{\epsilon}}e^{-1}\right|\geqslant\frac{1}{\underline{\alpha}(N^{\epsilon})}(N^{\epsilon})^{a^{\epsilon}+1-N^{\epsilon}}e^{-1}=(N^{\epsilon})^{1+a^{\epsilon}-a^0}e^{-1}\geqslant(N^{\epsilon})^{\frac{1}{2}}e^{-1}>K,$$

using that $|a^{\epsilon}-a^0|\leqslant\frac{1}{2}$ for all $N^{\epsilon}\gg 1$. A similar result holds for (42) for all $0<1-\alpha^{\epsilon}\leqslant 1-\overline{\alpha}(N^{\epsilon})$. We leave out the details in this case.

Consider now items 2a and 3a regarding $\alpha^{\epsilon}\to 0$. We then have by (41) (which is continuous and monotone with respect to $\bar{x}_2\in[-1,0)$) and (39) that for any $0<\alpha^{\epsilon}\leqslant\underline{\alpha}(N^{\epsilon})$, the equation $|\overline{m}^{\epsilon}(\bar{x})|=\frac{1}{2}|S^0_{\infty}|K$ has a solution $\bar{x}_-\in(-\epsilon,0)$. The sign of $\overline{m}^{\epsilon}(\bar{x}_-)$ is determined by $(-1)^{N^{\epsilon}}s\bar{x}_-^{N^{\epsilon}}$, cf (39) and (41). From $\{\bar{y}=\pm\frac{1}{2}|S^0_{\infty}|K\}$, we undo the scaling (30) and return to (20). Then the proof of items 2a and 3a is completed by using the backward flow. Indeed, W^{ws} aligns itself with one side of W^{ss} in this case and we can therefore just use W^{ss} as a guide for the backward flow up until W^{ss}'s transverse intersection with $\{y=\pm c\}$, see figure 5. The case $\alpha^{\epsilon}\to 1$ (items 2b and 3b) is similar and we therefore leave out further details. □

3.1. Overview

We prove theorem 3.2 in section 4. Theorem 3.5 is proven in section 5, see also section 5.5 where lemma 3.4 is proven. The strategy of the proof of theorem 3.5 follows [16] insofar that we write $\bar{y}=\overline{m}^{\epsilon}(\bar{x})$ as a finite sum $\bar{y}=\sum_{k=2}^{N^{\epsilon}}\overline{m}_k^{\epsilon}\bar{x}^k$, up until 'before the resonance', plus a remainder $\overline{M}^{\epsilon}(\bar{x})=\mathcal{O}(\bar{x}^{N^{\epsilon}+1})$ that we solve by setting up a fixed-point equation using an integral operator $\mathcal{T}^{\epsilon}$, see lemma 5.17. A main difficult lies in estimating the growth of coefficients in the series expansion of $\bar{g}^{\epsilon}$ when composed with the finite sum $\bar{y}=\sum_{k=2}^{N^{\epsilon}}\overline{m}_k^{\epsilon}\bar{x}^k$ (with the number of terms going unbounded as $\epsilon\to 0$). This is covered by the novel lemma 5.7 (which does not depend upon hypothesis 2). Our treatment of $\overline{M}^{\epsilon}$ is also novel (and also does not rely on hypothesis 2) insofar that we view the integral operator $\mathcal{T}^{\epsilon}$ as a bounded operator on a certain Banach space $\mathcal{D}_{\delta}^{\epsilon}$ of analytic functions $H=H(\bar{x})$ with $H(\bar{x})=\mathcal{O}(\bar{x}^{N^{\epsilon}+1})$, see (131). We believe that these novel aspects are crucial for making conclusions regarding $W^{ws}\cap W^s$. In particular, such conclusions cannot be made from the results of [16] on their specific nonlinearity (to the best of our judgement).

4. The centre manifold W^c: the proof of theorem 3.2

In this section, we consider $\epsilon = 0$ and (27) in the equivalent form

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + y\left(1 + a^0 x\right) = g^0\left(x, y\right), \tag{43}$$

where

$$g^0\left(x, y\right) = f^0\left(x\right) + \mu h^0\left(x, y\right) = \sum_{k=2}^{\infty} f_k^0 x^k + \mu \sum_{k=1}^{\infty}\sum_{l=2}^{\infty} h_{k,l}^0 x^k y^l,$$

cf (25) and (26). Let

$$\widehat{m}^0\left(x\right) = \sum_{k=2}^{\infty} m_k^0 x^k, \tag{44}$$

denote the formal series expansion of the centre manifold $y = m^0(x)$. We define

$$w_k^0 := \Gamma\left(k + a^0\right) \quad \forall k \geqslant 2, \tag{45}$$

and a norm

$$\|y\| = \sup_{k\geqslant 2} \frac{|y_k|}{w_k^0}, \tag{46}$$

on the space of formal series

$$\mathcal{D}^0 = \left\{ y = \sum_{k=2}^{\infty} y_k x^k \,:\, y_k \in \mathbb{R}\ \forall k \geqslant 2 \right\}.$$

Notice that (45) is well-defined by virtue of hypothesis 1 and that $\mathcal{D}^0$ is a Banach space (due to the sequence space l^∞ being Banach). For any $C > 0$, we also define

$$\mathcal{B}^C := \left\{ y \in \mathcal{D}^0 \,:\, \|y\| \leqslant C \right\}, \tag{47}$$

as the closed ball of radius C. Moreover, for any $y(x) = \sum_{k=2}^{\infty} y_k x^k \in \mathcal{D}^0$, the composition $g^0(x, y(x))$ of $y(x)$ with the analytic function g^0 is itself a formal series. The results below (see proposition 4.2) show that the associated operator

$$\mathcal{G}^0 : \mathcal{D}^0 \to \mathcal{D}^0, \quad \mathcal{G}^0\left[y\right]\left(x\right) = g^0\left(x, y\left(x\right)\right) = \sum_{k=2}^{\infty} \mathcal{G}^0\left[y\right]_k x^k,$$

is well-defined. The expression also defines $\mathcal{G}^0[y]_k$. $\mathcal{H}^0[y]$ and $\mathcal{H}^0[y]_k$ are similarly defined through the composition $h^0(x, y(x))$ of $y(x)$ with h^0:

$$\mathcal{H}^0\left[y\right]\left(x\right) = h^0\left(x, y\left(x\right)\right) = \sum_{k=2}^{\infty} \mathcal{H}^0\left[y\right]_k x^k.$$

By (24), we have $\mathcal{H}^0[y]_k = 0$ for $k = 2, 3$ and 4 and therefore

$$\begin{cases} \mathcal{G}^0[y]_2 &= f_2^0, \\ \mathcal{G}^0[y]_3 &= f_3^0, \\ \mathcal{G}^0[y]_4 &= f_4^0, \\ \mathcal{G}^0[y]_k &= f_k^0 + \mu \mathcal{H}^0[y]_k, \quad k \geqslant 5. \end{cases} \tag{48}$$

Finally, for any $l \in \mathbb{N}$ and any $y \in \mathcal{D}^0$, we define $(y^l)_k$ as the coefficients of y^l:

$$y(x)^l =: \sum_{k=2l}^{\infty} (y^l)_k x^k.$$

One can express $(y^l)_k$ in terms of $y_2, \ldots, y_{k-2}$ using the multinomial theorem, but we will not make use of this. It will also follow from proposition 4.2 that $y^l \in \mathcal{D}^0$.

Lemma 4.1. *The following holds*

$$\mathcal{H}^0[y]_k = \sum_{l=2}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{j=2l}^{k-1} h_{k-j,l}^0 (y^l)_j \quad \forall k \geqslant 5. \tag{49}$$

Proof. We use the expansion of h^0 in (25) and Cauchy's product rule:

$$\sum_{k=0}^{\infty} q_k \sum_{l=0}^{\infty} p_l = \sum_{k=0}^{\infty} \sum_{j=0}^{k} q_{k-j} p_j. \tag{50}$$

We have

$$\begin{aligned} \mathcal{H}^0[y](x) = h^0(x, y(x)) = \sum_{l=2}^{\infty} \sum_{k=1}^{\infty} h_{k,l}^0 x^k y(x)^l &= \sum_{l=2}^{\infty} \left(\sum_{k=1}^{\infty} h_{k,l}^0 x^k \right) \left(\sum_{j=2l}^{\infty} (y^l)_j x^j \right) \\ &= \sum_{l=2}^{\infty} \sum_{k=2l+1}^{\infty} \left(\sum_{j=2l}^{k-1} h_{k-j,l}^0 (y^l)_j \right) x^k \\ &= \sum_{k=5}^{\infty} \sum_{l=2}^{\lfloor \frac{k-1}{2} \rfloor} \left(\sum_{j=2l}^{k-1} h_{k-j,l}^0 (y^l)_j \right) x^k. \end{aligned}$$

□

Proposition 4.2. *Let $y \in \mathcal{B}^C$. Then $\mathcal{G}^0[y] \in \mathcal{D}^0$. In particular, there is a constant $K = K(a^0, \rho, C)$ such that*

$$|\mathcal{G}^0[y]_k| \leqslant B\rho^{-k} + \mu K w_{k-2}^0 \quad \forall k \geqslant 5. \tag{51}$$

Moreover, $y \mapsto \mathcal{H}^0[y]$ is C^1 (in the sense of Fréchet) and

$$\left(D\left(\mathcal{H}^0[y]\right)(z)\right)(x) = \sum_{k=5}^{\infty} \left(D\left(\mathcal{H}^0[y]\right)(z)\right)_k x^k, \quad |\left(D\left(\mathcal{H}^0[y]\right)(z)\right)_k| \leqslant K w_{k-2}^0 \|z\| \quad \forall z \in \mathcal{D}^0, \tag{52}$$

recall the definition of $\|\cdot\|$ *in (46).*

We prove this proposition in section 4.1 below. First we need some intermediate results.

Lemma 4.3. *Consider* w_k^0 *defined in (45) for all* $k \in \mathbb{N}\setminus\{1\}$ *and suppose* $a^0 > -2$ *(hypothesis 1). Then the following holds.*

1. *Convolution estimate: there exists a* $C = C(a^0) > 0$ *such that*

$$\sum_{j=2}^{k-2} w_j^0 w_{k-j}^0 \leqslant C w_{k-2}^0 \quad \forall k \geqslant 4.$$

2. *Let* $\rho > 0$*. Then there exists a* $C = C(a^0, \rho) > 0$ *such that*

$$\sum_{j=2}^{k-2} \rho^{j-k+2} w_j^0 \leqslant C w_{k-2}^0 \quad \forall k \geqslant 4.$$

3. *Let* $\xi > 0$*. Then there exists a* $C = C(a^0, \xi) > 0$ *such that*

$$\sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \xi^{l-2} w_{k-2(l-1)}^0 \leqslant C w_{k-2}^0 \quad \forall k \geqslant 4.$$

Proof. We prove the items 1–3 successively in the following.

Proof of item 1. We first notice that

$$\sum_{j=2}^{k-2} w_j^0 w_{k-j}^0 \leqslant 2 \sum_{j=2}^{\lfloor \frac{k}{2} \rfloor} w_j^0 w_{k-j}^0,$$

with $k \geqslant 4$. The result follows once we have shown that

$$2 \sum_{j=2}^{\lfloor \frac{k}{2} \rfloor} w_j^0 w_{k-j}^0 \leqslant C w_{k-2}^0 \quad \forall k \geqslant 4, \tag{53}$$

for some $C = C(a^0)$. We believe that this result, which is a result on gamma functions, is known, but for completeness we will present a simple proof that will form the basis for proofs of similar statements later on.

The starting point for this approach is to define $\Phi_1^0(j)$ for $j \in [2, k-2]$ by

$$w_j^0 w_{k-j}^0 = \exp\left(\Phi_1^0(j)\right). \tag{54}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}j}\Phi_1^0(j) = \phi\left(j + a^0\right) - \phi\left(k - j + a^0\right), \quad \frac{\mathrm{d}^2}{\mathrm{d}j^2}\Phi_1^0(j) = \phi'\left(j + a^0\right) + \phi'\left(k - j + a^0\right),$$

using (176). Since the digamma function $\phi(z)$ is strictly increasing for $z > 0$, see (177), and since $a^0 > -2$ (recall hypothesis 1), we conclude that $\Phi_1^0(j), j \in [2, k-2]$, is convex, having a single minimum at $j = \frac{k}{2}$. We therefore have that

$$\Phi_1^0(j) \leqslant Q_1^0(j-2) + P_1^0 \quad \forall j \in \left[2, \tfrac{k}{2}\right], \tag{55}$$

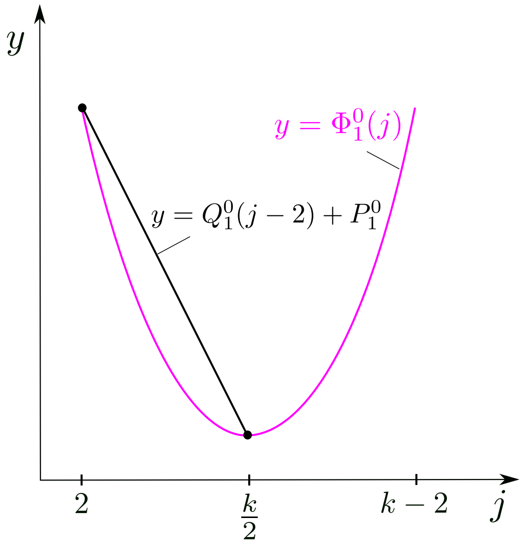

Figure 7. Graph of the function Φ_1^0 (magenta) and the secant $Q_1^0(j-2)+P_1^0$ (in black), see (54) and (56). Since Φ_1^0 is convex, (55) holds.

where

$$\exp\left(P_1^0\right) = w_2^0 w_{k-2}^0 \quad \text{and} \quad Q_1^0 = \frac{1}{\frac{k}{2}-2}\log\frac{\left(w_{k/2}^0\right)^2}{w_2^0 w_{k-2}^0} < 0; \tag{56}$$

in particular equality holds in (55) for $j=2$ and $j=\frac{k}{2}$ so that also

$$\exp\left(Q_1^0\left(\frac{k}{2}-2\right)+P_1^0\right) = w_{k/2}^0.$$

We illustrate the situation in figure 7. Then by (172), a simple calculation shows that

$$Q_1^0 = -\log 4 + o(1) \quad \text{and} \quad \left(w_{k/2}^0\right)^2 / w_{k-2}^0 \to 0 \quad \text{for} \quad k\to\infty. \tag{57}$$

Therefore

$$\begin{aligned}\sum_{j=2}^{\lfloor \frac{k}{2}\rfloor} w_j^0 w_{k-j}^0 &\leqslant w_2^0 w_{k-2}^0 + \int_2^\infty \mathrm{e}^{Q_1^0(j-2)+P_1^0}\mathrm{d}j\\ &\leqslant \left(1+\log^{-1}4\right) w_2^0 w_{k-2}^0\left(1+o_{k_0\to\infty}(1)\right),\end{aligned}$$

using (56) and (57), for all $k \geqslant k_0$ large enough. This finishes the proof of item 1.

Proof of item 2. We proceed as in the proof of item 1: let $w_j^0 = \exp(\Phi_2^0(j))$ for $j\in[2,k-2]$. Then Φ_2^0 is convex; in fact $\frac{\mathrm{d}}{\mathrm{d}j}\Phi_2^0(j) = \phi(j+a^0)$ (positive for $j\geqslant 4$ since $a^0>-2$), $\frac{d^2}{dj^2}\Phi_2^0(j) = \phi'(j+a^0)>0$, see (176) and (177). We conclude that

$$\Phi_2^0(j) \leqslant Q_2^0(j-2)+P_2^0 \quad \forall j\in[2,k-2], \tag{58}$$

where

$$\exp\left(P_2^0\right) = w_2^0 \quad \text{and} \quad Q_2^0 = \frac{1}{k-4}\log\frac{w_{k-2}^0}{w_2^0} > 0; \tag{59}$$

in particular equality holds in (58) for $j=2$ and $j=k-2$. By (172), we find that

$$Q_2^0 = \log k - 1 + o(1) \quad \text{for} \quad k \to \infty. \tag{60}$$

We can therefore estimate

$$\sum_{j=2}^{k-2} \rho^{j-k+2} w_j^0 \leqslant \rho^{-k+2} e^{-2Q_2^0 + P_2^0} \sum_{j=2}^{k-2} \left(\rho \mathrm{e}^{Q_2^0}\right)^j.$$

By (60), there is a $k_0 \gg 1$ such that

$$\rho e^{Q_2^0} \geqslant 2 \quad \forall k \geqslant k_0,$$

and therefore by estimating the geometric sum and using (59), we find that

$$\sum_{j=2}^{k-2} \rho^{j-k+2} w_j^0 \leqslant 2e^{Q_2^0(k-4)+P_2^0} = 2w_{k-2}^0 \quad \forall k \geqslant k_0.$$

It follows that

$$C := \sup_{k\geqslant 4} \left(\frac{1}{w_{k-2}^0} \sum_{j=2}^{k-2} \rho^{j-k+2} w_j^0 \right) < \infty,$$

is well-defined.

Proof of item 3. We use

$$w_j^0 \leqslant e^{Q_2^0(j-2)+P_2^0} \quad \forall j \in [2,k],$$

with Q_2^0 and P_2^0 defined in (59), to estimate

$$\sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \xi^{l-2} w_{k-2(l-1)}^0 \leqslant e^{Q_2^0 k + P_2^0} \xi^{-2} \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \left(\xi\, \mathrm{e}^{-2Q_2^0}\right)^l.$$

By (60), there is a $k_0 \gg 1$ such that

$$\xi\, e^{-2Q_2^0} \leqslant \frac{1}{2} \quad \forall k \geqslant k_0,$$

and therefore by estimating the geometric sum and using (59), we find that

$$\sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \xi^{l-2} w_{k-2(l-1)}^0 \leqslant 2e^{Q_2^0(k-4)+P_2^0} = 2w_{k-2}^0 \quad \forall k \geqslant k_0.$$

It follows that

$$C := \sup_{k\geqslant 4}\left(\frac{1}{w^0_{k-2}}\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\xi^{l-2}w^0_{k-2(l-1)}\right) < \infty,$$

is well-defined. ☐

Remark 6. The strategy used in the proof of lemma 4.3, based on the convexity of the functions $\Phi^0_i(j)$, see e.g. (54) and figure 7, will also be used for $\epsilon > 0$ below, see lemma 5.5.

Lemma 4.4. *If $G\in\mathcal{D}^0$ and $H\in\mathcal{D}^0$ then $GH =: \sum_{k=4}^\infty (GH)_k(\cdot)^k \in \mathcal{D}^0$. In particular, there is a constant $C = C(a^0)$ such that*

$$|(GH)_k| \leqslant C\|G\|\|H\|w^0_{k-2} \quad \forall k\geqslant 4. \tag{61}$$

Proof. Notice that (61) implies the first statement since

$$\frac{w^0_{k-2}}{w^0_k} = \frac{1}{(k-1+a^0)(k-2+a^0)} \quad \forall k\geqslant 4.$$

using (170). Next regarding (61), we use (50): $(GH)_k = \sum_{j=2}^{k-2} G_j H_{k-j} \implies$

$$|(GH)_k| \leqslant \|G\|\|H\|\sum_{j=2}^{k-2} w^0_j w^0_{k-j} \leqslant C\|G\|\|H\|w^0_{k-2},$$

by lemma 4.3 item 1. ☐

A consequence of this result is that

$$|(y^l)_k| \leqslant \|y\|^l C^{l-1} w^0_{k-2(l-1)} \quad \forall k\geqslant 2l, \tag{62}$$

for all $l\geqslant 2$. This follows by induction. Indeed, having already established the base case, $l = 2$, in lemma 4.4, we can proceed analogously for any l by writing

$$(y^l)_k = \sum_{j=2(l-1)}^{k-2} (y^{l-1})_j y_{k-j},$$

and using

$$\sum_{j=2}^{k-2(l-1)} w^0_j w^0_{k-2(l-2)-j} \leqslant C w^0_{k-2(l-1)},$$

cf lemma 4.3 item 1. We also emphasise the following:

$$(y^l)_k,\ k\geqslant 2l, \quad \text{only depends upon} \quad y_2,\cdots,y_{k-2(l-1)} \quad \forall l\in\mathbb{N}. \tag{63}$$

4.1. Proof of proposition 4.2

We now turn to the proof of proposition 4.2 (with $k \geqslant 5$). By (26), (48), (49) and lemma 4.4, we have

$$\begin{aligned}
|\mathcal{G}^0[y]_k| &\leqslant B\rho^{-k} + \mu \sum_{l=2}^{\lfloor \frac{k-1}{2} \rfloor} \|y\|^l \rho^{-l} C^{l-1} \sum_{j=2l}^{k-1} \rho^{j-k} w^0_{j-2(l-1)} \\
&\leqslant B\rho^{-k} + \mu \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \|y\|^l \rho^{-l} C^{l-1} \sum_{j=2l}^{k} \rho^{j-k} w^0_{j-2(l-1)};
\end{aligned}$$

the last estimate, due to

$$\sum_{l=2}^{\lfloor \frac{k-1}{2} \rfloor} (\cdots) \sum_{j=2l}^{k-1} (\cdots) \leqslant \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} (\cdots) \sum_{j=2l}^{k} (\cdots),$$

is not important, but it streamlines some estimates for $\epsilon = 0$ with similar ones for $\epsilon > 0$ later on (see e.g. (112)). We focus on the final term:

$$\mu \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \|y\|^l \rho^{-l} C^{l-1} \sum_{j=2l}^{k} \rho^{j-k} w^0_{j-2(l-1)}.$$

By lemma 4.3 item 2 with $k \to k - 2(l-2)$, we can conclude that

$$\sum_{j=2l}^{k} \rho^{j-k} w^0_{j-2(l-1)} \leqslant C w^0_{k-2(l-1)},$$

where $C > 0$ is large enough but independent of l and k. We are therefore left with

$$\mu \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \|y\|^l \rho^{-l} C^l w^0_{k-2(l-1)},$$

upon increasing $C > 0$ if necessary. This sum is bounded by $\mu K w_{k-2}$, with $K = K(\|y\|) > 0$, for all $k \geqslant 5$ by lemma 4.3 item 3. This completes the proof of (51).

The proof of (52) proceeds completely analogously. In particular, we find that

$$\left(D\left(\mathcal{H}^0[y]\right)(z)\right)_k = \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=2l}^{k-1} h^0_{k-j,l}\, l \left(y^{l-1} z\right)_j \quad \forall z(x) = \sum_{k=2}^{\infty} z_k x^k \in \mathcal{D}^0, k \geqslant 5, \tag{64}$$

using the binomial theorem, which is well-defined by lemma 4.4. We therefore leave out further details. $\square$

4.2. The formal expansion of the centre manifold

We are now ready to show that $\widehat{m}^0 \in \mathcal{D}^0$ (i.e. that it has finite $\mathcal{D}^0$-norm; existence of $\widehat{m}^0$ is well-known). For this purpose, we define the nonlinear operator $\mathcal{P}^0 : \mathcal{D}^0 \to \mathcal{D}^0$ by

$$\mathcal{P}^0(y)(x) = \sum_{k=2}^{\infty} (-1)^k w_k^0 \left(\sum_{j=2}^{k} \frac{(-1)^j \mathcal{G}^0[y]_j}{w_j^0} \right) x^k. \tag{65}$$

It follows from lemma 2.3 that $\widehat{m}^0(x) = \sum_{k=2}^{\infty} m_k^0 x^k$ is given recursively by

$$m_k^0 = (-1)^k w_k^0 S_k^0, \quad S_k^0 := \sum_{j=2}^{k} \frac{(-1)^j \mathcal{G}^0\left[\widehat{m}^0\right]_j}{w_j^0}, \tag{66}$$

where the right hand side only depends upon $m_2^0, \ldots, m_{k-3}^0$ (which is a simple consequence of (49) and (63)). Consequently, at the level of formal series, $\widehat{m}^0$ is a fixed-point of $\mathcal{P}^0$:

$$\mathcal{P}^0\left(\widehat{m}^0\right) = \widehat{m}^0.$$

Lemma 4.5. *Let*

$$F := B \sum_{j=2}^{\infty} \frac{\rho^{-j}}{w_j^0} < \infty,$$

Then there is a $\mu_0 > 0$ small enough, such that $\mathcal{P}^0 : \mathcal{B}^{2F} \to \mathcal{B}^{2F}$ is well-defined for all $0 \leqslant \mu < \mu_0$. Moreover, $\mathcal{P}^0(y; f_2^0, \mu)$ is C^1 with respect to y, f_2^0 and μ, specifically

$$D\mathcal{P}^0(y) = \mathcal{O}(\mu) \quad \forall y \in \mathcal{B}^{2F}, \tag{67}$$

so that $\mathcal{P}^0$ is a contraction on $\mathcal{B}^{2F}$ for all $0 \leqslant \mu \ll 1$.

Proof. We have

$$\begin{aligned} \|\mathcal{P}^0(y)\| &\leqslant \sum_{j=2}^{\infty} \frac{|\mathcal{G}^0[y]_j|}{w_j^0} \\ &\leqslant \sum_{j=2}^{\infty} \frac{|f_j^0|}{w_j^0} + \mu \sum_{j=5}^{\infty} \frac{|\mathcal{H}^0[y]_j|}{w_j^0} \\ &\leqslant F + \mathcal{O}(\mu) \leqslant 2F, \end{aligned} \tag{68}$$

for all $y \in \mathcal{B}^{2F}$, provided that $\mu > 0$ is small enough. Here we have used that

$$\sum_{j=2}^{\infty} \frac{|f_j^0|}{w_j^0} \leqslant F, \quad \sum_{j=5}^{\infty} \frac{|\mathcal{H}^0[y]_j|}{w_j^0} \leqslant K \sum_{j=5}^{\infty} (j-4)^{-2} < \infty, \quad K = K(F), \tag{69}$$

by proposition 4.2, see (51), and $a^0 > -2$, recall hypothesis 1. The statement regarding $\mathcal{P}^0(y; f_2^0, \mu)$ being C^1 follows from (52) and the linearity with respect to f_2^0 and μ. $\square$

Proposition 4.6. *There exists a* $\mu_0 > 0$ *sufficiently small, so that* $\widehat{m}^0 \in \mathcal{D}^0$ *for all* $\mu \in [0,\mu_0)$ *and*

$$\|\widehat{m}^0\| \leqslant 2F \quad \forall\, \mu \in [0,\mu_0). \tag{70}$$

Moreover, $\widehat{m}^0$ *is* C^1 *with respect to* $f_2^0 \in I$ *and* $\mu \in [0,\mu_0(I))$*, with* $I \subset \mathbb{R}$ *any fixed compact set.*

Proof. There are different ways to proceed (see also remark 7 below), but we will simply use the fact that $\mathcal{P}^0(y;f_2^0,\mu)$ is C^1 and a contraction for all $0 \leqslant \mu \ll 1$. Indeed, define $\mathcal{Q}^0(y,f_2^0,\mu) := y - \mathcal{P}^0(y;f_2^0,\mu)$, $y \in \mathcal{B}^{2F}$. Fixed-points of $\mathcal{P}^0$ then correspond to roots of $\mathcal{Q}^0$. Define $\widehat{m}_*^0 = \sum_{k=2}^\infty m_{*k}^0 x^k$ with m_{*k}^0 given by lemma 2.3 (the y-linear case for $\mu = 0$):

$$m_{*k}^0 = (-1)^k \Gamma\left(k+a^0\right) S_k^0, \quad S_k^0 = \sum_{j=2}^k \frac{(-1)^j f_j^0}{\Gamma\left(j+a^0\right)}.$$

Then by (67), we have that:

$$\mathcal{Q}^0\left(\widehat{m}_*^0, f_2^0, 0\right) = 0, \quad D_y\mathcal{Q}^0\left(\widehat{m}_*^0, f_2^0, 0\right) = \mathrm{Id}_{\mathcal{D}^0} \quad \forall f_2^0 \in I.$$

The result therefore follows by the implicit function theorem. □

Remark 7. As (66) defines m_k^0 recursively, it is clearly also possible to prove (70) by induction. By definition, (70) is equivalent with

$$\left|\frac{m_k^0}{w_k^0}\right| \leqslant 2F \quad \forall k \geqslant 2. \tag{71}$$

The statement is clearly true for all $k = 2,\ldots,4$ (base case). For the induction step, suppose that the result holds true for any $k \geqslant 4$. Then analogously to (68), we find that (66), proposition 4.2 and $a^0 > -2$ imply that

$$\left|\frac{m_{k+1}^0}{w_{k+1}^0}\right| = |S_{k+1}^0| \leqslant B\sum_{j=2}^\infty \frac{\rho^{-j}}{w_j^0} + \mu K \sum_{j=5}^\infty (j-4)^2 \leqslant 2F,$$

for all $0 \leqslant \mu < \mu_0(F)$, as desired. Here we have used that S_{k+1} is a finite sum that only depends upon $m_2^0,\ldots,m_{k-2}^0$ (where (71) holds true by the induction hypothesis).

For any $0 \leqslant \mu < \mu_0$, we have $\widehat{m}^0 \in \mathcal{D}^0$. We then define

$$S_\infty^0 := \lim_{k\to\infty} S_k^0 = \sum_{j=2}^\infty \frac{(-1)^j \mathcal{G}^0\left[\widehat{m}^0\right]_j}{w_j^0}, \tag{72}$$

see (66).

Lemma 4.7. *Consider the assumptions of proposition 4.6. Then the series* S_∞^0 *is absolutely convergent and* $|S_\infty^0| \leqslant 2F$.

Proof. We have

$$|S_\infty^0| \leqslant \sum_{j=2}^\infty \frac{\left|\mathcal{G}^0\left[\widehat{m}^0\right]_j\right|}{w_j^0} \leqslant 2F,$$

by (70). □

In turn, if $S_\infty^0 \neq 0$ then

$$m_k^0 = (-1)^k (1 + o(1)) S_\infty^0 w_k^0 \quad \text{for} \quad k \to \infty, \tag{73}$$

cf (66) and (72), and there are constants $0 < C_1 < C_2$ such that

$$C_1 (k-1)! k^{a^0} \leqslant |m_k^0| \leqslant C_2 (k-1)! k^{a^0}, \tag{74}$$

for all k large enough. Here we have used (173):

$$\Gamma\left(k + a^0\right) = \Gamma(k)(1 + o(1)) k^{a^0} = (k-1)!(1 + o(1)) k^{a^0}.$$

In this way, we obtain our first result.

Lemma 4.8. *If $S_\infty^0 \neq 0$ then $\widehat{m}^0 \in \mathcal{D}^0$ is not convergent for any $x \neq 0$ and the centre manifold of $(x, y) = (0, 0)$ for (43) is therefore not analytic.*

Remark 8. We expect that the converse:

'if $S_\infty^0 = 0$ holds, then the centre manifold is analytic',

is true in general (recall lemma 2.4), but leave this for future work.

Lemma 4.9. *$S_\infty^0 = S_\infty^0(f_2^0, \mu)$ is C^1 with respect to f_2^0 for all $0 \leqslant \mu \ll 1$. In particular,*

$$\frac{\partial S_\infty^0}{\partial f_2^0} = \frac{1}{w_2^0} + \mathcal{O}(\mu) \neq 0.$$

Proof. Having already established the C^1-smoothness of $\widehat{m}^0 = \widehat{m}^0(f_2^0, \mu)$ the result follows from differentiation of (72) with respect to f_2^0 (using (64)). □

Theorem 3.2 item 1 follows from lemma 4.8, see also (73) with $m_k^0 = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} m^0(0)$, $w_k^0 = \Gamma(k + a^0)$. Finally, lemma 4.9 is precisely the statement in theorem 3.2 item 2.

5. The analytic weak-stable manifold W^{ws}: the proof of theorem 3.5

To study (29) and the analytic weak-stable manifold for all $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$, we use the scalings (30), repeated here for convenience:

$$x = \epsilon \bar{x}, \quad y = \epsilon \bar{y}.$$

In the $(\bar{x}, \bar{y})$-coordinates, (20) becomes the following singularly perturbed system:

$$\begin{aligned} \dot{\bar{x}} &= \epsilon \bar{x} (\bar{x} - 1), \\ \dot{\bar{y}} &= -\bar{y} (1 + \epsilon a^\epsilon \bar{x}) + \epsilon \bar{g}^\epsilon (\bar{x}, \bar{y}). \end{aligned} \tag{75}$$

or alternatively in the form

$$\epsilon \bar{x} (\bar{x} - 1) \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} + \bar{y} (1 + \epsilon a^\epsilon \bar{x}) = \epsilon \bar{g}^\epsilon (\bar{x}, \bar{y}), \tag{76}$$

relevant for invariant manifold solutions, where

$$\begin{aligned}\overline{g}^\epsilon(\overline{x},\overline{y}) &:= \epsilon^{-2}g^\epsilon(\epsilon\overline{x},\epsilon\overline{y}) = \overline{f}^\epsilon(\overline{x}) + \mu\overline{h}^\epsilon(\overline{x},\overline{y}),\\ \overline{f}^\epsilon(\overline{x}) &:= \epsilon^{-2}f^\epsilon(\epsilon\overline{x}),\\ \overline{h}^\epsilon(\overline{x},\overline{y}) &:= \epsilon^{-2}h^\epsilon(\epsilon\overline{x},\epsilon\overline{y}).\end{aligned} \tag{77}$$

Here we have also defined $\overline{f}^\epsilon$ and $\overline{h}^\epsilon$. By (25), we obtain the absolutely convergent power series expansion of $\overline{f}^\epsilon$ and $\overline{h}^\epsilon$:

$$\overline{f}^\epsilon(\overline{x}) = \sum_{k=2}^\infty f_k^\epsilon \epsilon^{k-2}\overline{x}^k,\quad \overline{h}^\epsilon(\overline{x},\overline{y}) = \sum_{k=2}^\infty h_{k,1}^\epsilon \epsilon^{k-1}\overline{x}^k\overline{y} + \sum_{k=1}^\infty\sum_{l=2}^\infty h_{k,l}^\epsilon \epsilon^{k+l-2}\overline{x}^k\overline{y}^l. \tag{78}$$

For all $\epsilon^{-1}\notin\mathbb{N}$, $0<\epsilon\ll 1$, $(x,y)=(0,0)$ is a nonresonant hyperbolic node of (75) (the eigenvalues being $-\epsilon$ and -1). Consequently, there is an analytic weak-stable manifold:

$$W^{ws}:\quad \overline{y} = \overline{m}^\epsilon(\overline{x}),\quad \overline{m}^\epsilon(\overline{x}) = \sum_{k=2}^\infty \overline{m}_k^\epsilon \overline{x}^k,\quad \overline{x}\in(-\delta,\delta), \tag{79}$$

with $\delta=\delta(\epsilon)>0$, see e.g. [7, theorem 2.14], which solves (76). Now, for any series $\overline{y}(\overline{x}) = \sum_{k=2}^\infty \overline{y}_k\overline{x}^k$, we define $\overline{\mathcal{G}}^\epsilon[y]$ and $\overline{\mathcal{G}}^\epsilon[y]_k$ as above by composition with the analytic function $\overline{g}^\epsilon$:

$$\overline{\mathcal{G}}^\epsilon[y](x) := \overline{g}^\epsilon(\overline{x},\overline{y}(\overline{x})) = \sum_{k=2}^\infty \overline{\mathcal{G}}^\epsilon[y]_k\overline{x}^k.$$

Again, $\overline{\mathcal{H}}^\epsilon[y]$ and $\overline{\mathcal{H}}^\epsilon[y]_k$ are defined in the same way by composition with the analytic function $\overline{h}^\epsilon$, recall (77). We have $\overline{\mathcal{H}}^\epsilon[y]_k = 0$ for $k=2$ and 3 and therefore

$$\begin{cases}\overline{\mathcal{G}}^\epsilon[y]_2 &= f_2^\epsilon,\\ \overline{\mathcal{G}}^\epsilon[y]_3 &= f_3^\epsilon\epsilon,\\ \overline{\mathcal{G}}^\epsilon[y]_k &= f_k^\epsilon\epsilon^{k-2} + \mu\overline{\mathcal{H}}^\epsilon[y]_k,\quad k\geqslant 4.\end{cases} \tag{80}$$

Finally, we define $(\overline{y}^l)_k$, $k\geqslant 2l$, by

$$\overline{y}(\overline{x})^l =: \sum_{k=2l}^\infty \left(\overline{y}^l\right)_k \overline{x}^k,$$

for all $\overline{y} = \sum_{k=2}^\infty \overline{y}_k\overline{x}^k$, $l\in\mathbb{N}$.

Lemma 5.1. *The following holds:*

$$\overline{\mathcal{H}}^\epsilon[y]_k = \sum_{j=2}^{k-2} h_{k-j,1}^\epsilon \epsilon^{k-j-1}\overline{y}_j + \sum_{l=2}^{\lfloor\frac{k-1}{2}\rfloor}\sum_{j=2l}^{k-1} h_{k-j,l}^\epsilon \epsilon^{k-j+l-2}\left(\overline{y}^l\right)_j,\quad k\geqslant 4. \tag{81}$$

(The last sum is zero for $k=4$.)

Proof. The proof is identical to the proof of lemma 4.1 and further details are therefore left out. □

Lemma 5.2. *Suppose that* $\epsilon^{-1} \notin \mathbb{N}$, $0 < \epsilon \ll 1$ *and let* (79) *denote the analytic weak-stable manifold. Then the* $\overline{m}_k^\epsilon$*'s satisfy the recursion relation:*

$$(1-\epsilon k)\,\overline{m}_k^\epsilon + \epsilon\,(k-1+a^\epsilon)\,\overline{m}_{k-1}^\epsilon = \epsilon \overline{\mathcal{G}}^\epsilon\,[\overline{m}^\epsilon]_k \quad \forall k \geqslant 2; \tag{82}$$

here we define $\overline{m}_1^\epsilon = 0$. *In particular, the right side of* (82) *only depends upon* $\overline{m}_2^\epsilon, \ldots, \overline{m}_{k-2}^\epsilon$.

Proof. Simple calculation. □

Lemma 5.3. *For* $\epsilon^{-1} \notin \mathbb{N}$, $0 < \epsilon \ll 1$, *define*

$$\overline{w}_k^\epsilon := \frac{\Gamma\left(\epsilon^{-1} - k\right)\Gamma\left(k + a^\epsilon\right)}{\epsilon \Gamma\left(\epsilon^{-1}\right)} \quad \forall k \geqslant 2, \tag{83}$$

and

$$\overline{S}_k^\epsilon := \sum_{j=2}^{k} \frac{(-1)^j\,\epsilon \overline{\mathcal{G}}^\epsilon\,[\overline{m}^\epsilon]_j}{\overline{w}_j^\epsilon\,(1-\epsilon j)} \quad \forall k \geqslant 2.$$

Then $\overline{S}_k^\epsilon$ *depends upon* $\overline{m}_2^\epsilon, \ldots, \overline{m}_{k-2}^\epsilon$ *for each* $k \geqslant 4$ *and* $\overline{m}_k^\epsilon$ *satifies*

$$\overline{m}_k^\epsilon = (-1)^k\,\overline{w}_k^\epsilon \overline{S}_k^\epsilon \quad \forall k \geqslant 2. \tag{84}$$

Proof. The result follows from induction on k, with the base case being $k = 2$, upon using (82) and the recursion relation

$$(1-\epsilon k)\,\overline{w}_k^\epsilon = \epsilon\,(k-1+a^\epsilon)\,\overline{w}_{k-1}^\epsilon,$$

for the $\overline{w}_k^\epsilon$'s in the induction step. □

Lemma 5.4. *Write*

$$\overline{m}_k^\epsilon =: \epsilon^{k-1} m_k^\epsilon, \tag{85}$$

and let $\widehat{m}^0(x) = \sum_{k=2}^\infty m_k^0 x^k$ *denote the formal series expansion of the centre manifold for* $\epsilon = 0$, *recall* (66). *Then for any fixed k,*

$$m_k^\epsilon \to m_k^0,$$

as $\epsilon \to 0$.

Proof. Inserting (85) into (82), it is straightforward to obtain

$$m_k^\epsilon\,(1-\epsilon k) + (k-1+a^\epsilon)\,m_{k-1}^\epsilon = \mathcal{G}^\epsilon\,[y]_k \to m_k^0 + \left(k-1+a^0\right) m_{k-1}^0 = \mathcal{G}^0\left[\widehat{m}^0\right]_k,$$

as $\epsilon \to 0$. (Here $\mathcal{G}^\epsilon[y]$ is the power series defined by composition $g^\epsilon(\cdot, y)$ of a series y with the analytic function g^ϵ (without bars).) The result then follows from induction on k. □

5.1. Growth properties of $\overline{m}_k^\epsilon$

We now study the formal series (79) and the growth properties of $\overline{m}_k^\epsilon$, $k=2,\ldots N^\epsilon$. For this purpose, the following lemma, on the properties of the $\overline{w}_k^\epsilon$'s, defined in (83), will be crucial.

Lemma 5.5. *Suppose that $a^0>-2$, that $\epsilon^{-1}\notin\mathbb{N}$ and write*

$$\epsilon^{-1}=:N^\epsilon+\alpha^\epsilon,\quad N^\epsilon:=\lfloor\epsilon^{-1}\rfloor,\quad \alpha^\epsilon\in(0,1),\tag{86}$$

Then the following can be said about $\overline{w}_k^\epsilon$, defined in (83), for all $2\leqslant k\leqslant N^\epsilon$:

1. *For fixed $k\in\mathbb{N}\setminus\{1\}$:*

$$\overline{w}_k^\epsilon=\epsilon^{k-1}(1+o(1))\Gamma(k+a^\epsilon),$$

as $\epsilon\to0$.

2. *Lower bound of $\overline{w}_k^\epsilon(1-\epsilon k)$:*

$$\overline{w}_k^\epsilon(1-\epsilon k)\geqslant\Gamma(k+a^\epsilon)\epsilon^{k-1}\quad\forall 2\leqslant k\leqslant N^\epsilon+1.$$

3. *Convolution estimate: there is a $C=C(a^0)$ such that*

$$\sum_{j=2}^{k-2}\overline{w}_j^\epsilon\overline{w}_{k-j}^\epsilon\leqslant C\overline{w}_2^\epsilon\overline{w}_{k-2}^\epsilon\quad\forall 4\leqslant k\leqslant N^\epsilon+1,\tag{87}$$

and

$$\sum_{j=k-(N^\epsilon-1)}^{N^\epsilon-1}\overline{w}_j^\epsilon\overline{w}_{k-j}^\epsilon\leqslant C\overline{w}_{N^\epsilon-1}^\epsilon\overline{w}_{k-(N^\epsilon-1)}^\epsilon\quad\forall N^\epsilon+1\leqslant k\leqslant 2(N^\epsilon-1).\tag{88}$$

4. *Define*

$$\begin{aligned}Q_4^\epsilon&:=\frac{1}{N^\epsilon-3}\log\frac{\overline{w}_{N^\epsilon-1}^\epsilon}{\overline{w}_2^\epsilon},\\ P_4^\epsilon&:=\log(\overline{w}_2^\epsilon)\end{aligned}\tag{89}$$

Then

$$\begin{aligned}Q_4^\epsilon&=\frac{1}{N^\epsilon-3}\left((a^\epsilon+1-\alpha^\epsilon)\log N^\epsilon+\log\frac{\Gamma(1+\alpha^\epsilon)}{\Gamma(2+a^\epsilon)}+o(1)\right),\\ P_4^\epsilon&=\log\epsilon+\log\Gamma(2+a^\epsilon)+o(1)\end{aligned}\tag{90}$$

and

$$\overline{w}_k^\epsilon\leqslant e^{Q_4^\epsilon(k-2)+P_4^\epsilon}\quad\forall 2\leqslant k\leqslant N^\epsilon-1,\tag{91}$$

for all $0<\epsilon\ll1$. In particular,

$$\sum_{k=2}^{N^\epsilon-1}\overline{w}_k^\epsilon\delta^k\leqslant\sum_{k=2}^{N^\epsilon-1}e^{Q_4^\epsilon(k-2)+P_4^\epsilon}\delta^k\leqslant\delta^2C\epsilon\quad\forall 0<\delta\leqslant\frac{3}{4}.$$

for some $C>0$ and all $0<\epsilon\ll1$.

5. *For fixed* $\epsilon^{-1} \notin \mathbb{N}$,

$$\overline{w}_k^\epsilon = \frac{(-1)^{N^\epsilon - k}\Gamma(\alpha^\epsilon)\Gamma(1-\alpha^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})}\frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}$$
$$= \mathcal{O}(1)k^{\epsilon^{-1}+a^\epsilon-1}, \tag{92}$$

with respect to $k \to \infty$.

6. *Let* $\xi > 0$. *Then there is a constant* $C = C(a^\epsilon, \xi)$ *such that*

$$\sum_{j=2}^{k-2}\left(\xi^{-1}\epsilon\right)^{k-2-j}\overline{w}_j^\epsilon \leqslant C\overline{w}_{k-2}^\epsilon \quad \forall 4 \leqslant k \leqslant N^\epsilon + 1, \tag{93}$$

for all $0 < \epsilon \ll 1$.

7. *Let* $\xi > 0$. *Then there is a constant* $C = C(a^\epsilon, \xi)$ *such that*

$$\sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\left(\xi^{-1}\epsilon\right)^{l-2}\left(\overline{w}_2^\epsilon\right)^{l-1}\overline{w}_{k-2(l-1)}^\epsilon \leqslant C\overline{w}_2^\epsilon\overline{w}_{k-2}^\epsilon \quad \forall 4 \leqslant k \leqslant N^\epsilon + 1, \tag{94}$$

for all $0 < \epsilon \ll 1$.

Proof. We prove the items 1–7 successively in the following.

Proof of item 1. For fixed $k \in \mathbb{N}$, we have

$$\overline{w}_k^\epsilon = \frac{\Gamma\left(\epsilon^{-1}-k\right)}{\epsilon\Gamma(\epsilon^{-1})}\Gamma(k+a^\epsilon) = \frac{\Gamma\left(\epsilon^{-1}\right)\epsilon^{k-1}}{\Gamma(\epsilon^{-1})}(1+o(1))\Gamma(k+a^\epsilon) = \mathcal{O}\left(\epsilon^{k-1}\right),$$

using (173), (83) and the definition of the gamma function.

Proof of item 2. We calculate

$$\frac{\overline{w}_k^\epsilon(1-\epsilon k)}{\Gamma(k+a^\epsilon)} = \frac{\Gamma\left(\epsilon^{-1}-k+1\right)}{\Gamma(\epsilon^{-1})} = \frac{1}{\Pi_{j=1}^{k-1}(\epsilon^{-1}-j)} = \epsilon^{k-1}\Pi_{j=1}^{k-1}\frac{1}{1-j\epsilon} \geqslant \epsilon^{k-1},$$

using (170) and $1-(k-1)\epsilon \geqslant \alpha^\epsilon > 0$ for $2 \leqslant k \leqslant N^\epsilon + 1$.

Proof of item 3. We first focus on (87) and notice from item 1 that the claim holds true for all $4 \leqslant k \leqslant k_0$ with $k_0 > 0$ fixed and all $0 < \epsilon \ll 1$. We therefore consider $k_0 < k \leqslant N^\epsilon + 1$ with $k_0 > 0$ fixed large. We write

$$\sum_{j=2}^{k-2}\overline{w}_j^\epsilon\overline{w}_{k-j}^\epsilon \leqslant 2\sum_{j=2}^{\lfloor \frac{k}{2}\rfloor}\overline{w}_j^\epsilon\overline{w}_{k-j}^\epsilon =: 2\sum_{j=2}^{\lfloor \frac{k}{2}\rfloor}\mathrm{e}^{\Phi_{31}^\epsilon(j)}.$$

By proceeding as in the proof of lemma 4.4, a simple computation, using (176) and (177), shows that $\Phi_{31}^\epsilon(j), j \in [2, k-2]$, is convex, having a unique minimum at $j = \frac{k}{2}$. Therefore

$$\Phi_{31}^\epsilon(j) \leqslant Q_{31}^\epsilon(j-2) + P_{31}^\epsilon, \tag{95}$$

where Q_{31}^ϵ and P_{31}^ϵ are chosen such that

$$Q_{31}^\epsilon = \frac{\Phi_{31}^\epsilon\left(\frac{k}{2}\right) - \Phi_{31}^\epsilon(2)}{\frac{k}{2}-2} = \frac{1}{\frac{k}{2}-2}\log\frac{\left(\overline{w}_{\frac{k}{2}}^\epsilon\right)^2}{\overline{w}_2^\epsilon\overline{w}_{k-2}^\epsilon}, \quad P_{31}^\epsilon = \Phi_{31}^\epsilon(2).$$

In particular, equality holds for $j=2$ and $j=\frac{k}{2}$ in (95) and consequently

$$\left(\overline{w}^{\epsilon}_{\frac{k}{2}}\right)^2=\mathrm{e}^{Q^{\epsilon}_{31}\left(\frac{k}{2}-2\right)+P^{\epsilon}_{31}},\quad \overline{w}^{\epsilon}_2\overline{w}^{\epsilon}_{k-2}=\mathrm{e}^{P^{\epsilon}_{31}}.$$

Using (83) and (172), a simple calculation shows that

$$Q^{\epsilon}_{31}<\frac{1}{\frac{k}{2}-2}\log\frac{\Gamma\left(\frac{k}{2}+a^{\epsilon}\right)^2}{\Gamma\left(2+a^{\epsilon}\right)\Gamma\left(k-2+a^{\epsilon}\right)}=-\log 4\left(1+o_{k_0\to\infty}\left(1\right)\right),$$

for all $k_0<k\leqslant N^{\epsilon}+1$, uniformly in $0<\epsilon\ll 1$. Then proceeding as in the proof of lemma 4.4, we have

$$\begin{aligned}\sum_{j=2}^{k-2}\overline{w}^{\epsilon}_j\overline{w}^{\epsilon}_{k-j}&\leqslant 2\overline{w}^{\epsilon}_2\overline{w}^{\epsilon}_{k-2}+\int_2^{\infty}\mathrm{e}^{Q^{\epsilon}_{31}(j-2)+P^{\epsilon}_{31}}\mathrm{d}j\\&\leqslant 2\left(1+\log 4\right)\overline{w}^{\epsilon}_2\overline{w}^{\epsilon}_{k-2}\left(1+o\left(1\right)\right),\end{aligned}$$

which completes the proof of (87).

The inequality (88) is proven in a similar way. First, we put $k=2(N^{\epsilon}-1)-p$ and use (83) and (173) to obtain

$$\overline{w}^{\epsilon}_{N^{\epsilon}-1-p+j}\overline{w}^{\epsilon}_{N^{\epsilon}-1-j}=\Gamma\left(\alpha^{\epsilon}+1+p-j\right)\Gamma\left(\alpha^{\epsilon}+1+j\right)\left(N^{\epsilon}\right)^{2(a^{\epsilon}-\alpha^{\epsilon})-p}\left(1+o\left(1\right)\right).$$

Therefore

$$\begin{aligned}\sum_{j=k-(N^{\epsilon}-1)}^{N^{\epsilon}-1}\overline{w}^{\epsilon}_j\overline{w}^{\epsilon}_{k-j}&=\sum_{j=0}^{p}\overline{w}^{\epsilon}_{N^{\epsilon}-1-p+j}\overline{w}^{\epsilon}_{N^{\epsilon}-1-j}\\&=\left(N^{\epsilon}\right)^{2(a^{\epsilon}-\alpha^{\epsilon})-p}\sum_{j=0}^{p}\Gamma\left(\alpha^{\epsilon}+1+p-j\right)\Gamma\left(\alpha^{\epsilon}+1+j\right)\left(1+o\left(1\right)\right).\end{aligned}$$

Here

$$\sum_{j=0}^{p}\Gamma\left(\alpha^{\epsilon}+1+p-j\right)\Gamma\left(\alpha^{\epsilon}+1+j\right)\leqslant C\Gamma\left(\alpha^{\epsilon}+1\right)\Gamma\left(\alpha^{\epsilon}+1+p\right),$$

cf (53) and therefore (88) holds true for all $2(N^{\epsilon}-1)-p\leqslant k\leqslant 2(N^{\epsilon}-1)$ and any $p>0$ provided that $0<\epsilon\ll 1$.

We therefore proceed to consider $N^{\epsilon}+1\leqslant k\leqslant 2(N^{\epsilon}-1)-p$ with $p>0$ fixed large. We write

$$\sum_{j=k-(N^{\epsilon}-1)}^{N^{\epsilon}-1}\overline{w}^{\epsilon}_j\overline{w}^{\epsilon}_{k-j}\leqslant 2\sum_{j=k-(N^{\epsilon}-1)}^{\lfloor\frac{k}{2}\rfloor}\overline{w}^{\epsilon}_j\overline{w}^{\epsilon}_{k-j}:=2\sum_{j=k-(N^{\epsilon}-1)}^{\lfloor\frac{k}{2}\rfloor}e^{\Phi^{\epsilon}_{32}(j)}.$$

As above, $\Phi^{\epsilon}_{32}(j), j\in[2,k-2]$, is convex, having a unique minimum at $j=\frac{k}{2}$, so that

$$\Phi^{\epsilon}_{32}\left(j\right)\leqslant Q^{\epsilon}_{32}\left(j-2\right)+P^{\epsilon}_{32},\tag{96}$$

where Q_{32}^{ϵ} and P_{32}^{ϵ} are now chosen such that

$$Q_{32}^{\epsilon} = \frac{\Phi_{32}\left(\frac{k}{2}\right) - \Phi_{32}\left(k-(N^{\epsilon}-1)\right)}{N^{\epsilon}-1-\frac{k}{2}} = \frac{1}{N^{\epsilon}-1-\frac{k}{2}} \log \frac{\left(\overline{w}_{\frac{k}{2}}^{\epsilon}\right)^2}{\overline{w}_{k-(N^{\epsilon}-1)}^{\epsilon}\overline{w}_{N^{\epsilon}-1}^{\epsilon}},$$

$$P_{32}^{\epsilon} = \Phi^{\epsilon}\left(k-(N^{\epsilon}-1)\right).$$

Equality holds in (96) for $j = k-(N^{\epsilon}-1)$ and $j=\frac{k}{2}$. Now, using (83) and (172) a simple calculation shows that

$$Q_{32}^{\epsilon} < \frac{1}{N^{\epsilon}-1-\frac{k}{2}} \log \frac{\Gamma\left(\epsilon^{-1}-\frac{k}{2}\right)^2}{\Gamma\left(1+\alpha^{\epsilon}\right)\Gamma\left(\epsilon^{-1}-\left(k-(N^{\epsilon}-1)\right)\right)} \leqslant -\log 4\left(1+o_{p\to\infty}(1)\right),$$

for all $N^{\epsilon}+1 \leqslant k \leqslant 2(N^{\epsilon}-1)-p$, uniformly in $0<\epsilon\ll 1$. We can now complete the proof by proceeding in the exact same way that we did in the proof of (87).

Proof of item 4. First, we write

$$\overline{w}_k^{\epsilon} = e^{\Phi_4^{\epsilon}(k)},$$

where

$$\Phi_4^{\epsilon}(k) = \log \frac{\Gamma\left(\epsilon^{-1}-k\right)\Gamma\left(k+a^{\epsilon}\right)}{\epsilon\Gamma\left(\epsilon^{-1}\right)}.$$

Again, $\Psi_4^{\epsilon}(k)$ is convex on $k\in[2,N^{\epsilon}-1]$ (having a minimum at $k=k_m(\epsilon):=\frac{1}{2\epsilon}-\frac{a^{\epsilon}}{2}$). Next, Q_4^{ϵ} and P_4^{ϵ}, defined by (89), are chosen such that

$$Q_4^{\epsilon} = \frac{\Psi_4^{\epsilon}(N^{\epsilon}-1)-\Psi_4^{\epsilon}(2)}{N^{\epsilon}-3}, \quad P_4^{\epsilon} = \Psi_4^{\epsilon}(2),$$

specifically

$$\Psi_4^{\epsilon}(k) \leqslant Q_4^{\epsilon}(k-2)+P_4^{\epsilon},$$

for all $k\in[2,N^{\epsilon}-1]$ with equality for $k=2$ and $k=N^{\epsilon}-1$:

$$\overline{w}_2^{\epsilon} = \mathrm{e}^{P_4^{\epsilon}}, \quad \overline{w}_{N^{\epsilon}-1}^{\epsilon} = \mathrm{e}^{Q_4^{\epsilon}(N^{\epsilon}-3)+P_4^{\epsilon}}. \tag{97}$$

Moreover, $Q_4^{\epsilon}=o(1)$, see (90) which we prove below. Consequently, for all $0<\delta\leqslant\frac{3}{4}$, we have

$$\begin{aligned}\sum_{k=2}^{N^{\epsilon}-1}\overline{w}_k^{\epsilon}\delta^k &\leqslant \sum_{k=2}^{N^{\epsilon}-1}\mathrm{e}^{Q_4^{\epsilon}(k-2)+P_4^{\epsilon}}\delta^k\\ &\leqslant \mathrm{e}^{P_4^{\epsilon}}\delta^2+\int_2^{\infty}\mathrm{e}^{Q_4^{\epsilon}(k-2)+P_4^{\epsilon}}\delta^k\mathrm{d}k\\ &\leqslant C\delta^2\mathrm{e}^{P_4^{\epsilon}}.\end{aligned}$$

Here we have used that

$$\int_2^{\infty}\mathrm{e}^{a(k-2)}\delta^k\mathrm{d}k = \frac{1}{\log\delta^{-1}-a}\delta^2 \quad \forall 0<\delta<\mathrm{e}^{-a}.$$

To complete the proof of item 4, we just have to prove the asymptotics in (90). The asymptotics of P_4^ϵ follows from item 1, so we focus on Q_4^ϵ. For this, we use Stirling's approximation in the form (173) for $N^\epsilon \gg 1$:

$$\begin{aligned} Q_4^\epsilon &= \frac{1}{N^\epsilon - 3} \log \frac{\overline{w}^\epsilon_{N^\epsilon - 1}}{\overline{w}_2^\epsilon} \\ &= \frac{1}{N^\epsilon - 3} \log \left(\frac{\Gamma(1+\alpha^\epsilon)}{\Gamma(2+a^\epsilon)} \frac{\Gamma(N^\epsilon - 1 + a^\epsilon)}{\Gamma(N^\epsilon + \alpha^\epsilon - 2)} \right) \\ &= \frac{1}{N^\epsilon - 3} \log \left(\frac{\Gamma(1+\alpha^\epsilon)}{\Gamma(2+a^\epsilon)} (1+o(1)) (N^\epsilon)^{a^\epsilon + 1 - \alpha^\epsilon} \right), \end{aligned} \tag{98}$$

using (83), $\epsilon^{-1} = N^\epsilon + \alpha$ and

$$\frac{\Gamma(N^\epsilon - 1 + a^\epsilon)}{\Gamma(N^\epsilon + \alpha^\epsilon - 2)} = (1+o(1)) \frac{(N^\epsilon)^{a^\epsilon - 1}}{(N^\epsilon)^{\alpha^\epsilon - 2}},$$

in the last equality of (98).

Proof of item 5. For (5), we use the reflection formula (174) and Sterling's approximation in the form (173) for $k \to \infty$.

Proof of item 6. It is easy to verify the claim for all $4 \leqslant k \leqslant k_0$ for any $k_0 > 0$ fixed and $0 < \epsilon \ll 1$ by using item 1. We therefore consider $k_0 < k \leqslant N^\epsilon + 1$ with $k_0 > 0$ fixed large and write

$$\overline{w}_j^\epsilon =: \mathrm{e}^{\Phi_6^\epsilon(j)}.$$

Again, Φ_6^ϵ is convex for any $j \in [2, N^\epsilon - 1]$ and therefore

$$\Phi_6^\epsilon(j) \leqslant Q_6^\epsilon(j-2) + P_6^\epsilon,$$

where Q_6^ϵ and P_6^ϵ are chosen such that equality holds for $j = 2$ and $j = k-2$:

$$Q_6^\epsilon = \frac{1}{k-4} \log \frac{\overline{w}^\epsilon_{k-2}}{\overline{w}^\epsilon_2}, \quad \mathrm{e}^{P_6^\epsilon} = \overline{w}_2^\epsilon. \tag{99}$$

By the convexity of Φ_6^ϵ it follows that Q_6^ϵ is increasing. Therefore by item 1 and (173)

$$\begin{aligned} Q_6^\epsilon &\geqslant \log \epsilon + \mathcal{O}(\epsilon) + \log \left(\frac{\Gamma(k_0 - 2 + a^\epsilon)}{\Gamma(2+a^\epsilon)} \right)^{\frac{1}{k_0 - 4}} \\ &= \log \epsilon + \mathcal{O}(\epsilon) + \log k_0 (1 + o_{k_0 \to \infty}(1)), \end{aligned}$$

for all $k_0 \leqslant k \leqslant N^\epsilon - 1$. In turn, we can assume that

$$\xi \epsilon^{-1} e^{Q_6^\epsilon} \geqslant 2 \quad \forall k \in [k_0, N^\epsilon - 1].$$

This allow us to estimate the sum as a geometric sum:

$$\begin{aligned}\sum_{j=2}^{k-2}\left(\xi^{-1}\epsilon\right)^{k-2-j}\overline{w}_j^\epsilon &\leqslant \left(\xi^{-1}\epsilon\right)^{k-2}\mathrm{e}^{P_6^\epsilon}\sum_{j=2}^{k-2}\left(\xi\epsilon^{-1}\mathrm{e}^{Q_6^\epsilon}\right)^j\\ &\leqslant 2\left(\xi^{-1}\epsilon\right)^{k-2}\mathrm{e}^{P_6^\epsilon}\left(\xi\epsilon^{-1}\mathrm{e}^{Q_6^\epsilon}\right)^{k-2}\\ &\leqslant 2\mathrm{e}^{Q_6^\epsilon(k-2)+P_6^\epsilon}\\ &= 2\overline{w}_{k-2}^\epsilon.\end{aligned}$$

Proof of item 7. It is easy to verify the claim for all $4\leqslant k<k_0$ for any $k_0>0$ and $0<\epsilon\ll 1$ by using item 1. We therefore consider $k_0\leqslant k\leqslant N^\epsilon+1$ with $k_0>0$ fixed large and write

$$\overline{w}_j^\epsilon =: \mathrm{e}^{\Phi_6^\epsilon(j)},$$

as in the proof of item 6, with

$$\Phi_6^\epsilon(j)\leqslant Q_6^\epsilon(j-2)+P_6^\epsilon,$$

for all $j\in[2,k-2]$ with equality for $j=2$ and $j=k-2$. We may assume that $k_0>0$ is such that

$$\left(\xi^{-1}\epsilon e^{-2Q_6^\epsilon+P_6^\epsilon}\right)\leqslant\frac{1}{2}\quad\forall k\in[k_0,N^\epsilon+1],$$

for all $0<\epsilon\ll 1$. In this way, we estimate can estimate the sum as a geometric sum

$$\begin{aligned}\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\left(\xi^{-1}\epsilon\right)^{l-2}\left(\overline{w}_2^\epsilon\right)^{l-1}\overline{w}_{k-2(l-1)}^\epsilon &\leqslant \left(\xi\epsilon^{-1}\right)^2\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\left(\xi^{-1}\epsilon\right)^l\mathrm{e}^{P_6^\epsilon(l-1)}\mathrm{e}^{Q_6^\epsilon(k-2l)+P_6^\epsilon}\\ &\leqslant \left(\xi\epsilon^{-1}\right)^2\mathrm{e}^{Q_6^\epsilon k}\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\left(\xi^{-1}\epsilon\mathrm{e}^{-2Q_6^\epsilon+P_6^\epsilon}\right)^l\\ &\leqslant 2\left(\xi\epsilon^{-1}\right)^2\mathrm{e}^{Q_6^\epsilon k}\left(\xi^{-1}\epsilon\mathrm{e}^{-2Q_6^\epsilon+P_6^\epsilon}\right)^2\\ &= 2\overline{w}_2^\epsilon\overline{w}_{k-2}^\epsilon,\end{aligned}$$

for all $k_0<k\leqslant N^\epsilon+1$. Here we have used (99) in the last equality. □

In contrast to the analysis of the centre manifold for $\epsilon=0$, we are in the present case of $\epsilon>0$ only interested in estimating the partial sum of (79):

$$\overline{y}=\sum_{k=2}^{N^\epsilon-1}\overline{m}_k^\epsilon\overline{x}^k,\quad m_k^\epsilon=(-1)^k\overline{w}_k^\epsilon\overline{S}_k^\epsilon,\quad \overline{S}_k^\epsilon=\sum_{j=2}^{k}\frac{(-1)^j\epsilon\overline{\mathcal{G}}^\epsilon\left[\overline{m}^\epsilon\right]_j}{\overline{w}_j^\epsilon(1-\epsilon j)},$$

where

$$N^\epsilon=\lfloor\epsilon^{-1}\rfloor,$$

recall (86). (We will deal with the remainder later, see section 5.3). We therefore define the semi-norm

$$\|\sum_{k=2}^{\infty}\overline{y}_k\overline{x}^k\| := \sup_{k\in[2,N^\epsilon-1]}\frac{|\overline{y}_k|}{\overline{w}_k^\epsilon}, \tag{100}$$

on the set of formal series $\overline{y}=\sum_{k=2}^{\infty}\overline{y}_k\overline{x}^k$.

Lemma 5.6. *Consider* $\overline{y}(\overline{x})=\sum_{k=2}^{N^\epsilon-1}\overline{y}_k\overline{x}^k$ *and define* $(\overline{y}^l)_k, k=2l,\ldots,l(N^\epsilon-1)$ *by*

$$\overline{y}(\overline{x})^l =: \sum_{k=2l}^{l(N^\epsilon-1)}\left(\overline{y}^l\right)_k\overline{x}^k. \tag{101}$$

Then there exists a $C=C(a^0)>0$ *such that for any* $l\in\mathbb{N}\setminus\{1\}$ *and all* $1\leqslant p\leqslant l$ *the following holds true:*

$$\begin{aligned}|\left(\overline{y}^l\right)_k| &\leqslant \|y\|^l\binom{l-1}{p-1}C^{l-1}\left(\overline{w}_2^\epsilon\right)^{l-p}\left(\overline{w}_{N^\epsilon-1}^\epsilon\right)^{p-1}\left(\overline{w}_{k-(p-1)(N^\epsilon-1)-2(l-p)}^\epsilon\right),\\ &\forall k\in\left[(p-1)\left(N^\epsilon-1\right)+2\left(l-p+1\right),p\left(N^\epsilon-1\right)+2\left(l-p\right)\right],\end{aligned} \tag{102}$$

for all $0<\epsilon\ll 1$. *Here*

$$\binom{l-1}{p-1}$$

denotes the binomial coefficient for any $1\leqslant p\leqslant l$.

In particular, for $p=1$*:*

$$\begin{aligned}|\left(\overline{y}^l\right)_k| &\leqslant \|y\|^lC^{l-1}\left(\overline{w}_2^\epsilon\right)^{l-1}\overline{w}_{k-2(l-1)}^\epsilon,\\ &\forall k\in\left[2l,N^\epsilon-1+2\left(l-1\right)\right],\end{aligned} \tag{103}$$

for all $0<\epsilon\ll 1$.

Proof. The claim is proven by induction, with the base case being $l=2$, $p=1$ and $p=2$.

The base case: $(l,p)=(2,1),(2,2)$. For $l=2$, we have by Cauchy's product formula:

$$|\left(\overline{y}^2\right)_k|\leqslant\|y\|^2\sum_{j=\max(2,k-(N^\epsilon-1))}^{\min(k-2,N^\epsilon-1)}\overline{w}_j^\epsilon\overline{w}_{k-j}^\epsilon.$$

We first consider $p=1$: $4\leqslant k\leqslant(N^\epsilon-1)+2=N^\epsilon+1$. Then by item 3 of lemma 5.5, see (87), we conclude that

$$|\left(\overline{y}^2\right)_k|\leqslant\|y\|^2C\overline{w}_2^\epsilon\overline{w}_{k-2}^\epsilon.$$

Next, for $p=2$:

$$|\left(\overline{y}^2\right)_k| \leqslant \|y\|^2 \sum_{j=k-(N^\epsilon-1)}^{N^\epsilon-1} \overline{w}_j^\epsilon \overline{w}_{k-j}^\epsilon \leqslant \|y\|^2 C\overline{w}_{N^\epsilon-1}^\epsilon \overline{w}_{k-(N^\epsilon-1)}^\epsilon,$$

using (88).

Induction step. The induction proceeds in two steps: We assume that the claim is true for all $l\in\mathbb{N}\setminus\{1\}$ and all $1\leqslant p\leqslant l$. We then first proof that it is true for $l+1$, $1\leqslant p\leqslant l$. Subsequently, we consider $p=l+1$.

We assume that (102) holds true. Then by using Cauchy's product formula we find that

$$\left(\overline{y}^{l+1}\right)_k = \sum_{j=\max(2l,k-(N^\epsilon-1))}^{\min(k-2,l(N^\epsilon-1))} \left(\overline{y}^l\right)_j \overline{y}_{k-j}.$$

For $p=1$ and $k\in[2(l+1),N^\epsilon-1+2l]$, we find

$$\begin{aligned}
|\left(\overline{y}^{l+1}\right)_k| &\leqslant \|y\|^{l+1} C^{l-1}\left(\overline{w}_2^\epsilon\right)^{l-1} \sum_{j=2l}^{k-2} \overline{w}_{j-2(l-1)}^\epsilon \overline{w}_{k-j}^\epsilon \\
&\leqslant \|y\|^{l+1} C^{l-1}\left(\overline{w}_2^\epsilon\right)^{l-1} \sum_{j=2}^{k-2l} \overline{w}_j^\epsilon \overline{w}_{(k-2(l-1))-j}^\epsilon \\
&\leqslant \|y\|^{l+1} C^l \left(\overline{w}_2^\epsilon\right)^l \overline{w}_{k-2l}^\epsilon,
\end{aligned}$$

using (87), which proves (102) with $l\to l+1$ and $p=1$. Next, for $2\leqslant p\leqslant l$, we find completely analogously that

$$\begin{aligned}
|\left(\overline{y}^{l+1}\right)_k| \leqslant& \sum_{j=k-(N^\epsilon-1)}^{k-2} |\left(\overline{y}^l\right)_j||\overline{y}_{k-j}| \\
\leqslant& \sum_{j=k-(N^\epsilon-1)}^{(p-1)(N^\epsilon-1)+2(l-p+1)} |\left(\overline{y}^l\right)_j||\overline{y}_{k-j}| + \sum_{j=(p-1)(N^\epsilon-1)+2(l-p+1)}^{k-2} |\left(\overline{y}^l\right)_j||\overline{y}_{k-j}|,
\end{aligned}$$

for

$$k\in\left[(p-1)\left(N^\epsilon-1\right)+2\left(l-p+1\right),p\left(N^\epsilon-1\right)+2\left(l-p+1\right)\right].$$

Therefore by (102) (for (l,p) and $(l,p)\to(l,p-1)$):

$$\begin{aligned}
|\left(\overline{y}^{l+1}\right)_k| \leqslant& \|\overline{y}\|^{l+1}\binom{l-1}{p-2} C^{l-1}\left(\overline{w}_2^\epsilon\right)^{l-p+1}\left(\overline{w}_{N^\epsilon-1}^\epsilon\right)^{p-2} \\
&\times \sum_{j=k-(N^\epsilon-1)}^{(p-1)(N^\epsilon-1)+2(l-p+1)} \overline{w}_{j-(p-2)(N^\epsilon-1)-2(l-p+1)}^\epsilon \overline{w}_{k-j}^\epsilon \\
&+\|y\|^{l+1}\binom{l-1}{p-1} C^{l-1}\left(\overline{w}_2^\epsilon\right)^{l-p}\left(\overline{w}_{N^\epsilon-1}^\epsilon\right)^{p-1}
\end{aligned}$$

$$\times \sum_{j=(p-1)(N^\epsilon-1)+2(l-p+1)}^{k-2} \overline{w}^\epsilon_{j-(p-1)(N^\epsilon-1)-2(l-p)} \overline{w}^\epsilon_{k-j}$$
$$\leqslant \|y\|^{l+1} C^l \left(\overline{w}^\epsilon_2\right)^{l-p+1} \left(\overline{w}^\epsilon_{N^\epsilon-1}\right)^{p-1} \left(\binom{l-1}{p-2} + \binom{l-1}{p-1}\right)$$
$$\times \overline{w}^\epsilon_{k-(p-1)(N^\epsilon-1)-2(l-p+1)},$$

using (87) and (88) to estimate the two sums. Then as

$$\binom{l-1}{p-2} + \binom{l-1}{p-1} = \binom{l}{p-1} \tag{104}$$

the claim follows.

We are left with proving that the claim holds true for $p = l+1$ and

$$k \in [l(N^\epsilon - 1) + 2, (l+1)(N^\epsilon - 1)],$$

where

$$|\left(\overline{y}^{l+1}\right)_k| \leqslant \sum_{j=k-(N^\epsilon-1)}^{l(N^\epsilon-1)} |\left(\overline{y}^l\right)_j| |\overline{y}_{k-j}|.$$

By the induction assumption, we have

$$|\left(\overline{y}^l\right)_k| \leqslant \|\overline{y}\|^l C^{l-1} \left(\overline{w}^\epsilon_{N^\epsilon-1}\right)^{l-1} \overline{w}^\epsilon_{k-(l-1)(N^\epsilon-1)},$$

for all

$$k \in [(l-1)(N^\epsilon - 1) + 2, l(N^\epsilon - 1)],$$

see (102) with $p = l$. Therefore

$$|\left(\overline{y}^{l+1}\right)_k| \leqslant \|\overline{y}\|^{l+1} C^{l-1} \left(\overline{w}^\epsilon_{N^\epsilon-1}\right)^{l-1} \sum_{j=k-(N^\epsilon-1)}^{l(N^\epsilon-1)} \overline{w}^\epsilon_{j-(l-1)(N^\epsilon-1)} \overline{w}^\epsilon_{k-j}$$
$$\leqslant \|\overline{y}\|^{l+1} C^l \left(\overline{w}^\epsilon_{N^\epsilon-1}\right)^l \overline{w}^\epsilon_{k-l(N^\epsilon-1)},$$

using (88). This proves (102) with $l \to l+1$ and $p = l+1$ and completes the proof. □

By using lemma 5.5 item 4, we obtain the following bound on $(\overline{y}^l)_k$

Lemma 5.7. *Consider* $\overline{y}(\overline{x}) = \sum_{k=2}^{N^\epsilon-1} \overline{y}_k \overline{x}^k$ *and recall the definition of* $(\overline{y}^l)_k$ *in (101). Then there is a new* $C > 0$ *such that*

$$|\left(\overline{y}^l\right)_k| \leqslant \|\overline{y}\|^l C^{l-1} e^{(-2Q_4^\epsilon + P_4^\epsilon)l} e^{Q_4^\epsilon k} \quad \forall 2l \leqslant k \leqslant l(N^\epsilon - 1), \tag{105}$$

for all $0 < \epsilon \ll 1$. *Here* Q_4^ϵ *and* P_4^ϵ *are defined in (89).*

Proof. We will use (102), repeated here for convenience:

$$\begin{aligned}|\left(\overline{y}^l\right)_k| \leqslant \|y\|^l \binom{l-1}{p-1} C^{l-1} \underline{\left(\overline{w}_2^\epsilon\right)^{l-p}\left(\overline{w}_{N^\epsilon-1}^\epsilon\right)^{p-1}\overline{w}_{k-(p-1)(N^\epsilon-1)-2(l-p)}^\epsilon},\\ \forall k \in \left[(p-1)(N^\epsilon-1)+2(l-p+1), p(N^\epsilon-1)+2(l-p)\right],\end{aligned} \tag{106}$$

where $1 \leqslant p \leqslant l$. Using (89) we have

$$\overline{w}_2^\epsilon = \mathrm{e}^{P_4^\epsilon}, \quad \overline{w}_{N^\epsilon-1}^\epsilon = \mathrm{e}^{Q_4^\epsilon(N^\epsilon-3)+P_4^\epsilon} \quad \text{and} \quad \overline{w}_k^\epsilon \leqslant \mathrm{e}^{Q_4^\epsilon(k-2)+P_4^\epsilon} \quad \forall 2 \leqslant k \leqslant N^\epsilon+1,$$

and we can therefore estimate the underlined factor in (106) as follows:

$$\underline{\left(\overline{w}_2^\epsilon\right)^{l-p}\left(\overline{w}_{N^\epsilon-1}^\epsilon\right)^{p-1}\overline{w}_{k-(p-1)(N^\epsilon-1)-2(l-p)}^\epsilon} \leqslant \mathrm{e}^{P_4^\epsilon(l-p)}\mathrm{e}^{(Q_4^\epsilon(N^\epsilon-3)+P_4^\epsilon)(p-1)}\mathrm{e}^{(Q_4^\epsilon(k-\{\cdots\}-2)+P_4^\epsilon)},$$

where $\{\cdots\} = (p-1)(N^\epsilon-1)+2(l-p)$. By simplifying, we obtain

$$\underline{\left(\overline{w}_2^\epsilon\right)^{l-p}\left(\overline{w}_{N^\epsilon-1}^\epsilon\right)^{p-1}\overline{w}_{k-(p-1)(N^\epsilon-1)-2(l-p)}^\epsilon} \leqslant e^{(-2Q_4^\epsilon+P_4^\epsilon)l}\mathrm{e}^{Q_4^\epsilon k}. \tag{107}$$

Subsequently, we use

$$\binom{l-1}{p-1} \leqslant \sum_{q=0}^{l-1}\binom{l-1}{q} = 2^{l-1}, \tag{108}$$

for all $1 \leqslant p \leqslant l$. Therefore (105) follows from (106), (107) and (108). $\square$

Lemma 5.8. *Recall the definition of the semi-norm $\|\cdot\|$ in (100) and suppose that $\|\overline{y}\| \leqslant C$ with $C > 0$. Then there is a $\overline{K} = \overline{K}(C) > 0$, independent of μ and ϵ, such that*

$$\begin{cases}|\epsilon\overline{\mathcal{G}}^\epsilon[\overline{y}]_2| & \leqslant B\rho^{-2}\epsilon,\\ |\epsilon\overline{\mathcal{G}}^\epsilon[\overline{y}]_3| & \leqslant B\rho^{-3}\epsilon^2,\\ |\epsilon\overline{\mathcal{G}}^\epsilon[\overline{y}]_k| & \leqslant B\rho^{-k}\epsilon^{k-1}+\mu\epsilon^2\overline{K}\overline{w}_{k-2}^\epsilon \quad \forall 4 \leqslant k \leqslant N^\epsilon+1,\end{cases}$$

for all $0 < \epsilon \ll 1$.

Proof. We use (80):

$$|\overline{\mathcal{G}}^\epsilon[y]_k| \leqslant |\bar{f}_k^\epsilon| + \mu|\overline{\mathcal{H}}^\epsilon[y]_k| \quad \forall k \geqslant 2.$$

The first term on the right hand side is directly estimated by (26):

$$|\bar{f}_k^\epsilon| \leqslant B\rho^{-k}\epsilon^{k-2},$$

for all $k \in \mathbb{N}\setminus\{1\}$. We therefore focus on the second term, which vanishes for $k=2$ and $k=3$. By using (26), (81),

$$|\overline{y}_j| \leqslant \|\overline{y}\|\overline{w}_j^\epsilon \quad \forall j \in [2, N^\epsilon-1],$$

and (103), we obtain

$$\begin{aligned}|\overline{\mathcal{H}}^{\epsilon}[y]_k| &\leqslant \sum_{j=2}^{k-2} \rho^{-k+j-1}\epsilon^{k-j-1}\|\overline{y}\|\overline{w}_j^{\epsilon}\\ &\quad + \sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\sum_{j=2l}^{k} \rho^{-k+j-l}\epsilon^{k-j+l-2}\|\overline{y}\|^l C^{l-1}\left(\overline{w}_2^{\epsilon}\right)^{l-1}\overline{w}_{j-2(l-1)}^{\epsilon}\\ &= \|\overline{y}\|\rho^{-1}\epsilon\sum_{j=2}^{k-2}\left(\rho^{-1}\epsilon\right)^{k-2-j}\overline{w}_j^{\epsilon}\\ &\quad + \sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\rho^{-l}\epsilon^{l-2}\|\overline{y}\|^l C^{l-1}\left(\overline{w}_2^{\epsilon}\right)^{l-1}\sum_{j=2}^{k-2(l-1)}\left(\rho^{-1}\epsilon\right)^{k-2(l-1)-j}\overline{w}_j^{\epsilon},\end{aligned}$$

for all $4\leqslant k\leqslant N^{\epsilon}+1$. We now use (93) and (94), respectively:

$$\begin{aligned}|\overline{\mathcal{H}}^{\epsilon}[y]_k| &\leqslant \|\overline{y}\|\rho^{-1}\epsilon C\overline{w}_{k-2}^{\epsilon}\\ &\quad + \sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\rho^{-l}\epsilon^{l-2}\|\overline{y}\|^l C^l\left(\overline{w}_2^{\epsilon}\right)^{l-1}\overline{w}_{k-2(l-1)}^{\epsilon}\\ &\leqslant \|\overline{y}\|\rho^{-1}\epsilon C\overline{w}_{k-2}^{\epsilon}\\ &\quad + \rho^{-2}\|\overline{y}\|^2C^2\sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\left(\rho^{-1}\|\overline{y}\|C\epsilon\right)^{l-2}\left(\overline{w}_2^{\epsilon}\right)^{l-1}\overline{w}_{k-2(l-1)}^{\epsilon}\\ &\leqslant \overline{K}\epsilon\overline{w}_{k-2}^{\epsilon},\end{aligned}$$

with $\overline{K}=\overline{K}(\|y\|,a^0,\rho)>0$ large enough. Here we have used that $\overline{w}_2^{\epsilon}=\mathcal{O}(\epsilon)$ cf lemma 5.5 item 1. □

This leads to the following important estimate:

Lemma 5.9. *Fix $C>0$ and define*

$$F = B\sum_{j=2}^{\infty}\frac{\rho^{-j}}{\Gamma\left(j+a^0\right)}.$$

Then the following holds for all $0\leqslant\mu<\mu_0$ with $\mu_0>0$ small enough:

$$\left|\sum_{j=2}^{k}\frac{(-1)^j\epsilon\overline{\mathcal{G}}^{\epsilon}[\overline{y}]_j}{\overline{w}_j^{\epsilon}(1-\epsilon j)}\right|\leqslant 2F\quad\forall 2\leqslant k\leqslant N^{\epsilon},\ \|\overline{y}\|\leqslant C,$$

for all $0<\epsilon\ll 1$.

Proof. Let $\overline{K}=\overline{K}(C)>0$ be the constant in lemma 5.8. We then estimate

$$\left|\sum_{j=2}^{k}\frac{(-1)^j\epsilon\overline{\mathcal{G}}^{\epsilon}[\overline{y}]_j}{\overline{w}_j^{\epsilon}(1-\epsilon j)}\right|\leqslant B\sum_{j=2}^{N^{\epsilon}}\frac{\rho^{-j}\epsilon^{j-1}}{\overline{w}_j^{\epsilon}(1-\epsilon j))}+\mu\overline{K}\sum_{j=4}^{N^{\epsilon}}\frac{\epsilon^2\overline{w}_{j-2}^{\epsilon}}{\overline{w}_j^{\epsilon}(1-\epsilon j)},$$

for all $2 \leqslant k \leqslant N^\epsilon$ using lemma 5.8. Then by the definition of $\overline{w}_k^\epsilon$ (83) and (170), we have

$$\frac{\overline{w}_{j-2}^\epsilon}{\overline{w}_j^\epsilon} = \frac{\Gamma\left(\epsilon^{-1}-j+2\right)\Gamma\left(j-2+a^\epsilon\right)}{\Gamma\left(\epsilon^{-1}-j\right)\Gamma\left(j+a^\epsilon\right)}$$
$$= \frac{\left(\epsilon^{-1}-j+1\right)\left(\epsilon^{-1}-j\right)}{\left(j-1+a^\epsilon\right)\left(j-2+a^\epsilon\right)}.$$

Therefore by lemma 5.5 item 2 we find that

$$\left|\sum_{j=2}^{k}\frac{(-1)^j\,\epsilon\overline{\mathcal{G}}^\epsilon\,[\overline{y}]_j}{\overline{w}_j^\epsilon\,(1-\epsilon j)}\right| \leqslant B\sum_{j=2}^{N^\epsilon}\frac{\rho^{-j}}{\Gamma\left(j+a^\epsilon\right)} + \mu\overline{K}\sum_{j=4}^{N^\epsilon}\frac{\left(1-\epsilon\left(j-1\right)\right)\left(1-\epsilon j\right)}{\left(j-1+a^\epsilon\right)\left(j-2+a^\epsilon\right)\left(1-\epsilon j\right)}$$
$$\leqslant B\sum_{j=2}^{\infty}\frac{\rho^{-j}}{\Gamma\left(j+a^\epsilon\right)} + \mu\overline{K}\sum_{j=4}^{\infty}\frac{1}{\left(j-1+a^\epsilon\right)\left(j-2+a^\epsilon\right)}. \tag{109}$$

The result now follows. □

Following lemma 5.3, we have that $\overline{y} = \overline{m}^\epsilon(\overline{x})$ (as a power series) is a fixed-point of the nonlinear operator $\mathcal{P}^\epsilon$ defined by

$$\mathcal{P}^\epsilon\left(\overline{y}\right) = \sum_{k=2}^{\infty}(-1)^k\overline{w}_k^\epsilon\sum_{j=2}^{k}\frac{(-1)^j\epsilon\overline{\mathcal{G}}^\epsilon\,[\overline{y}]_j}{\overline{w}_j^\epsilon(1-\epsilon j))}x^k. \tag{110}$$

By lemmas 5.8 and 5.9, we have that there is a $\mu_0 > 0$ such that for all $0 \leqslant \mu < \mu_0$ the following estimate holds:

$$\|\mathcal{P}^\epsilon\left(\overline{y}\right)\| \leqslant 2F \quad \forall \|\overline{y}\| \leqslant 2F,\, 0 < \epsilon \ll 1,$$

with respect to the semi-norm $\|\cdot\|$ (that only involves the finite sum) defined in (100). Then by proceeding as in remark 7 (using induction on k), we directly obtain the following:

Proposition 5.10. *There is a $\mu_0 > 0$, such that for all $0 \leqslant \mu < \mu_0$ the following holds true:*

1. *The analytic weak-stable manifold satisfies the following estimate*

$$\|\overline{m}^\epsilon\| \leqslant 2F,$$

for all $0 < \epsilon \ll 1$.

2. *The numbers*

$$\overline{S}_k^\epsilon := \sum_{j=2}^{k}\frac{(-1)^j\,\epsilon\overline{\mathcal{G}}^\epsilon\,[\overline{m}^\epsilon]_j}{\overline{w}_j^\epsilon\,(1-\epsilon j)}, \quad 2 \leqslant k \leqslant N^\epsilon,$$

are uniformly bounded with respect to $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$.

Lemma 5.11. *Let $0 \leqslant \mu < \mu_0$ with $\mu_0 > 0$ small enough so that proposition 5.10 applies and so that the series S_∞^0 from lemma 4.7 is well-defined and absolutely convergent. Then*

$$\overline{S}_{N^\epsilon}^\epsilon \to S_\infty^0 \quad \textit{for} \quad N^\epsilon \to \infty.$$

Proof. The proof is elementary, but since this result is crucial to the whole construction, we provide the full details:

For simplicity, we write

$$\overline{\mathcal{G}}^{\epsilon}\left[\overline{m}^{\epsilon}\right]_j =: \overline{\mathcal{G}}_j^{\epsilon}, \quad \mathcal{G}^0\left[\widehat{m}^0\right]_j =: \mathcal{G}_j^0,$$

in the following. By lemma 5.9, $|\overline{S}_{N^\epsilon}^{\epsilon}| \leqslant 2F$. Moreover, $S_\infty^0 = \sum_{j=2}^{\infty} \frac{(-1)^j \mathcal{G}_j^0}{w_j^0}$ is absolutely convergent, recall lemma 4.7.

For fixed j we have (recall item 1 of lemma 5.5)

$$\overline{w}_j^{\epsilon} = \Gamma\left(j+a^0\right)\epsilon^{j-1}\left(1+o\left(1\right)\right) = w_j^0 \epsilon^{j-1}\left(1+o\left(1\right)\right).$$

Moreover, by lemma 5.4 we have

$$\epsilon^{1-j}\epsilon\overline{\mathcal{G}}_j^{\epsilon} \to \mathcal{G}_j^0,$$

and therefore

$$\frac{\left(-1\right)^j \epsilon\overline{\mathcal{G}^{\epsilon}}_j}{\overline{w}_j^{\epsilon}\left(1-\epsilon j\right)} \to \frac{\left(-1\right)^j \mathcal{G}_j^0}{\Gamma\left(j+a^0\right)}, \tag{111}$$

as $\epsilon \to 0$ (fixed j).

Next, we estimate

$$\begin{aligned}
\left|\sum_{j=2}^{N^\epsilon} \frac{\left(-1\right)^j \epsilon\overline{\mathcal{G}^{\epsilon}}_j}{\overline{w}_j^{\epsilon}\left(1-\epsilon j\right)} - \sum_{j=2}^{\infty} \frac{\left(-1\right)^j \mathcal{G}_j^0}{w_j^0}\right| &\leqslant \left|\sum_{j=2}^{J}\left(\frac{\left(-1\right)^j \epsilon\overline{\mathcal{G}_j^{\epsilon}}}{\overline{w}_j^{\epsilon}\left(1-\epsilon j\right)} - \frac{\left(-1\right)^j \mathcal{G}_j^0}{w_j^0}\right)\right| \\
&\quad + \left|\sum_{j=J+1}^{N^\epsilon} \frac{\left(-1\right)^j \epsilon\overline{\mathcal{G}^{\epsilon}}_j}{\overline{w}_j^{\epsilon}\left(1-\epsilon j\right)}\right| + \left|\sum_{j=J+1}^{\infty} \frac{\left(-1\right)^j \mathcal{G}_j^0}{w_j^0}\right| \\
&\leqslant \sum_{j=2}^{J}\left|\frac{\left(-1\right)^j \epsilon\overline{\mathcal{G}^{\epsilon}}_j}{\overline{w}_j^{\epsilon}\left(1-\epsilon j\right)} - \frac{\left(-1\right)^j \mathcal{G}_j^0}{w_j^0}\right| \\
&\quad + \sum_{j=J+1}^{\infty}\left(\frac{B\rho^{-j}}{\Gamma\left(j+a^{\epsilon}\right)} + \frac{\mu\overline{K}}{\left(j-1+a^{\epsilon}\right)\left(j-2+a^{\epsilon}\right)}\right) \\
&\quad + \sum_{j=J+1}^{\infty}\left(\frac{B\rho^{-j}}{\Gamma\left(j+a^{0}\right)} + \frac{\mu K}{\left(j-1+a^{0}\right)\left(j-2+a^{0}\right)}\right),
\end{aligned} \tag{112}$$

for any $2 \leqslant J \leqslant N^\epsilon$, using $w_j^0 = \Gamma(j+a^0)$, lemma 5.8 (see also (109)) and proposition 4.2 (see also (69)). Consequently, we have

$$\left|\sum_{j=2}^{N^\epsilon} \frac{(-1)^j \epsilon \overline{\mathcal{G}^\epsilon}_j}{\overline{w}_j^\epsilon (1-\epsilon j)} - \sum_{j=2}^{\infty} \frac{(-1)^j \mathcal{G}_j^0}{w_j^0}\right| \leqslant \sum_{j=2}^{J} \left|\frac{(-1)^j \epsilon \overline{\mathcal{G}^\epsilon}_j}{\overline{w}_j^\epsilon (1-\epsilon j)} - \frac{(-1)^j \mathcal{G}_j^0}{w_j^0}\right|$$
$$+ 2\mu\left(\overline{K}+K\right) \sum_{j=J+1}^{\infty} \frac{1}{(j-2+a^0)(j-1+a^0)} \tag{113}$$
$$+ 3B \sum_{j=J+1}^{\infty} \frac{\rho^{-j}}{\Gamma(j+a^0)},$$

for all $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$. Now, for any $\upsilon > 0$, we take $J \gg 1$ (independent of $\epsilon > 0$) so that each of the last two convergent series on the right hand side of (113) are less than $\upsilon/3$. Subsequently, we then take $\epsilon > 0$ small enough so that the first term on the right hand side of (113) (using (111)) is less than $\upsilon/3$. In total, we have

$$\left|\sum_{j=2}^{N^\epsilon} \frac{(-1)^j \epsilon \overline{\mathcal{G}^\epsilon}_j}{\overline{w}_j^\epsilon (1-\epsilon j)} - \sum_{j=2}^{\infty} \frac{(-1)^j \mathcal{G}_j^0}{w_j^0}\right| \leqslant \upsilon,$$

and the result follows. □

5.2. Estimating the finite sum

Let $j^n[H]$ denote the nth-order Taylor jet/partial sum of $H(\overline{x}) = \sum_{k=2}^{\infty} H_k \overline{x}^k$:

$$j^n[H] := \sum_{k=2}^{n} H_k (\cdot)^k \quad \forall n \in \mathbb{N}. \tag{114}$$

Moreover, we define the nth-order remainder by

$$r^n[H] = (I - j^n)[H] := \sum_{k=n+1}^{\infty} H_k (\cdot)^k \quad \forall n \in \mathbb{N}. \tag{115}$$

Lemma 5.12. *Consider the partial sum*

$$j^{N^\epsilon - 1}[\overline{m}^\epsilon](\overline{x}) = \sum_{k=2}^{N^\epsilon - 1} \overline{m}_k^\epsilon \overline{x}^k,$$

of the series $\overline{m}^\epsilon(\overline{x}) = \sum_{k=2}^{\infty} \overline{m}_k^\epsilon \overline{x}^k$. *Then there is a constant* $C > 0$ *such that*

$$|j^{N^\epsilon - 1}[\overline{m}^\epsilon](\overline{x})| \leqslant C\epsilon \quad \forall \overline{x} \in \left[-\frac{3}{4}, \frac{3}{4}\right], \tag{116}$$

for all $0 < \epsilon \ll 1$.

Proof. The estimate (116) follows from item 4 of lemma 5.5 with $\delta = \frac{3}{4}$. □

Lemma 5.13. *For any* $\overline{D} > 0$*, we consider*

$$\overline{g}^\epsilon\left(\overline{x}, j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x}) + q\right) \quad \forall \overline{x} \in \left[-\frac{3}{4}, \frac{3}{4}\right], q \in \left(-\overline{D}, \overline{D}\right). \tag{117}$$

It is well-defined for all $0 < \epsilon \ll 1$ *and has the following absolutely convergent power series expansion*

$$\overline{g}^\epsilon\left(\overline{x}, j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x}) + q\right) = \overline{g}_0^\epsilon(\overline{x}) + \overline{x}^2\overline{g}_1^\epsilon(\overline{x})\, q + \overline{x}\sum_{l=2}^\infty \overline{g}_l^\epsilon(\overline{x})\, q^l, \tag{118}$$

with

$$\overline{g}_0^\epsilon(\overline{x}) = \sum_{k=2}^\infty \overline{\mathcal{G}}^\epsilon\left[j^{N^\epsilon-1}[\overline{m}^\epsilon]\right]_k \overline{x}^k.$$

Moreover, we have the following estimates (Q_4^ϵ *is defined in* (89))*:*

$$|\overline{\mathcal{G}}^\epsilon\left[j^{N^\epsilon-1}[\overline{m}^\epsilon]\right]_k| \leqslant C\left(\overline{w}_k^\epsilon\right)^2 e^{Q_4^\epsilon(k-4)} \quad \forall k \geqslant N^\epsilon + 1; \tag{119}$$

specifically, for $k = N^\epsilon + 1$*:*

$$|\overline{\mathcal{G}}^\epsilon\left[j^{N^\epsilon-1}[\overline{m}^\epsilon]\right]_{N^\epsilon+1}| \leqslant C\overline{w}_2^\epsilon \overline{w}_{N^\epsilon-1}^\epsilon, \tag{120}$$

and

$$|\overline{g}_l^\epsilon(\overline{x})| \leqslant \mu C\overline{D}^{-l+1} \quad \forall l \geqslant 1, \tag{121}$$

for all $0 < \epsilon \ll 1$, $\overline{x} \in [-\frac{3}{4}, \frac{3}{4}]$*. Here* $C > 0$ *is some constant that is independent of* $\overline{D}$ *and* ϵ*.*

Proof. The expansion of (117) follows from composition of analytic functions. For the property of the convergence radius in (121), we use the binomial theorem to obtain

$$\overline{g}_l^\epsilon(\overline{x}) = \mu\sum_{n=l}^\infty \left(\sum_{m=1}^\infty h_{m,n}^\epsilon \epsilon^{m-1}\overline{x}^m\right)\epsilon^{n-1}\binom{n}{l}\left(j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x})\right)^{n-l}, \quad l \geqslant 2, \tag{122}$$

cf (77) and (78), and use (116), (108) and (26). This gives

$$|\overline{g}_l^\epsilon(\overline{x})| \leqslant \frac{3}{2}\mu\rho^{-1}\sum_{n=l}^\infty \epsilon^{n-1}\rho^{-n}2^n(C\epsilon)^{n-l} \leqslant 3\mu\rho^{-1}\left(2\rho^{-1}\right)^l\epsilon^{l-1} \leqslant 6\mu\rho^{-2}\overline{D}^{-l+1},$$

for all $\overline{x} \in [-\frac{3}{4}, \frac{3}{4}]$, $\overline{D} < (2\rho^{-1}\epsilon)^{-1}$ and $0 < \epsilon \ll 1$, upon estimating the geometric sums.

Next, we notice that (120) follows from (119) upon using (97). We therefore turn to proving (119). For this purpose, we use (80) and focus on estimating

$$\overline{\mathcal{H}}^\epsilon\left[j^{N^\epsilon-1}[\overline{m}^\epsilon]\right]_k.$$

By (81), (26), $\|\overline{m}^\epsilon\| \leqslant 2F$ in the seminorm (100) (cf proposition 5.10) and lemma 5.7, we obtain that

$$
\begin{aligned}
|\overline{\mathcal{H}}^\epsilon\left[j^{N^\epsilon-1}\left[\overline{m}^\epsilon\right]\right]_k| \leqslant 2F &\sum_{j=2}^{\min(k-2,N^\epsilon-1)} \rho^{-k+j-1}\epsilon^{k-j-1}\overline{w}_j^\epsilon \\
&+\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\sum_{j=2l}^{\min(k,l(N^\epsilon-1))} \rho^{-k+j-l}\epsilon^{k-j+l-2}(2F)^l C^{l-1}\mathrm{e}^{(-2Q_4^\epsilon+P_4^\epsilon)l}\mathrm{e}^{Q_4^\epsilon j} \\
&\leqslant 2F\rho^{-3}\epsilon\left(\rho^{-1}\epsilon\right)^{k-(N^\epsilon+1)}\sum_{j=2}^{N^\epsilon-1}\left(\rho^{-1}\epsilon\right)^{N^\epsilon-1-j}\overline{w}_j^\epsilon \\
&+\mathrm{e}^{Q_4^\epsilon k}\epsilon^{-2}C^{-1}\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\left(2\rho^{-1}\epsilon FC\mathrm{e}^{-2Q_4^\epsilon+P_4^\epsilon}\right)^l\sum_{j=2l}^{\min(k,l(N^\epsilon-1))}\left(\rho^{-1}\epsilon\mathrm{e}^{-Q_4^\epsilon}\right)^{k-j}.
\end{aligned}
$$

Here

$$
0<\rho^{-1}\epsilon\mathrm{e}^{-Q_4^\epsilon}\ll 1,\quad 0<2\rho^{-1}\epsilon FC\mathrm{e}^{-2Q_4^\epsilon+P_4^\epsilon}\ll 1,
$$

for all $0<\epsilon\ll 1$, recall (90). But then, by estimating the geometric series and using $\exp(P_4^\epsilon)=\overline{w}_2^\epsilon$ (see (97)), we conclude that

$$
|\overline{\mathcal{H}}^\epsilon\left[j^{N^\epsilon-1}\left[\overline{m}^\epsilon\right]\right]_k| \leqslant \overline{C}\mathrm{e}^{Q_4^\epsilon k}\mathrm{e}^{-4Q_4^\epsilon+2P_4^\epsilon}=\overline{C}\left(\overline{w}_2^\epsilon\right)^2\mathrm{e}^{Q_4^\epsilon(k-4)},
$$

for some $\overline{C}>0$ large enough. This gives the desired estimates (upon $\overline{C}\to C$). □

We now turn to estimating $j^{N^\epsilon}[\overline{m}^\epsilon]$; in contrast to $j^{N^\epsilon-1}[\overline{m}^\epsilon]$, it is not uniformly bounded with respect to $\alpha^\epsilon\in(0,1)$.

Lemma 5.14. *Suppose that* $S_\infty^0\neq 0$. *Then*

$$
|\overline{m}_{N^\epsilon}^\epsilon\overline{x}^{N^\epsilon}|\leqslant(1+o(1))\,|S_\infty^0|\overline{w}_{N^\epsilon}^\epsilon\delta^{N^\epsilon}\quad\forall\overline{x}\in[-\delta,\delta],\tag{123}
$$

for all $0<\epsilon\ll 1$. *Moreover, fix any* $K>0$ *and suppose for* $N^\epsilon\gg 1$ *and* $\alpha^\epsilon\in(0,1)$ *that*

$$
\delta\leqslant\min\left(\frac{3}{4},\left(\frac{K}{2|S_\infty^0||\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}}\right)^{\frac{1}{N^\epsilon}}\right).\tag{124}
$$

(For fixed α^ϵ*, the expression on the right hand side of (124) converges to* $\frac{3}{4}$ *for* $N^\epsilon\to\infty$*). Then*

$$
|j^{N^\epsilon}\left[\overline{m}^\epsilon\right](\overline{x})|\leqslant K\quad\forall\overline{x}\in[-\delta,\delta].\tag{125}
$$

Proof. We estimate

$$
\begin{aligned}
|\overline{m}_{N^\epsilon}^\epsilon\overline{x}^{N^\epsilon}| &\leqslant|\overline{S}_{N^\epsilon}^\epsilon|\overline{w}_{N^\epsilon}^\epsilon\delta^{N^\epsilon} \\
&=(1+o(1))\,|S_\infty^0|\frac{\Gamma(\alpha^\epsilon)\Gamma(N^\epsilon+a^\epsilon)}{\Gamma(\epsilon^{-1}-1)}\delta^{N^\epsilon} \\
&=(1+o(1))\,|S_\infty^0|\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}\delta^{N^\epsilon},
\end{aligned}\tag{126}
$$

using (84), (170), $S_\infty^0 \neq 0$ and Stirling's approximation (in the form (173)) on the factor

$$\frac{\Gamma(N^\epsilon + a^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})} = \frac{\Gamma(N^\epsilon + a^\epsilon)}{(1-\epsilon)\Gamma(\epsilon^{-1}-1)} = (1+o(1))\frac{\Gamma(N^\epsilon)(N^\epsilon)^{a^\epsilon}}{\Gamma(N^\epsilon)(N^\epsilon)^{\alpha^\epsilon - 1}} = (1+o(1))(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}, \tag{127}$$

for $N^\epsilon \to \infty$; in particular the $o(1)$-terms in (126) are uniform with respect to α^ϵ. Using (124), we have

$$(1+o(1))|S_\infty^0|\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}\delta^{N^\epsilon} \leqslant \frac{1}{2}K(1+o(1)).$$

The result then follows from $j^{N^\epsilon}[\overline{m}^\epsilon](\overline{x}) = j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x}) + \overline{m}^\epsilon_{N^\epsilon}\overline{x}^{N^\epsilon}$. □

If we take $\overline{D} > C > 0$ and $0 < \epsilon \ll 1$, then it follows from lemma 5.12 (upon setting $q = \overline{m}^\epsilon_{N^\epsilon}\overline{x}^{N^\epsilon}$) that

$$\overline{g}^\epsilon\left(x, j^{N^\epsilon}[\overline{m}^\epsilon](\overline{x})\right) = \overline{g}^\epsilon\left(x, j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x}) + \overline{m}^\epsilon_{N^\epsilon}\overline{x}^{N^\epsilon}\right),$$

is well-defined for all $\overline{x} \in [\delta, \delta]$ with $\delta > 0$ satisfying (124).

5.3. The operator $\mathcal{T}^\epsilon$

Define $H \mapsto \mathcal{T}^\epsilon[H]$ by

$$\begin{aligned}\mathcal{T}^\epsilon[H](\overline{x}) &:= \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}\int_0^{\overline{x}} \frac{(1-v)^{\epsilon^{-1}+a^\epsilon-1}}{v^{\epsilon^{-1}+1}} H(v)\,\mathrm{d}v \\ &:= \frac{|\overline{x}|^{\alpha^\epsilon}\overline{x}^{N^\epsilon}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}\int_0^{\overline{x}} \frac{(1-v)^{\epsilon^{-1}+a^\epsilon-1}}{|v|^{\alpha^\epsilon}v^{N^\epsilon+1}} H(v)\,\mathrm{d}v \quad \forall -1 < \overline{x} < 1.\end{aligned} \tag{128}$$

It is well-defined on analytic functions H with $j^{N^\epsilon}[H] = 0$, see also [16, section 7].

Lemma 5.15. *Suppose that $\epsilon^{-1} \notin \mathbb{N}$. Then the following statements hold true:*

1. *For any analytic H with $j^{N^\epsilon}[H] = 0$, $G = \mathcal{T}^\epsilon[H]$ is the unique solution of*

$$\epsilon\overline{x}(1-\overline{x})\frac{\mathrm{d}G}{\mathrm{d}\overline{x}} - (1+\epsilon a^\epsilon\overline{x})G = \epsilon H \quad \text{and} \quad j^{N^\epsilon}[G] = 0. \tag{129}$$

2. *$\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]$ has an absolutely convergent power series representation for $-1 < \overline{x} < 1$:*

$$\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) = \frac{\Gamma(1-\alpha^\epsilon)}{\Gamma(N^\epsilon+1+a^\epsilon)}\sum_{k=N^\epsilon+1}^\infty \frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\overline{x}^k. \tag{130}$$

Proof. To prove item 1, we define $\mathcal{J}(\overline{x}) := \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}$ and subsequently $\mathcal{I}(\overline{x}) := \int_0^{\overline{x}} \frac{1}{\mathcal{J}(v)v(1-v)}H(v)\mathrm{d}v$. Then

$$\mathcal{T}^\epsilon[H] = \mathcal{J}\mathcal{I} \quad \text{and} \quad \mathcal{J}(\overline{x})I'(\overline{x}) = \frac{1}{\overline{x}(1-\overline{x})}H(\overline{x}).$$

Moreover,

$$\begin{aligned}\mathcal{J}'(\overline{x}) &= \mathcal{J}(\overline{x})\left(\epsilon^{-1}\overline{x}^{-1} + \left(\epsilon^{-1} + a^{\epsilon}\right)(1-\overline{x})^{-1}\right)\\ &= \mathcal{J}(\overline{x})\frac{1+\epsilon a^{\epsilon}\overline{x}}{\epsilon\overline{x}(1-\overline{x})},\end{aligned}$$

and therefore

$$\epsilon\overline{x}(1-\overline{x})\mathcal{T}^{\epsilon}[H]'(\overline{x}) = (1+\epsilon a^{\epsilon}\overline{x})\mathcal{J}(\overline{x})\mathcal{I}(\overline{x}) + \epsilon H(\overline{x}).$$

Consequently,

$$\epsilon\overline{x}(1-\overline{x})\mathcal{T}^{\epsilon}[H]'(\overline{x}) - (1+\epsilon a^{\epsilon}\overline{x})\mathcal{T}^{\epsilon}[H](\overline{x}) = \epsilon H(\overline{x}),$$

as desired.

Next, to prove item 2, we use item 1 and the fact that the solution is unique. Then lemma 5.3 with

$$\epsilon\overline{\mathcal{G}}^{\epsilon}[\overline{m}^{\epsilon}]_k = \begin{cases}-\epsilon & \text{for } k = N^{\epsilon}+1,\\ 0 & \text{else,}\end{cases}$$

allow us to write $\mathcal{T}^{\epsilon}\left[(\cdot)^{N^{\epsilon}+1}\right](\overline{x}) = \sum_{k=N^{\epsilon}+1}^{\infty}\overline{m}_k^{\epsilon}\overline{x}^k$ as an absolutely convergent power series; notice the change of sign when comparing (129) and (76). In particular, we find that

$$\overline{S}_k^{\epsilon} = \frac{(-1)^{N^{\epsilon}+1}}{\overline{w}_{N^{\epsilon}+1}^{\epsilon}(1-\alpha^{\epsilon})} \quad \forall k \geqslant N^{\epsilon}+1 \,(\text{zero otherwise}),$$

and therefore

$$\overline{m}_k^{\epsilon} = \frac{1}{1-\alpha^{\epsilon}}\frac{(-1)^k\overline{w}_k^{\epsilon}}{(-1)^{N^{\epsilon}+1}\overline{w}_{N^{\epsilon}+1}^{\epsilon}} \quad \forall k \geqslant N^{\epsilon}+1 \,(\text{zero otherwise}),$$

by (84). Subsequently, we then use item 5 of lemma 5.5 to write

$$\begin{aligned}\frac{(-1)^k\overline{w}_k^{\epsilon}}{(-1)^{N^{\epsilon}+1}\overline{w}_{N^{\epsilon}+1}^{\epsilon}} &= \frac{\Gamma(2-\alpha^{\epsilon})}{\Gamma(N^{\epsilon}+1+a^{\epsilon})}\frac{\Gamma(k+a^{\epsilon})}{\Gamma(k+1-\epsilon^{-1})}\\ &= \frac{(1-\alpha^{\epsilon})\Gamma(1-\alpha^{\epsilon})}{\Gamma(N^{\epsilon}+1+a^{\epsilon})}\frac{\Gamma(k+a^{\epsilon})}{\Gamma(k+1-\epsilon^{-1})}.\end{aligned}$$

This gives the desired expression for $\mathcal{T}^{\epsilon}\left[(\cdot)^{N+1}\right]$ in item 2. □

We will view $\mathcal{T}^{\epsilon}$ on the Banach space $\mathcal{D}_{\delta}^{\epsilon}$ of analytic functions $H:[0,\delta]\to\mathbb{R}$ with $|H(\overline{x})\overline{x}^{-N^{\epsilon}-1}|$ bounded at $\overline{x}=0$. More specifically, we define

$$\mathcal{D}_{\delta}^{\epsilon} := \{H:[0,\delta]\to\mathbb{R} \text{ analytic} : \|\|H\|\|_{\delta} < \infty\},$$

with the Banach norm

$$\|\|H\|\|_{\delta} := \sup_{\overline{x}\in(0,\delta]}|H(\overline{x})|\frac{\mathcal{T}\left[(\cdot)^{N^{\epsilon}+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^{\epsilon}+1}\right](\overline{x})}; \tag{131}$$

here we have used that $\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x}) = \mathcal{O}(\bar{x}^{N^\epsilon+1})$ as $\bar{x}\to 0$ and that $\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x}) > 0$ for all $\bar{x}\in(0,1)$, cf lemma 5.15 item 2.

The case $\bar{x} < 0$ has to be treated slightly different (we will have to take $0 \leqslant -\bar{x} \leqslant \delta_2\epsilon$); we will consider this case at the end of section 5.6 below.

A nice property of the Banach norm (131) is highlighted in the following Lemma.

Lemma 5.16. *Define*

$$\|H\|_\delta := \sup_{\bar{x}\in[0,\delta]} |H(\bar{x})|. \tag{132}$$

Then the following estimate holds:

$$\|H\|_\delta \leqslant |||H|||_\delta \quad \forall H\in\mathcal{D}_\delta^\epsilon.$$

Proof. The proof is elementary. Indeed, for any $\bar{x}\in(0,\delta]$ we have

$$|H(\bar{x})| \leqslant \left(|H(\bar{x})| \frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})}\right) \times \frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)} \leqslant |||H|||_\delta \times \frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}, \tag{133}$$

and therefore

$$|H(\bar{x})| \leqslant |||H|||_\delta \times 1, \tag{134}$$

since $\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})$ is an increasing function of $\bar{x}\in[0,1)$. As $H(0)=0$ the inequality (134) holds for all $\bar{x}\in[0,\delta]$, completing the proof. □

Notice also the (obvious) fact that

$$|||H|||_{\delta'} \leqslant |||H|||_\delta,$$

for any $0<\delta'<\delta$. This also holds with $|||$ replaced by $\|$, recall (132). We will use these properties without further mention in the following. The following set of equalities

$$\|\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]\|_\delta = |||\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]|||_\delta = \mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta) \quad \forall 0<\delta<1, \tag{135}$$

are consequences of $\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})$ being an increasing function of $\bar{x}\in[0,1)$, and they will also be important.

It turns out that

$$0<\delta\leqslant\frac{3}{4}, \tag{136}$$

will be adequate for our purposes.

In the following (see e.g. item 7), we will use $(\cdot,\overline{Y})$ to denote the composition $\bar{x}\mapsto(\bar{x},\overline{Y}(\bar{x}))$ for given analytic functions $\overline{Y}:\bar{x}\mapsto\overline{Y}(\bar{x})$.

Lemma 5.17. *Fix any $\delta_2>0$, suppose that $\epsilon^{-1}\notin\mathbb{N}$ and that (136) holds. Then there exists a $K_2=K_2(\delta_2,a^0)$ such that the following holds true.*

1. *Define $\sigma_\epsilon:[-\delta_2\epsilon,\frac{3}{4}]\to\mathbb{R}_+$ by*

$$\sigma_\epsilon(\bar{x}):=\begin{cases}1 & \forall 0\leqslant|\bar{x}|\leqslant\delta_2\epsilon\\ \left(\bar{x}^{-1}\delta_2\epsilon\right)^{1-\alpha^\epsilon} & \forall\delta_2\epsilon<\bar{x}\leqslant\frac{3}{4},\end{cases}\tag{137}$$

so that

$$\left(\frac{4}{3}\delta_2\epsilon\right)^{1-\alpha^\epsilon}\leqslant\sigma_\epsilon(\bar{x})\leqslant 1\quad\forall\bar{x}\in\left[-\delta_2\epsilon,\frac{3}{4}\right].\tag{138}$$

Then there are constants $0<C_1<C_2$, $C_i=C_i(\delta_2,a^0)$, such that the following holds for all $0<\epsilon\ll 1$, $\epsilon^{-1}\notin\mathbb{N}$:

$$C_1\left(\frac{|\bar{x}|}{1-\bar{x}}\right)^{N^\epsilon+1}\sigma_\epsilon(\bar{x})\leqslant(1-\alpha^\epsilon)\left|\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})\right|\leqslant C_2\left(\frac{|\bar{x}|}{1-\bar{x}}\right)^{N^\epsilon+1}\sigma_\epsilon(\bar{x}),\tag{139}$$

for all $-\delta_2\epsilon\leqslant\bar{x}\leqslant\frac{3}{4}$.

2. *Asymptotics for $\bar{x}=\mathcal{O}(\epsilon)$: For any $\bar{x}=\epsilon\bar{x}_2$, $\bar{x}_2\in[-\delta_2,\delta_2]$, the following asymptotics hold true*

$$\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\epsilon\bar{x}_2)=\frac{1}{1-\alpha^\epsilon}(\epsilon\bar{x}_2)^{N^\epsilon+1}\left[1+\bar{x}_2\int_0^1 \mathrm{e}^{(1-v)\bar{x}_2}v^{1-\alpha^\epsilon}\mathrm{d}v+\mathcal{O}(\epsilon)\right]\quad\forall\bar{x}_2\in[-\delta_2,\delta_2],$$

with $\mathcal{O}(\epsilon)$ being uniform with respect to $\alpha^\epsilon\in(0,1)$.

3. *$\mathcal{T}^\epsilon:\mathcal{D}_\delta^\epsilon\to\mathcal{D}_\delta^\epsilon$ is a bounded operator. In particular, let*

$$|||\mathcal{T}^\epsilon|||_\delta:=\sup_{|||H|||_\delta=1}|||\mathcal{T}^\epsilon[H]|||_\delta,$$

denote the operator norm. Then

$$|||\mathcal{T}^\epsilon|||_\delta\leqslant\frac{K_2}{1-\alpha^\epsilon}\left(1+\log\sigma_\epsilon(\delta)^{-1}\right).$$

4. *The following holds for any $i\in\mathbb{N}$:*

$$|||\mathcal{T}^\epsilon\left[(\cdot)^iH\right)]|||_\delta\leqslant\frac{K_2\delta^i}{i}|||H|||_\delta\quad\forall H\in\mathcal{D}_\delta^\epsilon.\tag{140}$$

5. *The following holds for any $l\in\mathbb{N}$:*

$$|||\mathcal{T}^\epsilon\left[(\cdot)^{lN^\epsilon+1}\right]|||_\delta\leqslant\delta^{(l-1)N^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta).\tag{141}$$

6. *Suppose that $E,R>0$, $0<\delta<R$ and consider*

$$H(\bar{x})=\sum_{k=N^\epsilon+1}^{\infty}H_k\bar{x}^k\quad\forall\bar{x}\in[0,\delta],$$

with $|H_k| \leqslant ER^k$. *Then*

$$|||\mathcal{T}^\epsilon[H]|||_\delta \leqslant \frac{ER^{N^\epsilon+1}}{1-R\delta}\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta),$$

for all $0<\epsilon\ll 1$.

7. *Let* $C := C_2C_1^{-1} > 1$, *with* $C_i > 0$ *defined in (139),* $E, R > 0$, *and suppose that* $\overline{Y} \in \mathcal{D}_\delta^\epsilon$ *and*

$$H(\bar{x},\overline{Y}) = \sum_{l=2}^{\infty} H_l(\bar{x})\overline{Y}^l \quad \forall \bar{x}\in[0,\delta], 0\leqslant |||\overline{Y}|||_\delta < R^{-1}, \tag{142}$$

with

$$\|H_l\|_\delta \leqslant ER^{l-1} \quad \forall l\geqslant 2,$$

recall (132). Then

$$|||\mathcal{T}^\epsilon\left[H(\cdot,\overline{Y})\right]|||_\delta \leqslant 4\epsilon EK_2CR|||\overline{Y}|||_\delta^2 \quad \forall 0\leqslant |||\overline{Y}|||_\delta < \frac{1}{2}(CR)^{-1},$$

uniformly in $\alpha^\epsilon\in(0,1)$. *In particular,* $\overline{Y}\mapsto \mathcal{T}^\epsilon\left[h(\cdot,\overline{Y})\right]$ *is* C^1 *and for all* $0<\epsilon\ll 1$, *it is a contraction:*

$$|||D_{\overline{Y}}(\mathcal{T}^\epsilon\left[H(\cdot,\overline{Y})\right](\overline{Z})|||_\delta \leqslant \mathcal{O}(\epsilon)|||\overline{Z}|||_\delta \quad \forall \overline{Z}\in\mathcal{D}_\delta^\epsilon, |||\overline{Y}|||_\delta < \frac{1}{2}(CR)^{-1}. \tag{143}$$

Proof. We prove the items 1–7 successively in the following.

Proof of item 1. The result follows from [16, lemma 7.2], see [16, equation (7.10)], and it is based on the integral representation for $\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]$:

$$\begin{aligned}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\bar{x}) &= \frac{\bar{x}^{\epsilon^{-1}}}{(1-\bar{x})^{\epsilon^{-1}+a^\epsilon}}\int_0^{\bar{x}}(1-v)^{\epsilon^{-1}+a^\epsilon-1}v^{-\alpha^\epsilon}\mathrm{d}v\\ &= \frac{|\bar{x}|^{\alpha^\epsilon}\bar{x}^{N^\epsilon}}{(1-\bar{x})^{\epsilon^{-1}+a^\epsilon}}\int_0^{\bar{x}}(1-v)^{\epsilon^{-1}+a^\epsilon-1}|v|^{-\alpha^\epsilon}\mathrm{d}v\end{aligned} \tag{144}$$

For completeness, we include the details (which will also be important later): Firstly, for $\bar{x} = \epsilon\bar{x}_2\in[-\epsilon\delta_2,\epsilon\delta_2]$, $\delta_2>0$ fixed, we use:

$$(1-\bar{x})^{\epsilon^{-1}} = \mathrm{e}^{\epsilon^{-1}\log(1-\bar{x})} = \mathrm{e}^{-\bar{x}_2}\left(1+\mathcal{O}\left(\epsilon\bar{x}_2^2\right)\right) = \mathcal{O}(1) \gtrless \begin{cases}\mathrm{e}^{-2\delta_2}\\ \mathrm{e}^{2\delta_2}\end{cases} \tag{145}$$

for all $0<\epsilon\ll 1$. Consequently, for $\bar{x}\in[-\epsilon\delta_2,\epsilon\delta_2]$

$$\begin{aligned}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\bar{x}) &= \left(\frac{\bar{x}}{1-\bar{x}}\right)^{N^\epsilon+1}(1+o(1))|\bar{x}|^{\alpha^\epsilon}\bar{x}^{-1}\int_0^{\bar{x}}\mathcal{O}(1)|v|^{-\alpha^\epsilon}\mathrm{d}v\\ &= \left(\frac{\bar{x}}{1-\bar{x}}\right)^{N^\epsilon+1}(1+o(1))\frac{\mathcal{O}(1)}{1-\alpha^\epsilon},\end{aligned}$$

where $0 < C_1 < \mathcal{O}(1) < C_2$. Next, for $\delta_2\epsilon < \overline{x} \leqslant \frac{3}{4}$, we use a separate set of estimates:

$$\begin{aligned}\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})| &\leqslant \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}\int_0^1 (1-v)^{\epsilon^{-1}+a^\epsilon-1}v^{-\alpha^\epsilon}\mathrm{d}v\\ &= \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}\frac{\Gamma\left(\epsilon^{-1}+a^\epsilon\right)\Gamma\left(1-\alpha^\epsilon\right)}{\Gamma\left(\epsilon^{-1}+a^\epsilon+1-\alpha^\epsilon\right)}\\ &= \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}(1+o(1))\Gamma(1-\alpha^\epsilon)\epsilon^{1-\alpha^\epsilon}\\ &\leqslant C_2\left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^\epsilon+1}\left(\overline{x}^{-1}\delta_2\epsilon\right)^{1-\alpha^\epsilon}(1-\alpha^\epsilon)^{-1},\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})| &\geqslant \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}\int_0^{\delta_2\epsilon} (1-v)^{\epsilon^{-1}+a^\epsilon-1}v^{-\alpha^\epsilon}\mathrm{d}v\\ &\geqslant C_1\left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^\epsilon+1}\frac{\left(\overline{x}^{-1}\delta_2\epsilon\right)^{1-\alpha^\epsilon}}{1-\alpha^\epsilon}.\end{aligned}$$

for some $C_1 = C_1(\delta_2, a^0)$ small enough, cf (145). Here we have also used (175), (173) and (171).

Proof of item 2. For item 2, we use (144) with the substitution $v = \overline{x}\widetilde{v}$ and (145) with $\overline{x} = \epsilon\overline{x}_2 \in [-\epsilon\delta_2, \epsilon\delta_2]$. This gives

$$\begin{aligned}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) &= \overline{x}^{N^\epsilon+1}\mathrm{e}^{\overline{x}_2}(1+\mathcal{O}(\epsilon))\int_0^1 \mathrm{e}^{-(1+\epsilon(a^\epsilon-1))\widetilde{v}\overline{x}_2}\left(1+\mathcal{O}\left(\epsilon\widetilde{v}^2\right)\right)\widetilde{v}^{-\alpha^\epsilon}\mathrm{d}\widetilde{v}\\ &= \overline{x}^{N^\epsilon+1}\mathrm{e}^{\overline{x}_2}(1+\mathcal{O}(\epsilon))\left(\int_0^1 \mathrm{e}^{-(1+\epsilon(a^\epsilon-1))v\overline{x}_2}v^{-\alpha^\epsilon}\mathrm{d}v+\mathcal{O}(\epsilon)\right),\end{aligned}$$

with each $\mathcal{O}(\epsilon)$ uniform with respect to α^ϵ. We now use integration by parts on the remaining integral:

$$\begin{aligned}\int_0^1 \mathrm{e}^{-(1+\epsilon(a^\epsilon-1))t\overline{x}_2}v^{-\alpha^\epsilon}\mathrm{d}v &= \frac{\mathrm{e}^{-(1+\epsilon(a^\epsilon-1))\overline{x}_2}}{1-\alpha^\epsilon}\Bigg[1+(1+\epsilon(a^\epsilon-1))\overline{x}_2\\ &\qquad\times\int_0^1 \mathrm{e}^{(1+\epsilon(a^\epsilon-1))(1-v)\overline{x}_2}v^{1-\alpha^\epsilon}\mathrm{d}v\Bigg]\\ &= \frac{\mathrm{e}^{-\overline{x}_2}}{1-\alpha^\epsilon}\left[1+\overline{x}_2\int_0^1 \mathrm{e}^{(1-v)\overline{x}_2}v^{1-\alpha^\epsilon}\mathrm{d}v+\mathcal{O}(\epsilon)\right].\end{aligned}$$

This completes the proof.

Proof of item 3. We estimate using (128), (133) and (139)

$$
\begin{aligned}
\left|\mathcal{T}^\epsilon[H](\bar{x})\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})}\right| &\leqslant \frac{\bar{x}^{\epsilon^{-1}}}{(1-\bar{x})^{\epsilon^{-1}+a^\epsilon}}\frac{1}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})}\\
&\quad\times\int_0^{\bar{x}}\frac{(1-v)^{\epsilon^{-1}+a^\epsilon-1}}{v^{\epsilon^{-1}+1}}\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](v)\,\mathrm{d}v\|\!|H|\!\|_\delta\\
&\leqslant C_2C_1^{-1}\frac{\bar{x}^{\epsilon^{-1}}}{(1-\bar{x})^{\epsilon^{-1}+a^\epsilon}}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{N^\epsilon+1}\sigma_\epsilon(\bar{x})^{-1}\\
&\quad\times\int_0^{\bar{x}}\frac{(1-v)^{\epsilon^{-1}+a^\epsilon-1}}{v^{\epsilon^{-1}+1}}\left(\frac{v}{1-v}\right)^{N^\epsilon+1}\sigma_\epsilon(v)\,\mathrm{d}v\|\!|H|\!\|_\delta\\
&\leqslant C_2C_1^{-1}(1-\bar{x})^{1-\alpha^\epsilon-a^\epsilon}\frac{\bar{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\bar{x})}\\
&\quad\times\int_0^{\bar{x}}(1-v)^{\alpha^\epsilon+a^\epsilon-2}v^{-\alpha^\epsilon}\sigma_\epsilon(v)\,\mathrm{d}v\|\!|H|\!\|_\delta\\
&\leqslant K_2\frac{\bar{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\bar{x})}\int_0^{\bar{x}}v^{-\alpha^\epsilon}\sigma_\epsilon(v)\,\mathrm{d}v\|\!|H|\!\|_\delta\quad\forall 0<\bar{x}\leqslant\delta\leqslant\frac{3}{4},
\end{aligned}
$$

for some $K_2=K_2(\delta_2,a^0)>0$. Here we have used uniform bounds on

$$
(1-\bar{x})^{1-\alpha^\epsilon-a^\epsilon}\quad\text{and}\quad(1-\bar{x})^{\alpha^\epsilon+a^\epsilon-2}\quad\text{for}\quad\bar{x}\in\left[0,\frac{3}{4}\right].
$$

Due to (137), we estimate $0<\bar{x}\leqslant\delta_2\epsilon$ and $\delta_2\epsilon<\bar{x}\leqslant\delta$ separately. The former gives an estimate

$$
\left|\mathcal{T}^\epsilon[H](\bar{x})\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})}\right|\leqslant\frac{K_2}{1-\alpha^\epsilon}\|\!|H|\!\|_\delta\quad\forall 0<\bar{x}\leqslant\delta_2\epsilon,
$$

directly. We therefore consider $\delta_2\epsilon<\bar{x}\leqslant\delta\leqslant\frac{3}{4}$ and find

$$
\begin{aligned}
\frac{\bar{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\bar{x})}\int_0^{\bar{x}}v^{-\alpha^\epsilon}\sigma_\epsilon(v)\,\mathrm{d}v &= \frac{1}{(\delta_2\epsilon)^{1-\alpha^\epsilon}}\int_0^{\delta_2\epsilon}v^{-\alpha^\epsilon}\mathrm{d}v+\int_{\delta_2\epsilon}^{\bar{x}}v^{-1}\mathrm{d}v\\
&=\frac{1}{1-\alpha^\epsilon}-\log\left(\bar{x}^{-1}\delta_2\epsilon\right)\\
&=\frac{1}{1-\alpha^\epsilon}\left(1+\log\sigma_\epsilon(\bar{x})^{-1}\right).
\end{aligned}
$$

This completes the proof.

Proof of item 4. Proceeding as in the proof of item 3, we find

$$
\left|\mathcal{T}^\epsilon\left[(\cdot)^iH\right](\bar{x})\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})}\right|\leqslant K_2\frac{\bar{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\bar{x})}\int_0^{\bar{x}}v^{i-\alpha^\epsilon}\sigma_\epsilon(v)\,dv\|\!|H|\!\|_\delta\quad\forall 0<\bar{x}\leqslant\delta. \tag{146}
$$

As above, we estimate $0 < \bar{x} \leqslant \delta_2\epsilon$ and $\delta_2\epsilon < \bar{x} \leqslant \delta$ separately. In the former case, we directly obtain that

$$\left|\mathcal{T}^\epsilon\left[(\cdot)^i H\right](\bar{x})\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})}\right| \leqslant \frac{K_2\bar{x}^i}{i+1-\alpha^\epsilon}|||H|||_\delta$$
$$\leqslant \frac{K_2\bar{x}^i}{i}|||H|||_\delta \quad \forall 0 < \bar{x} \leqslant \delta_2\epsilon.$$

We are therefore left with $\delta_2\epsilon < \bar{x} \leqslant \delta$ where

$$\frac{\bar{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\bar{x})}\int_0^{\bar{x}} v^{i-\alpha^\epsilon}\sigma_\epsilon(v)\,\mathrm{d}v \leqslant \int_0^{\bar{x}} v^{i-1}\mathrm{d}v = \frac{\bar{x}^i}{i} \quad \forall \epsilon\delta_2 < \bar{x} \leqslant \delta,$$

Here we have used that

$$\frac{1}{(\delta_2\epsilon)^{1-\alpha^\epsilon}}\int_0^{\delta_2\epsilon} v^{i-\alpha^\epsilon}\mathrm{d}v \leqslant \int_0^{\delta_2\epsilon} v^{i-1}\mathrm{d}v.$$

This completes the proof.

Proof of item 5. This case is easy:

$$\left|\mathcal{T}^\epsilon\left[(\cdot)^{lN^\epsilon+1}\right](\bar{x})\right| \leqslant \delta^{(l-1)N^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\bar{x}) \quad \forall 0 \leqslant \bar{x} \leqslant \delta,$$

and therefore

$$|||\mathcal{T}^\epsilon\left[(\cdot)^{lN^\epsilon+1}\right]|||_\delta \leqslant \delta^{(l-1)N^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta).$$

Proof of item 6. We have

$$|H(\bar{x})| \leqslant ER^{N^\epsilon+1}\bar{x}^{N^\epsilon+1}\sum_{k=0}^\infty R^k\delta^k = \frac{ER^{N^\epsilon+1}\bar{x}^{N^\epsilon+1}}{1-R\delta} \quad \forall 0 \leqslant \bar{x} \leqslant \delta,$$

and consequently

$$|\mathcal{T}^\epsilon[H](\bar{x})| \leqslant \frac{ER^{N^\epsilon+1}}{1-R\delta}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\bar{x}) \quad \forall 0 < \bar{x} \leqslant \delta.$$

The result follows.

Proof of item 7. Now, for item 7 we use the linearity of $\mathcal{T}^\epsilon$ and first estimate each of the terms of the sum $\sum_{l\geqslant 2}\mathcal{T}^\epsilon\left[H_l(\cdot)\overline{Y}^l\right]$. By (128), (133) and (139), we find that

$$\left|\mathcal{T}^\epsilon\left[\overline{Y}^l\right](\overline{x})\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}\right|\leqslant K_2\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}v^{-\alpha^\epsilon}\sigma_\epsilon(v)\left(\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](v)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}\right)^{l-1}\mathrm{d}v\|\!|\overline{Y}|\!\|_\delta^l$$
$$\leqslant K_2C^{l-1}\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}\mathrm{d}v\|\!|\overline{Y}|\!\|_\delta^l\quad\forall 0<\overline{x}\leqslant\delta, \tag{147}$$

with $C=C_2/C_1$. We claim that

$$\|\!|\mathcal{T}^\epsilon\left[\overline{Y}^l\right]|\!\|_\delta\leqslant\frac{2\epsilon}{l-1}K_2C^{l-1}\|\!|\overline{Y}|\!\|_\delta^l. \tag{148}$$

In order to prove this, we only have to show that

$$\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}\mathrm{d}v\leqslant\frac{2\epsilon}{l-1}\quad\forall 0<\overline{x}\leqslant\delta, \tag{149}$$

cf (147). Consider first the simplest case $0<\overline{x}\leqslant\delta\leqslant\delta_2\epsilon$. Then $\sigma_\epsilon(\overline{x})=\sigma_\epsilon(v)=\sigma_\epsilon(\delta)=1$, for all $0\leqslant\overline{x}\leqslant v$, and we have

$$\begin{aligned}\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}\mathrm{d}v&=\frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon}\left(\frac{\overline{x}}{\delta}\right)^{(l-1)(N^\epsilon+1)}\\&\leqslant\frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon}\\&\leqslant\frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon-l(1-\alpha^\epsilon)}\\&=\frac{1}{(l-1)(N^\epsilon+\alpha^\epsilon)}\\&=\frac{\epsilon}{l-1},\end{aligned}$$

and (149) follows. We are left with $0<\overline{x}\leqslant\delta_2\epsilon<\delta$ and $\delta_2\epsilon<\overline{x}\leqslant\delta$. For the former, we have $\sigma_\epsilon(\overline{x})=\sigma_\epsilon(v)=1$, $\sigma_\epsilon(\delta)=(\delta^{-1}\delta_2\epsilon)^{1-\alpha^\epsilon}$ and

$$\begin{aligned}&\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}\mathrm{d}v\\&\quad=\overline{x}^{\alpha^\epsilon-1}\int_0^{\overline{x}}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}}{\delta^{(l-1)(N^\epsilon+1)}\left(\delta^{-1}\delta_2\epsilon\right)^{(l-1)(1-\alpha^\epsilon)}}\mathrm{d}v\\&\quad=\frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon}\left(\frac{\overline{x}}{\delta}\right)^{(l-1)(N^\epsilon+\alpha^\epsilon)}\left(\frac{\overline{x}}{\delta_2\epsilon}\right)^{(l-1)(1-\alpha^\epsilon)}\\&\quad\leqslant\frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon},\\&\quad\leqslant\frac{\epsilon}{l-1},\end{aligned}$$

and (149) follows. We finally consider $\delta_2\epsilon<\overline{x}\leqslant\delta$:

$$\begin{aligned}
\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}\mathrm{d}v &= \frac{1}{(\delta_2\epsilon)^{1-\alpha^\epsilon}}\int_0^{\delta_2\epsilon}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}}{\delta^{(l-1)(N^\epsilon+\alpha^\epsilon)}(\delta_2\epsilon)^{(l-1)(1-\alpha^\epsilon)}}\mathrm{d}v\\
&\quad+\int_{\delta_2\epsilon}^{\overline{x}}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon-l(1-\alpha^\epsilon)}}{\delta^{(l-1)(N^\epsilon+\alpha^\epsilon)}}\mathrm{d}v\\
&\leqslant\frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon}+\frac{1}{(l-1)(N^\epsilon+\alpha^\epsilon)}\\
&\leqslant\frac{2}{(l-1)(N^\epsilon+\alpha^\epsilon)}\\
&=\frac{2\epsilon}{l-1},
\end{aligned}$$

and (149) follows. Here we have used $\delta\geqslant\delta_2\epsilon$ in the denominator of the first integral on the right hand side. In turn, we obtain (148) and therefore

$$\begin{aligned}
\left\|\left|\mathcal{T}^\epsilon\left[\sum_{l=2}^\infty H_l(\cdot)\overline{Y}^l\right]\right\|\right|_\delta &\leqslant 2\epsilon EK_2C^{-1}R^{-1}\sum_{l=2}^\infty\left(CR|||\overline{Y}|||_\delta\right)^l\\
&=2\epsilon EK_2\frac{CR|||\overline{Y}|||_\delta^2}{1-CR|||\overline{Y}|||_\delta}\\
&\leqslant 4\epsilon EK_2CR|||\overline{Y}|||_\delta^2\quad\forall\,|||\overline{Y}|||_\delta<\frac{1}{2}(CR)^{-1}.
\end{aligned}$$

The fact that the mapping $\overline{Y}\mapsto\mathcal{T}^\epsilon[\sum_{l=2}^\infty H_l(\cdot)\overline{Y}^l]$ is C^1 and a contraction for all $\epsilon>0$ small enough follows from identical computations. Further details are therefore left out. ☐

The following will also be important: Define

$$\overline{U}^\epsilon(\overline{x}):=(N^\epsilon+a^\epsilon)\,\overline{w}^\epsilon_{N^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x});\tag{150}$$

this quantity corresponds to M_0 in [16, lemma 7.2].

Lemma 5.18. *Fix $\delta_2>0$ and suppose that $\epsilon^{-1}\notin\mathbb{N}$. Then we have the following statements regarding $\overline{U}$:*

1. *$\overline{U}$ has an absolutely convergent power series representation*

$$\overline{U}^\epsilon(\overline{x})=\frac{\Gamma(\alpha^\epsilon)\Gamma(1-\alpha^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})}\sum_{k=N^\epsilon+1}^\infty\frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\overline{x}^k\quad\forall-1<\overline{x}<1.\tag{151}$$

2. *For all $0<\epsilon\ll1$, $\epsilon^{-1}\notin\mathbb{N}$, $0<\delta\leqslant\frac{3}{4}$ the following estimate holds:*

$$\|\overline{U}^\epsilon\|_\delta=|||\overline{U}^\epsilon|||_\delta\gtreqless\frac{1}{1-\alpha^\epsilon}\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}\left(\frac{\delta}{1-\delta}\right)^{N^\epsilon+1}\sigma_\epsilon(\delta)\begin{cases}C_1,\\C_2.\end{cases}\tag{152}$$

Here $C_i=C_i(\delta_2,a^0)>0$, $i=1,2$.

3. *Asymptotics for* $\bar{x} = \mathcal{O}(\epsilon)$*: Let* $\bar{x} = \epsilon\bar{x}_2 \in [-\epsilon\delta_2, \epsilon\delta_2]$*,* $\delta_2 > 0$ *fixed. Then:*

$$\overline{U}^\epsilon(\epsilon\bar{x}_2) = \frac{\Gamma(\alpha^\epsilon)}{1-\alpha^\epsilon}(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}(\epsilon\bar{x}_2)^{N^\epsilon+1}\left[1+\bar{x}_2\int_0^1 e^{(1-v)\bar{x}_2}v^{1-\alpha^\epsilon}\mathrm{d}v + o(1)\right], \tag{153}$$

with o(1) uniform with respect to $\alpha^\epsilon \in (0,1)$.

Proof. We prove the items 1–3 successively in the following.

Proof of item 1. For (151) we use item 2 of lemma 5.15 and (92):

$$\begin{aligned}
&\overline{w}^\epsilon_{N^\epsilon} \times (N^\epsilon + a^\epsilon)\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\bar{x}) \\
&= \frac{\Gamma(\alpha^\epsilon)\Gamma(N^\epsilon+a^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})} \times \frac{(N^\epsilon+a^\epsilon)\Gamma(1-\alpha^\epsilon)}{\Gamma(N^\epsilon+1+a^\epsilon)}\sum_{k=N^\epsilon+1}^\infty \frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\bar{x}^k \\
&= \frac{\Gamma(\alpha^\epsilon)\Gamma(1+\alpha^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})}\sum_{k=N^\epsilon+1}^\infty \frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\bar{x}^k.
\end{aligned} \tag{154}$$

Proof of item 2. Next, regarding (152) we use (135), (83), (170), (173),

$$\begin{aligned}
\overline{U}^\epsilon &= \frac{\Gamma(\alpha^\epsilon)\Gamma(N^\epsilon+a^\epsilon+1)}{\epsilon\Gamma(\epsilon^{-1})}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right] = \frac{\Gamma(\alpha^\epsilon)\Gamma(N^\epsilon+a^\epsilon+1)}{(1-\epsilon)\Gamma(N^\epsilon+\alpha^\epsilon-1)}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right] \\
&= \Gamma(\alpha^\epsilon)(1+o(1))(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right],
\end{aligned} \tag{155}$$

and subsequently (139).

Proof of item 3. Finally, for (153) we use (155) and lemma 5.17 item 2:

$$\begin{aligned}
\overline{U}^\epsilon(\epsilon\bar{x}_2) &= \Gamma(\alpha^\epsilon)(1+o(1))(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\epsilon\bar{x}_2) \\
&= \Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}\frac{1}{1-\alpha^\epsilon}(\epsilon\bar{x}_2)^{N^\epsilon+1}\left[1+\bar{x}_2\int_0^1 \mathrm{e}^{(1-v)\bar{x}_2}v^{1-\alpha^\epsilon}\mathrm{d}v + o(1)\right].
\end{aligned}$$

□

5.4. Solving for the analytic weak-stable manifold

In the following, we write $\overline{m}^\epsilon$ as (recall the notation in (114) and (115))

$$\overline{m}^\epsilon = j^{N^\epsilon}[\overline{m}^\epsilon] + \overline{M}^\epsilon, \quad r^{N^\epsilon}[\overline{m}^\epsilon] =: \overline{M}^\epsilon; \tag{156}$$

we will use $\mathcal{T}^\epsilon$ to set up a fixed-point equation for $\overline{M}^\epsilon$. For this purpose, let

$$\overline{G}^\epsilon(\bar{x}, \overline{Y}) := \bar{g}^\epsilon\left(\bar{x}, j^{N^\epsilon}[\overline{m}^\epsilon](\bar{x}) + \overline{Y}\right) - \bar{g}^\epsilon\left(\bar{x}, j^{N^\epsilon}[\overline{m}^\epsilon](\bar{x})\right).$$

We clearly have

$$\overline{G}^\epsilon(\bar{x}, \overline{Y}) = \bar{x}^2\overline{G}^\epsilon_1(\bar{x})\overline{Y} + \sum_{l\geqslant 2}\overline{G}^\epsilon_l(\bar{x})\overline{Y}^l, \tag{157}$$

and for any $\overline{D}>0$:

$$\|\overline{G}_l^\epsilon\|_\delta \leqslant \mu C\overline{D}^{-l+1} \quad \forall l \in \mathbb{N},$$

with $C>0$ independent of μ and ϵ, provided that (124) hold and that $0<\epsilon\ll 1$. This is essentially identical to the computation leading to (122) (with $j^{N^\epsilon-1}$ replaced by j^{N^ϵ}), using $\|j^{N^\epsilon}[\overline{m}^\epsilon]\|_\delta \leqslant K$, see (125).

The following result corresponds to [16, lemma 7.3].

Lemma 5.19. *$\overline{Y}=M^\epsilon$ satisfies the fixed-point equation:*

$$\begin{aligned}\overline{Y}(\overline{x}) = &(N^\epsilon + a^\epsilon)(-1)^{N^\epsilon}\overline{w}_{N^\epsilon}^\epsilon \overline{S}_{N^\epsilon}^\epsilon \mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})\\ &-\mathcal{T}^\epsilon\left[r^{N^\epsilon}\left[\overline{g}^\epsilon\left(\cdot, j^{N^\epsilon}[\overline{m}^\epsilon]\right)\right]\right](\overline{x}) - \mathcal{T}^\epsilon\left[\overline{G}^\epsilon\left(\cdot,\overline{Y}\right)\right](\overline{x}).\end{aligned} \tag{158}$$

Proof. With $\overline{m}_k^\epsilon$ given by (84), we obtain

$$\begin{aligned}\epsilon\overline{x}(\overline{x}-1)\frac{d\overline{M}^\epsilon}{d\overline{x}} + (1+\epsilon a^\epsilon \overline{x})\overline{M}^\epsilon = &-\epsilon(N^\epsilon + a^\epsilon)\overline{m}_{N^\epsilon}^\epsilon \overline{x}^{N^\epsilon+1}\\ &+\epsilon\left(r^{N^\epsilon}\overline{g}^\epsilon\left(\overline{x}, j^{N^\epsilon}[\overline{m}^\epsilon](\overline{x})\right) + \overline{G}^\epsilon\left(\overline{x},\overline{M}^\epsilon\right)\right).\end{aligned}$$

Using $\overline{m}_{N^\epsilon}^\epsilon = (-1)^{N^\epsilon}\overline{w}_{N^\epsilon}^\epsilon \overline{S}_{N^\epsilon}^\epsilon$, (150) and lemma 5.15 item 1, the result follows; notice again the change of sign when comparing (129) and (76). □

We denote the right hand side of (158) by $\mathcal{F}(\overline{Y})(\overline{x})$:

$$\begin{aligned}\mathcal{F}\left(\overline{Y}\right) := &(N^\epsilon + a^\epsilon)(-1)^{N^\epsilon}\overline{w}_{N^\epsilon}^\epsilon \overline{S}_{N^\epsilon}^\epsilon \mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]\\ &-\mathcal{T}^\epsilon\left[r^{N^\epsilon}\left[\overline{g}^\epsilon\left(\cdot, j^{N^\epsilon}[\overline{m}^\epsilon]\right)\right]\right] - \mathcal{T}^\epsilon\left[\overline{G}^\epsilon\left(\cdot,\overline{Y}\right)\right],\end{aligned} \tag{159}$$

and define the closed ball

$$\mathcal{B}^C := \mathcal{D}_\delta^\epsilon \cap \{|||Y|||_\delta \leqslant C\},$$

of radius $C>0$.

Proposition 5.20. *Suppose that $S_\infty^0 \neq 0$ and that $\mu\in[0,\mu_0)$ with $\mu_0>0$ small enough. Then for any $K>0$, $\alpha^\epsilon\in(0,1)$, $N^\epsilon \gg 1$, there is an $0<\overline{\delta}\leqslant \frac{3}{4}$ such that for any $0<\delta\leqslant\overline{\delta}$ the following holds:*

1. *Boundedness of the 'leading order term':*

$$\|\overline{w}_{N^\epsilon}^\epsilon(\cdot)^{N^\epsilon}\|_\delta + \|\overline{U}^\epsilon\|_\delta \leqslant \frac{K}{|S_\infty^0|}, \tag{160}$$

2. *$\mathcal{F}:\mathcal{B}_\delta^{2K}\to\mathcal{B}_\delta^{2K}$, defined by (159), is a contraction.*

Proof. The first claim in item 1 is obvious as $\overline{U}^\epsilon(0)=0$, see also lemma 5.18 and (123). We therefore proceed to proof item 2 regarding $\mathcal{F}:\mathcal{B}_\delta^{2K}\to\mathcal{B}_\delta^{2K}$ being a contraction. For this purpose, we estimate each of the three terms on the right hand side (158) in the norm $|||\cdot|||_\delta$, recall (131).

The first term:

$$\left(N^\epsilon + a^\epsilon\right)(-1)^{N^\epsilon}\,\overline{w}^\epsilon_{N^\epsilon}\overline{S}^\epsilon_{N^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x}).$$

We write the first term as

$$(1+o(1))(-1)^{N^\epsilon}\,S^0_\infty\overline{U}^\epsilon(\overline{x}),$$

using lemma 5.11 and (150). Then by using (151) and the assumption on δ, see (160), we have

$$|||\left(N^\epsilon + a^\epsilon\right)(-1)^{N^\epsilon}\,\overline{w}^\epsilon_{N^\epsilon}\overline{S}^\epsilon_{N^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]|||_\delta \leqslant \frac{4}{3}K.$$

The second term:

$$-\mathcal{T}^\epsilon\left[r^{N^\epsilon}\left[\overline{g}^\epsilon\left(\cdot,j^{N^\epsilon}\left[\overline{m}^\epsilon\right]\right)\right]\right](\overline{x}).$$

For the second term, we first use lemma 5.13, setting

$$q = \overline{m}^\epsilon_{N^\epsilon}\overline{x}^{N^\epsilon}$$

in (118). Therefore

$$\begin{aligned} r^{N^\epsilon}\left[\overline{g}^\epsilon(\cdot,j^{N^\epsilon}\left[\overline{m}^\epsilon\right]))\right](\overline{x}) &= \sum_{k=N^\epsilon+1}^\infty \overline{\mathcal{G}}^\epsilon\left[j^{N^\epsilon-1}\left[\overline{m}^\epsilon\right]\right]_k\overline{x}^k + \overline{g}^\epsilon_1(\overline{x})\overline{m}^\epsilon_{N^\epsilon}\overline{x}^{N^\epsilon+2}\\ &\quad+\sum_{l=2}^\infty \overline{g}^\epsilon_l(\overline{x})(\overline{m}^\epsilon_{N^\epsilon})^l\overline{x}^{lN^\epsilon+1}, \end{aligned}$$

using (118). We now use item 6 of lemma 5.17:

$$|||\mathcal{T}^\epsilon\left[\sum_{k=N^\epsilon+1}^\infty \overline{\mathcal{G}}^\epsilon\left[j^{N^\epsilon-1}\left[\overline{m}^\epsilon\right]\right]_k(\cdot)^k\right]|||_\delta \leqslant \frac{ER^{N^\epsilon+1}}{1-R\delta}\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta).$$

Here

$$E = C\left(\overline{w}^\epsilon_2\right)^2e^{-4Q^\epsilon_4},\quad R = e^{Q^\epsilon_4},$$

cf (119) and (89). Consequently,

$$ER^{N^\epsilon+1} = C\left(\overline{w}^\epsilon_2\right)^2e^{Q^\epsilon_4(N^\epsilon-3)} = C\overline{w}^\epsilon_2\overline{w}^\epsilon_{N^\epsilon-1},$$

using (89) and therefore $R\delta \leqslant \frac{4}{5}$ for $0<\delta\leqslant\frac{3}{4}$ for all $0<\epsilon\ll 1$. We then arrive at

$$\begin{aligned} |||\mathcal{T}^\epsilon\left[\sum_{k=N^\epsilon+1}^\infty \overline{\mathcal{G}}^\epsilon\left[j^{N^\epsilon-1}\left[\overline{m}^\epsilon\right]\right]_k(\cdot)^k\right]|||_\delta &\leqslant 5C\overline{w}^\epsilon_2\overline{w}^\epsilon_{N^\epsilon-1}\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)\\ &\leqslant \epsilon^2\overline{C}K, \end{aligned}$$

with $\overline{C} > 0$ large enough, for all $0 < \epsilon \ll 1$. Here we have also used (152), (154), (160), $\Gamma(\alpha^\epsilon) > 1$ and $\overline{w}_2^\epsilon = \mathcal{O}(\epsilon)$ to estimate

$$5C\overline{w}_2^\epsilon \overline{w}_{N^\epsilon-1}^\epsilon \mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta) \leqslant \epsilon^2\overline{C}|S_\infty^0| \,|||\overline{U}^\epsilon|||_\delta \leqslant \epsilon^2\overline{C}K,$$

in the last inequality. Proceeding completely analogously, we obtain

$$|||\mathcal{T}^\epsilon\left[\overline{g}_1^\epsilon(\cdot)\overline{m}_{N^\epsilon}^\epsilon(\cdot)^{N^\epsilon+2}\right]|||_\delta \leqslant \mu C|S_\infty^0|\overline{w}_{N^\epsilon}^\epsilon \delta\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta) \leqslant \mu\epsilon\overline{C},$$

and by lemma 5.17 item 5

$$\begin{aligned}
|||\mathcal{T}^\epsilon\left[\sum_{l=2}^\infty \overline{g}_l^\epsilon(\cdot)\left(\overline{m}_{N^\epsilon}^\epsilon\right)^l(\cdot)^{lN^\epsilon+1}\right]|||_\delta &\leqslant \mu CK_2\sum_{l=2}^\infty \overline{D}^{-l+1}|\overline{m}_{N^\epsilon}|^l\delta^{N^\epsilon(l-1)}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta)\\
&= \mu CK_2\sum_{l=2}^\infty\left(\overline{D}^{-1}|\overline{m}_{N^\epsilon}|\delta^{N^\epsilon}\right)^{l-1}|\overline{m}_{N^\epsilon}|\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta)\\
&= \mu CK_2\frac{\overline{D}^{-1}|\overline{m}_{N^\epsilon}|\delta^{N^\epsilon}}{1-\overline{D}^{-1}|\overline{m}_{N^\epsilon}|\delta^{N^\epsilon}}|\overline{m}_{N^\epsilon}|\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta)\\
&\leqslant \mu\epsilon\overline{C},
\end{aligned}$$

for some $\overline{C} > 0$ independent of μ and ϵ. Here we have again used (160) and taken $\overline{D} > 0$ large enough. In total, we conclude that

$$|||-\mathcal{T}^\epsilon\left[r^{N^\epsilon}\left[\overline{g}^\epsilon\left(\cdot, j^{N^\epsilon}\left[\overline{m}^\epsilon\right]\right)\right]\right]|||_\delta \leqslant \mathcal{O}(\epsilon).$$

It follows that the second term is bounded by $\frac{1}{3}K$ for all $0 < \epsilon \ll 1$.
The third term:

$$-\mathcal{T}^\epsilon\left[\overline{G}^\epsilon\left(\cdot, \overline{Y}\right)\right](\overline{x}).$$

For the third term, we also use items 4 (with $i = 2$) and 7 of lemma 5.17. Specifically, by (157), we directly obtain

$$|||-\mathcal{T}^\epsilon\left[\overline{G}^\epsilon\left(\cdot, \overline{Y}\right)\right]|||_\delta = O(\mu) \leqslant \frac{1}{3}K,$$

for $|||Y|||_\delta \leqslant 2K$ by taking $\overline{D} > 0$ large enough.

In total, $\mathcal{F}: \mathcal{B}_\delta^{2K} \to \mathcal{B}_\delta^{2K}$ is well-defined when (160) holds and $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$ for $\mu > 0$ small enough. The fact that $\mathcal{F}$ is a contraction follows directly from using item 7 of lemma 5.17 on the third term, see (143). In fact, we find that the Lipschitz constant is $O(\mu)$ for all $0 < \epsilon \ll 1$ (independently of $\alpha^\epsilon \in (0,1)$). □

Proposition 5.21. *Assume that the conditions of proposition 5.20 hold true and that (160) holds. Let $\overline{M}^\epsilon$ denote the solution of the fixed-point equation (158) and define*

$$\overline{B}^{\epsilon}(\overline{x}) := j^{N^{\epsilon}-1}[\overline{m}^{\epsilon}](\overline{x}) - \mathcal{T}^{\epsilon}\left[r^{N^{\epsilon}}\left[\overline{g}^{\epsilon}\left(\cdot, j^{N^{\epsilon}}[\overline{m}^{\epsilon}]\right)\right]\right](\overline{x}) - \mathcal{T}^{\epsilon}\left[\overline{G}^{\epsilon}(\cdot, \overline{M}^{\epsilon})\right](\overline{x}),$$
$$\overline{V}^{\epsilon}(\overline{x}) := \overline{w}^{\epsilon}_{N^{\epsilon}}\overline{x}^{N^{\epsilon}} + \overline{U}^{\epsilon}(\overline{x}).$$

Then

$$\overline{m}^{\epsilon}(\overline{x}) = \overline{B}^{\epsilon}(\overline{x}) + (1+o(1))(-1)^{N^{\epsilon}} S^0_{\infty}\overline{V}^{\epsilon}(\overline{x}), \tag{161}$$

for all $0 \leqslant \overline{x} \leqslant \delta$*. Moreover, the following estimates hold true:*

$$\|\overline{B}^{\epsilon}\|_{\delta} = \mathcal{O}(\epsilon), \quad \|\overline{V}^{\epsilon}(\overline{x})\|_{\delta} \leqslant \frac{K}{|S_{\infty}|}. \tag{162}$$

Proof. (161) follows directly from (156) with $\overline{Y} = \overline{M}^{\epsilon}$ given by (158). Subsequently, the estimate for $\overline{B}^{\epsilon}$ follows from (132) and the proof of proposition 5.20 (see the second and third terms). Finally for the estimate for $\overline{V}^{\epsilon}$, we use that $\overline{U}^{\epsilon}$ is an increasing function of $\overline{x} \in [0, \delta]$ and

$$\|\overline{V}^{\epsilon}\|_{\delta} = \|\overline{w}^{\epsilon}_{N^{\epsilon}}(\cdot)^{N^{\epsilon}} + \overline{U}^{\epsilon}\|_{\delta} = \overline{w}^{\epsilon}_{N^{\epsilon}}\delta^{N^{\epsilon}} + \overline{U}^{\epsilon}(\delta) = \|\overline{w}^{\epsilon}_{N^{\epsilon}}(\cdot)^{N^{\epsilon}}\|_{\delta} + \|\overline{U}^{\epsilon}\|_{\delta} \leqslant \frac{K}{|S^0_{\infty}|},$$

using (160) in the final inequality. □

$\overline{V}^{\epsilon}$ has the following absolutely convergent power series expansion

$$\overline{V}^{\epsilon}(\overline{x}) = \frac{\Gamma(\alpha^{\epsilon})\Gamma(1-\alpha^{\epsilon})}{\epsilon\Gamma(\epsilon^{-1})}\sum_{k=N^{\epsilon}}^{\infty}\frac{\Gamma(k+a^{\epsilon})}{\Gamma(k+1-\epsilon^{-1})}\overline{x}^k \quad -1 < \overline{x} < 1. \tag{163}$$

This follows from (151).

Remark 9. The quantity $\overline{V}^{\epsilon}$ corresponds to Θ_{main} in [16, equation (7.25)].

5.5. *Completing the proof of lemma 3.4*

We prove the items 1–4 successively in the following.

Proof of item 1. The power series expansion (163) of $\overline{V}^{\epsilon}$ in (161) was proven in proposition 5.21. The absolute convergence of this power series expansion follows from (173):

$$\frac{\Gamma(k+a^{\epsilon})}{\Gamma(k+1-\epsilon^{-1})} = (1+o(1))k^{\epsilon^{-1}+a^{\epsilon}-1}. \tag{164}$$

Each term of the series $\overline{V}^{\epsilon}$ is positive for $\overline{x} > 0$, which implies (34).

Proof of item 2. For the lower bound, we use $\overline{w}^{\epsilon}_{N^{\epsilon}} > 0$, the definition of $\overline{V}^{\epsilon}$:

$$\overline{V}^{\epsilon}(\overline{x}) = \overline{w}^{\epsilon}_{N^{\epsilon}}\overline{x}^{N^{\epsilon}} + \overline{U}^{\epsilon}(\overline{x}) \geqslant \overline{U}^{\epsilon}(\overline{x}) \quad \forall 0 \leqslant \overline{x} < 1,$$

and the lower bound of $\overline{U}^{\epsilon}$ (see (152)):

$$\overline{V}^{\epsilon}(\overline{x}) \geqslant C_1\frac{1}{1-\alpha^{\epsilon}}\Gamma(\alpha^{\epsilon})(N^{\epsilon})^{a^{\epsilon}+2-\alpha^{\epsilon}}\left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^{\epsilon}+1}\sigma_{\epsilon}(\overline{x}) \quad \forall 0 \leqslant \overline{x} \leqslant \frac{3}{4}.$$

We take $\delta_2 = \frac{3}{4}$ in (138) and obtain

$$\sigma_{\epsilon}(\overline{x}) \geqslant \epsilon^{1-\alpha^{\epsilon}} \quad \forall 0 \leqslant \overline{x} \leqslant \frac{3}{4},$$

which together with

$$(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon} \geqslant \frac{1}{2}\epsilon^{\alpha^\epsilon-a^\epsilon-2} \quad \text{and} \quad \frac{1}{1-\alpha^\epsilon}\Gamma(\alpha^\epsilon) > 1 \quad \forall \alpha^\epsilon \in (0,1), 0<\epsilon \ll 1,$$

leads to

$$\overline{V}^\epsilon(\bar{x}) \geqslant \frac{1}{2}C_1\epsilon^{-a^\epsilon-1}\left(\frac{\bar{x}}{1-\bar{x}}\right)^{N^\epsilon+1} \qquad \forall 0 \leqslant \bar{x} \leqslant \frac{3}{4}.$$

We now use that $a^0 > -2$, recall hypothesis 1, to conclude that

$$\frac{1}{2}C_1\epsilon^{-a^\epsilon-1} \geqslant \epsilon \Longleftrightarrow \frac{1}{2}C_1 \geqslant \epsilon^{2+a^\epsilon} \quad \forall 0<\epsilon \ll 1.$$

Indeed, let $\nu = 2+a^0 > 0$. Then by taking $0<\epsilon\ll 1$, we have that $|a^\epsilon - a^0| \leqslant \frac{1}{2}\nu$ and hence

$$\epsilon^{2+a^\epsilon} \leqslant \epsilon^\nu \epsilon^{-\frac{1}{2}\nu} = \epsilon^{\frac{1}{2}\nu} \to 0 \quad \text{for} \quad \epsilon \to 0.$$

This completes the proof of item 2.

Proof of item 3. The divergence with respect to $\alpha^\epsilon \to 0^+$ and 1^- is a direct consequence of the factors $\Gamma(\alpha^\epsilon)\Gamma(1-\alpha^\epsilon)$ in the definition of $\overline{V}^\epsilon$, recall (171).

Proof of item 4. Finally, in order to obtain (36) we use the form in (161):

$$\overline{V}^\epsilon(\bar{x}) = \overline{w}^\epsilon_{N^\epsilon}\bar{x}^{N^\epsilon} + \overline{U}^\epsilon(\bar{x}),$$

(153) and

$$\overline{w}^\epsilon_{N^\epsilon}\bar{x}^{N^\epsilon} = (1+o(1))\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}\bar{x}^{N^\epsilon}.$$

This gives

$$\begin{aligned}
\overline{V}^\epsilon(\epsilon\bar{x}_2) &= (1+o(1))\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}(\epsilon\bar{x}_2)^{N^\epsilon} \\
&\quad + \frac{\Gamma(\alpha^\epsilon)}{1-\alpha^\epsilon}(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}(\epsilon\bar{x}_2)^{N^\epsilon+1}\left[1+\bar{x}_2\int_0^1 e^{(1-v)\bar{x}_2}v^{1-\alpha^\epsilon}\mathrm{d}v + o(1)\right] \\
&= (1+o(1))\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}(\epsilon\bar{x}_2)^{N^\epsilon} \\
&\quad \times \left(1+\frac{\bar{x}_2}{1-\alpha^\epsilon}\left[1+\bar{x}_2\int_0^1 e^{(1-v)\bar{x}_2}v^{1-\alpha^\epsilon}\mathrm{d}v + o(1)\right]\right).
\end{aligned}$$

5.6. *Completing the proof of theorem 3.5*

We consider I of the forms $[0,\delta]$, $0<\delta\leqslant\frac{3}{2}$, and $[-\delta,0]$, $0<\delta\leqslant\delta_2\epsilon$, separately in the following and prove the statement of theorem 3.5 in these cases. It will then follow that the statement is true for any $I\subset[-\delta_2\epsilon,\frac{3}{4}]$ satisfying (38).

The case $I=[0,\delta]$. If $I=[0,\delta]$, $\delta\leqslant\frac{3}{4}$, satisfies (38), then

$$\|\overline{V}^\epsilon\|_\delta = \|\overline{w}^\epsilon_{N^\epsilon}(\cdot)^{N^\epsilon}\|_\delta + \|\overline{U}^\epsilon\|_\delta \leqslant K.$$

We therefore apply proposition 5.20 with K replaced by $K|S_\infty^0|$. In this case, theorem 3.5 then follows from proposition 5.21, see (161) and (162).

The case $I = [-\delta, 0]$. Now, we consider case $I = [-\delta, 0]$ with $0 < \delta \leqslant \delta_2 \epsilon$. In this case, we adapt the space $\mathcal{D}_\delta^\epsilon$ and the norm $|||\cdot|||_\delta$ in the following way:

$$\mathcal{D}_\delta^\epsilon := \{H : [-\delta, 0] \to \mathbb{R} \text{ analytic} : |||H|||_\delta < \infty\},$$

where

$$|||H|||_\delta := \sup_{\bar{x} \in [-\delta, 0)} \left| H(u) \frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](-\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\bar{x})} \right|, \tag{165}$$

and importantly:

$$0 < \delta \leqslant \delta_2 \epsilon. \tag{166}$$

By (139), we have

$$C_1 \left(\frac{|\bar{x}|}{1-\bar{x}}\right)^{N^\epsilon+1} \leqslant (1-\alpha^\epsilon) \left|\mathcal{T}^\epsilon \left[(\cdot)^{N^\epsilon+1}\right](\bar{x})\right| \leqslant C_2 \left(\frac{|\bar{x}|}{1-\bar{x}}\right)^{N^\epsilon+1} \quad \forall -\delta_2\epsilon \leqslant \bar{x} \leqslant 0, \tag{167}$$

see (137).

Lemma 5.22. *Fix $\delta_2 > 0$ and suppose that (166) holds. Then the items 3–7 in lemma 5.17 also hold true with $|||\cdot|||_\delta$ given by (165).*

Proof. The proof of lemma 5.17 carries over since (167) holds. □

In this way, we obtain similar versions of proposition 5.20 (which only relies on the estimates in items 3–7 in lemma 5.17) and proposition 5.21 with

$$\|H\|_\delta := \sup_{u \in [-\delta, 0]} |H(u)|,$$

using (167) to estimate $\overline{V}^\epsilon$ in the sup-norm. Then, by proceeding as above for $I = [0, \delta]$, we complete the proof of theorem 3.5 in the case $I = [-\delta, 0]$, $0 < \delta \leqslant \delta_2 \epsilon$,

6. Discussion

In this paper, we have provided a detailed description of analytic weak-stable manifolds near analytic saddle-nodes (under a certain smallness assumption of the quantity $\mu = \sup u^\epsilon$ in the general normal form, see hypothesis 2). In further details, we have identified the quantity S_∞^0, with the property that $S_\infty^0 \neq 0$ implies the following (cf theorems 3.2 and 3.5):

(R1) The centre manifold is nonanalytic.
(R2) A certain flapping mechanism of the analytic weak-stable manifold W^{ws}.
(R3) W^{ws} does not intersect the unstable manifold of the saddle W^u for all $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$.

The quantity S_∞^0 is reminiscent of a Stokes constant, see e.g. [21]. Overall our approach is inspired by [16], proving statement (R2) for a specific (rational) system. In summary, [16] performs a blowup (scaling) transformation, writes the weak analytic manifold as a power series in the scaled coordinates, truncated at an order just below the resonant term, and then treats the remainder through a certain integral operator ($\mathcal{T}^\epsilon$). We also follow this strategy, but have brought the method into a form that is more in tune with dynamical systems theory (through normal forms, centre manifolds and fixed-point arguments). In this way, we established (R1)–(R3) as a generic phenomena for analytic saddle-nodes (albeit still within the context of hypotheses 1 and 2). We emphasise that (R3) does not follow from the results of [16] when applied to their specific nonlinearity. We also feel that we have obtained a deeper understanding of the underlying phenomena and also streamlined the method of [16] along the way. For an example of the latter, in our treatment of $\mathcal{T}^\epsilon$ we have used a Banach space of analytic functions $H(\bar{x}) = \mathcal{O}(\bar{x}^{N^\epsilon+1})$ and set up a fixed-point argument for the remainder; this approach does not depend upon hypothesis 2 (see discussion below).

We conjecture that hypotheses 1 and 2 can be removed so that our statements hold true for any analytic and generic unfolding of a saddle-node (by virtue of the normal form, see theorem 3.1). In fact, we believe that $a^0 \leqslant -2$ can be removed with only a few changes to our argument (we just have to handle the finite sums

$$\sum_{k\geqslant 2\,:\,k+a^0\leqslant 0} (\cdots),$$

separately).

Let us emphasise where hypothesis 2 is needed: it is used in the proof of $\widehat{m}^0 \in \mathcal{D}^0$, see proposition 4.6, and in the proof of the uniform boundedness of $\overline{m}^\epsilon$ in the semi-norm (100), see proposition 5.10.

At the same time, it is important to emphasise that *it is NOT needed in the treatment of the remainder, see proposition 5.20*. To see this, notice that in the proof of proposition 5.20, we only need that $\overline{G}$ has small Lipschitz-norm, see the estimate of the third term in the proof of proposition 5.20. To show this, we can first use the final condition of (26), see remark 3, to estimate the $\overline{Y}$-linear part of $\overline{G}$, and for the nonlinear part of $\overline{G}$ we can use lemma 5.17 item 7 (it is, in particular, $\mathcal{O}(\epsilon)$ in the norm $\|\!|\cdot|\!\|_\delta$ for all $\mu > 0$).

In other words, in order to remove hypothesis 2 we just need to find alternative proofs of propositions 4.6 and 5.10.

Let us focus on the former and

$$\widehat{m}^0(x) = \sum_{k=2}^\infty m_k^0 x^k \in \mathcal{D}^0 \Longleftrightarrow \sup_{k\geqslant 2} \frac{|m_k^0|}{w_k^0} < \infty.$$

In [16], the authors show that their corresponding sequence $\frac{m_k^0}{w_k^0}$, $k \in \mathbb{N}\setminus\{1\}$, is bounded by essentially setting up a majorant equation for S_k^0 defined by

$$m_k^0 =: (-1)^k w_k^0 S_k^0, \tag{168}$$

see [16, lemma 5.4]. It follows from (16) (with f_k^0 replaced by $\mathcal{G}_k^0 := \mathcal{G}^0[\widehat{m}^0]_k$) that S_k^0 satisfies the following recursion relation

$$S_k^0 = S_{k-1}^0 + \frac{(-1)^k \mathcal{G}_k^0}{w_k^0}, \quad w_k^0 = \Gamma\left(k + a^0\right),$$

with $\mathcal{G}_k^0$ depending on $S_2^0, \ldots, S_{k-3}^0$ (upon using (168)). The proof of [16, lemma 5.4] first consists of showing (using the majorant equation for S_k^0) that $|S_k^0| \leqslant K_0 C_0^k$ for some $K_0 > 0$, $C_0 > 1$ and all k. This is Step 3 of their proof. One can take $K_0 = 1, C_0 = e^n$ for some $n \in \mathbb{N}$.

Then in Step 4 of the proof of [16, lemma 5.4], the authors show that the exponential bound can be improved: There is some $\delta = \frac{1}{m} > 0$, $m \in \mathbb{N}$ large enough, and a $K_1 > 0$ large enough such that $|S_k^0| \leqslant K_1 (C_0 \mathrm{e}^{-\delta})^k = K_1 \mathrm{e}^{k\left(n - \frac{1}{m}\right)}$ for all k. Importantly, the authors of [16] show that this process can be iterated (with n and m fixed) in the following sense: For each $l \in \mathbb{N}$ with $n - \frac{l}{m} \geqslant 0$, there is a $K_l > 0$ such that

$$|S_k^0| \leqslant K_l \mathrm{e}^{k(n - l\delta)} = K_l \mathrm{e}^{k\left(n - \frac{l}{m}\right)},$$

for all k. Here $l \in \mathbb{N}$ is the number of applications of the improvement. Setting $l = nm$ then gives $|S_k^0| \leqslant K_{nm}$ (uniform bound) for all k as desired.

For our general normal form in theorem 3.1, it is straightforward to reproduce step 3 and the existence of C_0 from (49) (without using $\mu > 0$ small). However, we have not been able to reproduce the argument in step 4 in the general framework. We will pursue this further (along with alternative approaches to majorise S_k^0) in future work. In fact, at the time of writing, the first author of the present paper posted a preprint on arXiv, see [12], on the existence of (8) in the generic case (i.e. without the assumptions on μ and a^0). The author uses a separate approach based upon Borel-Laplace and the number S_∞^0 is connected with a singularity in the Borel plane. The preprint [12] does not address $S_\infty^0 = 0$, recall remark 8.

It is our belief that our results will find use in different areas of dynamical systems, in particular in the area of singularly perturbed systems where weak-stable manifolds play an important role. Here we are for example thinking of the weak canard of the folded node, see [3, 23]. Having said that, our approach is inherently planar and the minimal dimension of the folded node is three (with a two-dimensional critical manifold), so progress in this direction is therefore not just a simple incremental step. At the same time, we are confident that the overall approach of the paper (using power series expansion, identifying leading order terms and setting up fixed-point formulations) has potential (and is novel) in the context of the folded node. A natural starting point to extensions in higher dimensions, could also be to consider saddle-nodes in $(x, y) \in \mathbb{R}^{1+n}$; this would also be interesting at the level of $\epsilon = 0$ (where S_∞^0 would have to be reinterpreted). As other examples of future research directions, we mention extensions to different planar bifurcations with higher co-dimension, including the unfolding of the pitchfork. This would also require an extension of S_∞^0 to Poincaré rank $r = 2$ (where $\dot{x} = x^3$), which we believe is an interesting topic in itself. Here the results of [6] could be relevant. Finally, we would also like to explore connections in the future to the interesting results of Rousseau [19], see also [20] and references therein, on the analytic classification of unfoldings of saddle-nodes.

Data availability statement

No new data were created or analysed in this study.

Acknowledgment

The second author would like to acknowledge inspiring and pleasant discussions with Heinrich Freistühler in an early phase of this research project.

Appendix A. Basic properties of the gamma function

The gamma function $z \mapsto \Gamma(z)$, defined for $\mathrm{Re}(z) > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t, \tag{169}$$

will play a crucial role. We therefore collect a few well-known facts (see e.g. [18, chapter 5]) that will be used throughout the manuscript.

First, we recall that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$, which follows from $\Gamma(1) = 1$ and the basic property

$$\Gamma(z+1) = z\Gamma(z) \quad \forall\, \mathrm{Re}(z) > 0. \tag{170}$$

The gamma function can be analytically extended to the whole complex plane except zero and the negative integers (which are all simple poles); specifically,

$$\lim_{x\to 0} x\Gamma(x) = \Gamma(1) = 1, \tag{171}$$

In this paper, we will use Stirling's well-known formula:

$$\Gamma(x+1) = (1+o(1))\sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \tag{172}$$

for $x \to \infty$. The following form

$$\frac{\Gamma(x+b)}{\Gamma(x)} = (1+o(1))x^b, \tag{173}$$

for $b \in \mathbb{R}$ and $x \to \infty$, which can be obtained directly from (172), will also be needed. We will also use the reflection formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \forall z \notin \mathbb{Z}. \tag{174}$$

and the Euler integral of the first kind:

$$\int_0^1 (1-v)^{x-1} v^{y-1} \mathrm{d}v = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \forall x, y > 0. \tag{175}$$

Finally, the digamma function ϕ is defined as the logarithmic derivative of the gamma function:

$$\phi(z) := \frac{\Gamma'(z)}{\Gamma(z)}. \tag{176}$$

It has a unique positive zero at $z \approx 1.461\,6312\ldots$ and $\phi(z)$ is positive for all z-values larger than this number. It will be particularly important to us that ϕ is an increasing function of $z > 0$:

$$\phi'(z) > 0. \tag{177}$$

Appendix B. Proof of theorem 3.1

We first use [19, theorem 2.2], where the following normal form (with unfolding parameter λ) is provided

$$\begin{aligned}\dot{x} &= \lambda - x^2,\\ \dot{y} &= -y\left(1 - a^\lambda x\right) + g^\lambda\left(x,y\right),\end{aligned}$$

with

$$g^\lambda\left(x,y\right) = \left(\lambda - x^2\right) f^\lambda\left(x\right) + y^2 u^\lambda\left(x,y\right). \tag{178}$$

We focus on the singularity side of the bifurcation, i.e. $\lambda \geqslant 0$. The normal form is then analytic with respect to $\sqrt{\lambda} \geqslant 0$. Notice in comparison with [19, theorem 2.2] that we reverse the direction of time and have replaced their $(a(\lambda), o(y))$ by $(-a^\lambda, -y^2 u^\lambda(x,y))$, respectively. We then put

$$x =: -\widetilde{x} + \sqrt{\lambda},$$

so that

$$\begin{aligned}\dot{\widetilde{x}} &= \widetilde{x}\left(\widetilde{x} - 2\sqrt{\lambda}\right),\\ \dot{y} &= -y\left(1 - a^\lambda\sqrt{\lambda} + a^\lambda \widetilde{x}\right) + \widetilde{g}^\lambda\left(\widetilde{x},y\right),\end{aligned}$$

where $\widetilde{g}^\lambda$ is obtained from (178) and takes the same form (see (181)). We proceed to drop the tildes and then define (new tildes)

$$\epsilon := \frac{2\sqrt{\lambda}}{\kappa(\sqrt{\lambda})}, \quad \widetilde{x} := \frac{1}{\kappa(\sqrt{\lambda})} x. \tag{179}$$

where

$$\kappa(\sqrt{\lambda}) := 1 - a^\lambda \sqrt{\lambda},$$

which are all well-defined for all $0 \leqslant \lambda \ll 1$. We then obtain that

$$\begin{aligned}\widetilde{x}' &= \widetilde{x}\left(\widetilde{x} - \epsilon\right),\\ y' &= -y\left(1 + \widetilde{a}^\epsilon \widetilde{x}\right) + \widetilde{g}^\epsilon\left(\widetilde{x},y\right),\end{aligned} \tag{180}$$

after dividing the right hand side by $\kappa(\sqrt{\lambda})$. This corresponds to a reparameterisation of time. Here we have defined

$$\widetilde{a}^\epsilon = a^{\lambda(\epsilon)}, \quad \widetilde{g}^\epsilon\left(\widetilde{x},y\right) = \widetilde{x}\left(\widetilde{x} - \epsilon\right)\widetilde{f}^\epsilon\left(\widetilde{x}\right) + y^2 \widetilde{u}^\epsilon\left(x,y\right), \tag{181}$$

and used that the first equation in (179) can be (analytically) inverted for $\sqrt{\lambda} = \sqrt{\lambda}(\epsilon)$, $\sqrt{\lambda}(0) = 0$, $\frac{\mathrm{d}\sqrt{\lambda}}{\mathrm{d}\epsilon}(0) = \frac{1}{2}$. We then drop the tildes again.

Now, in order to achieve the desired normal form in theorem 3.1, we then apply three elementary transformations (T1)–(T3) to obtain (180) (with tildes dropped) with g^ϵ given by (21) and satisfying (23), repeated here for convenience:

$$g^\epsilon(x,y) = f^\epsilon(x) + u^\epsilon(x,y)$$
$$f^\epsilon(x) = \sum_{k=2}^{\infty} f_k^\epsilon x^k, \quad u^\epsilon(x,y) = \sum_{k=2}^{\infty} u_{k,1}^\epsilon x^k y + \sum_{k=1}^{\infty}\sum_{l=2}^{\infty} u_{k,l}^\epsilon x^k y^l, \tag{182}$$

and

$$|f_k^\epsilon| \leqslant B\rho^{-k}, \quad |u_{k,l}^\epsilon| \leqslant \mu\rho^{-k-l} \quad \text{and} \quad u_{k,1}^0 = 0 \quad \forall k,l \in \mathbb{N}, \epsilon \in [0,\epsilon_0), \tag{183}$$

respectively. The purposes of each of these successive transformations are:

(T1) Remove the x-linear term of g^ϵ from (181) ($-\epsilon x f^\epsilon(0)$) on the right hand side of the y-equation in (180).
(T2) Remove the y-linear term of the resulting nonlinearity $g^\epsilon = f^\epsilon + u^\epsilon$ obtained after application of (T1) for $\epsilon = 0$: $u_{k,1}^0 = 0$ for all $k \in \mathbb{N}\setminus\{1\}$, see the last condition in (183).
(T3) Remove the $x=0$ part of the resulting nonlinearity $g^\epsilon = f^\epsilon + u^\epsilon$ obtained after application of (T2): $u_{0,l}^\epsilon = 0$ for all $l \in \mathbb{N}\setminus\{1\}$ and all $\epsilon \in [0,\epsilon_0)$, see (182).

For (T1), we define $\widetilde{y} = y + \epsilon\frac{f^\epsilon(0)}{1-\epsilon}x$. This gives the following system

$$\begin{aligned} \dot{x} &= (x-\epsilon)x, \\ \dot{\widetilde{y}} &= -\widetilde{y}(1+\widetilde{a}^\epsilon x) + \widetilde{g}^\epsilon(x,\widetilde{y}), \end{aligned} \tag{184}$$

with

$$\widetilde{g}^\epsilon(x,\widetilde{y}) = \sum_{k=2}^{\infty}\widetilde{f}_k^\epsilon x^k + \sum_{k=2}^{\infty}\widetilde{u}_{k,1}^\epsilon x^k y + \sum_{k=0}^{\infty}\sum_{l=2}^{\infty}\widetilde{u}_{k,l}^\epsilon x^k y^l, \tag{185}$$

and

$$\widetilde{a}^\epsilon = a^\epsilon + 2\epsilon\frac{f^\epsilon(0)}{1-\epsilon}h^\epsilon(0,0).$$

This completes (T1). We drop the tildes.

For (T2), we introduce a new x-fibered diffeomorphism defined by

$$(x,y) \mapsto \widetilde{y} = \mathrm{e}^{-\psi(x)}y, \quad \psi(x) := \sum_{k=2}^{\infty}\frac{u_{k,1}^0}{k-1}x^{k-1} \Longrightarrow x^2\psi'(x) = \sum_{k=2}^{\infty}u_{k,1}^0 x^k.$$

A simple calculation then shows that in the new $(x,\widetilde{y})$-coordinates, we obtain a system of the form (184) with $\widetilde{g}^\epsilon$ given by (185), for a new $\widetilde{f}^\epsilon$ and a new $\widetilde{u}^\epsilon$ now satisfying $u_{k,1}^0 = 0$ for all $k \in \mathbb{N}$, upon dropping the tildes. This completes (T2).

Now, finally for (T3) we analytically linearise the $x=0$-subsystem: There is a locally defined analytic near-identity diffeomorphism $y \mapsto \widetilde{y} = \psi^\epsilon(y)$, $\psi^\epsilon(0) = 0$, $\frac{\mathrm{d}}{\mathrm{d}y}\psi^\epsilon(0) = 1$, depending analytically on $\epsilon \in [0,\epsilon_0)$, such that

$$\dot{y} = -y + \sum_{l=2}^{\infty}u_{0,l}^\epsilon y^l \Longrightarrow \dot{\widetilde{y}} = -\widetilde{y}.$$

In the coordinates $(x, \tilde{y})$, we therefore obtain the desired form (180) with g^ϵ given by (182) and satisfying (183) upon dropping the tildes a final time. In particular, the estimates in (183) follow from Cauchy's estimate for all $\rho > 0$ small enough.

ORCID iDs

Kristian Uldall Kristiansen https://orcid.org/0000-0001-6090-7649
Peter Szmolyan https://orcid.org/0000-0002-0595-9320

References

[1] Baldomá I, Castejón O and Seara T M 2013 Exponentially small heteroclinic breakdown in the generic Hopf-Zero singularity *J. Dyn. Differ. Equ.* **25** 335–92
[2] Balser W 1994 *From Divergent Power Series to Analytic Functions: Theory and Application of Multisummable Power Series* (Springer)
[3] Benoît E 1990 Canards et enlacements *Publ. Math. Inst. Hautes Etudes Sci.* **72** 63–91
[4] Bonckaert P and De Maesschalck P 2008 Gevrey normal forms of vector fields with one zero eigenvalue *J. Math. Anal. Appl.* **344** 301–21
[5] De Maesschalck P and Kenens K 2020 Gevrey asymptotic properties of slow manifolds *Nonlinearity* **33** 341–87
[6] De Maesschalck P and Kristiansen K U 2024 On k-summable normal forms of vector fields with one zero eigenvalue (arXiv:2410.20854)
[7] Dumortier F, Llibre J and Artes J C 2006 *Qualitative Theory of Planar Differential Systems* (Springer)
[8] Ecalle J 1981 *Les Fonctions Resurgentes, Vols. 1 & 3* vol 81 (Publ. Math. d'Orsay)
[9] Fenichel N 1979 Geometric singular perturbation theory for ordinary differential equations *J. Differ. Equ.* **31** 53–98
[10] Hayes M G, Kaper T J, Szmolyan P and Wechselberger M 2015 Geometric desingularization of degenerate singularities in the presence of fast rotation: a new proof of known results for slow passage through Hopf bifurcations *Indag. Math.* **27** 1184–203
[11] Jones C K R T 1995 Geometric singular perturbation theory *Dynamical Systems (Montecatini Terme)* (*Lecture Notes in Mathematics*) (Springer)
[12] Kristiansen K U 2024 Improved Gevrey-1 estimates of formal series expansions of center manifolds (arXiv:2410.08088)
[13] Loray F 2004 Versal deformation of the analytic saddle-node *Asterisque* **297** 167–87
[14] Martinet J and Ramis J-P 1983 Classification analytique des équations différentielles non linéaires résonnantes du premier ordre *Ann. Sci. Éc. Norm. Supér.* **16** 571–621
[15] Merle F, Raphaël P, Rodnianski I and Szeftel J 2022 On blow up for the energy super critical defocusing nonlinear Schrödinger equations *Invent. Math.* **227** 247–413
[16] Merle F, Raphaël P, Rodnianski I and Szeftel J 2022 On the implosion of a compressible fluid I: smooth self-similar inviscid profiles *Ann. Math.* **196** 567–778
[17] Merle F, Raphaël P, Rodnianski I and Szeftel J 2022 On the implosion of a compressible fluid II: singularity formation *Ann. Math.* **196** 779–889
[18] Olver F, Lozier D, Boisvert R and Clark C 2010 *The NIST Handbook of Mathematical Functions* (Cambridge University Press)
[19] Rousseau C 2005 Modulus of orbital analytic classification for a family unfolding a saddle-node *Mosc. Math. J.* **5** 245–68
[20] Rousseau C 2023 The equivalence problem in analytic dynamics for 1-resonance *Arnold Math. J.* **9** 1–39
[21] Stokes G G 1902 On the discontinuity of arbitrary constants that appear as multipliers of semi-convergent series. (A letter to the editor.) *Acta Math.* **26** 393–7
[22] van Strien S J 1979 Center manifolds are not C^∞ *Math. Z.* **166** 143–6
[23] Szmolyan P and Wechselberger M 2001 Canards in $\mathbb{R}^3$ *J. Differ. Equ.* **177** 419–53
[24] Voronin S M 1981 Analytic classification of germs of conformal-mappings $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with identity linear part *Funct. Anal. Appl.* **15** 1–13