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Published in:
IEEE Transactions on Signal Processing

Link to article, DOI:
10.1109/78.277812

Publication date:
1993

Document Version
Publisher's PDF, also known as Version of record

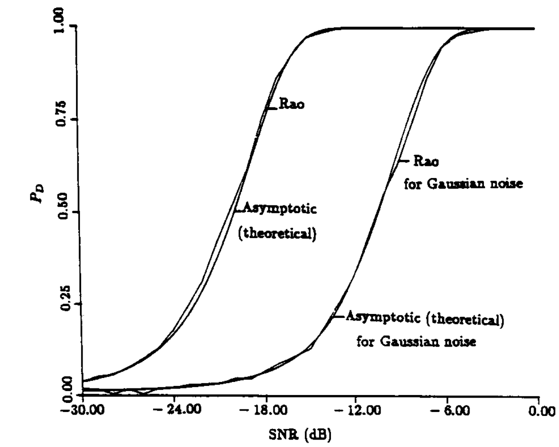

Fig. 8. Performance of the Gaussian and mixed-Gaussian Rao detectors for mixed-Gaussian noise process II.

as a result of the Gaussian assumption of the driving noise. It is also observed that the performance of the Gaussian Rao detector is similar to the predicted performance of a Gaussian detector in *Gaussian* noise having equivalent variance. This makes intuitive sense since the AR process parameters derived by the covariance method have an asymptotic distribution invariant to the underlying driving noise PDF. This makes the Gaussian Rao statistic invariant to the PDF also. Figs. 7 and 8 show a constant difference of approximately 10 dB between the performance of the Rao detector and the Gaussian Rao detector. The theoretical difference is $10 \log_{10} \sigma^2 I_f = 9.6$ dB.

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Three Different Criteria for the Design of Two-Dimensional Zero Phase FIR Digital Filters

Eyjolfur Gislason, Marnar Johansen, Knut Conradsen, Bjarne K. Ersbøll, and Søren Kruse Jacobsen

***Abstract*—A new error criterion for the design of FIR filters is proposed. Filters with relatively many free filter coefficients are designed using the Chebyshev, the WLS, and a new partitioned minimax error criterion and the performance of the filters is compared. A general and fast technique for the WLS design is also presented.**

I. Introduction

Techniques for designing two-dimensional digital filters have been investigated for several years and much effort has been spent on designing filters which are optimal in the Chebyshev (minimax) sense. By using the least squares error criterion for the design tasks, one typically gets an overshoot at the edge of the passband. This can be avoided using the Chebyshev criterion.

The computational difficulties with a minimax design has restricted its use in the literature to, e.g., 45 free filter coefficients in a 17×17 circularly symmetric filter [3]. The design of larger filters is very difficult due to computing time and ill-conditioning of matrices.

Algazi and Suk [2] present a nonanalytical solution to the WLS design problem, whereas Ahmad and Wang's [1] extremely fast algorithm is restricted to the nonweighted least squares design. Hu and Rabiner [5] use linear programming (LP) with an extremely long execution time for the Chebyshev design. Harris and Mersereau [4] compare a number of other algorithms for the optimal minimax design based on the single exchange ascent algorithm. Another approach to optimal FIR filter design is to use the l_p-norm as the error criterion. Lodge and Fahmy [8] use the method of parallel tangents and Lampropoulos and Fahmy [7] use an N-step Newton method. Charalambous [3] uses a minimax algorithm based on a least squares method with an embedded conjugate gradient algorithm.

In this paper we concentrate on designing higher order filters with at least 60 free filter coefficients as opposed to the 15 [5], 21 [4], and 45 [3] presented in the references mentioned.

We present a technique which extends the range of the LP approach [5] for the optimal Chebyshev design of filters with size up to 13×13 corresponding to 85 free filter coefficients. The criterion is a partitioned version of the Chebyshev criterion where a sum of local maxima is minimized rather than minimizing the maximum error over the entire frequency plane. This problem is also solved by LP. Using the new criteria filters with a maximum size of 17×17 have been designed (145 free filter coefficients).

We also present a weighted least squares design technique, which has not (to our knowledge) been presented before. The technique

Manuscript received September 25, 1990; revised September 21, 1992. This work was supported in part by the Danish Technical Research Council under Grant 5.26.09.07 (MOBS project).

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IEEE Log Number 9210942.

yields an analytical weighted least squares solution for *any* FIR filter (even with complex coefficients). This approach has successfully been used for filters up to 33 × 33 (544 free filter coefficients). The approach seems intuitively more appealing than the approach presented in [2] because an analytical solution is obtained without iterative operations.

II. Statement of the Problem

Let $H(\Omega_1, \Omega_2)$ be the transfer function of a two-dimensional FIR digital filter

$$H(\Omega_1, \Omega_2) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} a(k, l) e^{-j(k\Omega_1 + l\Omega_2)} \tag{1}$$

where N is the order of the filter and $a(k, l)$ are the, in general complex, filter coefficients. A more special form that encompasses the special cases of symmetry and antisymmetry is

$$H(\Omega_1, \Omega_2) = \sum_{i=1}^{K} a_i f_i(\Omega_1, \Omega_2) \tag{2}$$

where $a_i = a(k, l)$ and K is the number of free filter coefficients. The specifications of $f_i(\Omega_1, \Omega_2)$ determines the type of symmetry.

Let the desired amplitude characteristic $G(\Omega_1, \Omega_2)$ be sampled on a discrete set of frequency points $(\Omega_{1_p}, \Omega_{2_q}) \in \mathfrak{F}$, where $\mathfrak{F}$ is the finite set of frequency points, and indexes (p, q) identify the elements in $\mathfrak{F}$. We shall also consider a partitioning of $\mathfrak{F}$ in disjoint sets $\mathfrak{F}_i$. Restricting the domain of $H(\Omega_1, \Omega_2)$ to $\mathfrak{F}$, (2) becomes

$$H(\Omega_{1_p}, \Omega_{2_q}) = \sum_{i=1}^{K} a_i f_i(\Omega_{1_p}, \Omega_{2_q}) = \boldsymbol{c}_{pq}^T \boldsymbol{a} \tag{3}$$

with obvious definitions of $\boldsymbol{c}_{pq}$ and $\boldsymbol{a}$.

Let $G(\Omega_{1_p}, \Omega_{2_q}) = G_{pq}$ be the desired frequency response and let $W(\Omega_{1_p}, \Omega_{2_q}) = W_{pq}$ be an appropriate nonnegative real weighting function. An error function can then be defined as

$$E_{pq} = E(\Omega_{1_p}, \Omega_{2_q}) = W_{pq}[G_{pq} - \boldsymbol{c}_{pq}^T \boldsymbol{a}], \qquad (\Omega_{1_p}, \Omega_{2_q}) \in \mathfrak{F}. \tag{4}$$

If G_{pq} is real and center symmetrical, $G(\Omega_{1_p}, \Omega_{2_q}) = G(-\Omega_{1_p}, -\Omega_{2_q})$, the filter coefficients must be center symmetrical. In this case E_{pq} is real. If, on the other hand, G_{pq} is imaginary and center antisymmetrical, $G(\Omega_{1_p}, \Omega_{2_q}) = -G(-\Omega_{1_p}, -\Omega_{2_q})$, the filter coefficients must be center antisymmetrical. In this case E_{pq} is imaginary.

In this paper the following three criteria for determining the filter coefficients are considered.

- Find the filter coefficients $\boldsymbol{a}$ such that the Chebyshev-norm is minimized

$$\min_{\boldsymbol{a}} \{ \max_{(\Omega_{1_p}, \Omega_{2_q}) \in \mathfrak{F}} |E(\Omega_{1_p}, \Omega_{2_q})| \}. \tag{5}$$

- Find the filter coefficients $\boldsymbol{a}$ such that a partitioned minimax-norm is minimized

$$\min_{\boldsymbol{a}} \left\{ \sum_i \max_{(\Omega_{1_p}, \Omega_{2_q}) \in \mathfrak{F}} |E(\Omega_{1_p}, \Omega_{2_q})| \right\}. \tag{6}$$

- Find the filter coefficients $\boldsymbol{a}$ such that the squared error D is minimized

$$\min_{\boldsymbol{a}} \left\{ D = \sum_{(\Omega_{1_p}, \Omega_{2_q}) \in \mathfrak{F}} |E(\Omega_{1_p}, \Omega_{2_q})|^2 \right\}. \tag{7}$$

III. Linear Programming

In this section, two LP models for FIR filter design are formulated.

In the *partitioned minimax-norm*, the sum of the maximum errors in each subset $\mathfrak{F}_i$ is minimized. By minimizing the sum of some positive variables ξ_i, we can obtain an approximation to the desired characteristic $G(\Omega_{1_p}, \Omega_{2_q})$ as follows:

$$\min \sum_i \xi_i \tag{8}$$

$$E(\Omega_{1_p}, \Omega_{2_q}) \le \xi_i \;\forall (p, q) \in \mathfrak{F}_i \tag{9}$$

$$-E(\Omega_{1_p}, \Omega_{2_q}) \le \xi_i \;\forall (p, q) \in \mathfrak{F}_i \tag{10}$$

for all i. The LP approximation problem can be written

$$\min \sum_i \xi_i \tag{11}$$

subject to

$$-W_{pq} \boldsymbol{c}_{pq}^T \boldsymbol{a} - \xi_i \le -W_{pq} G_{pq} \;\forall (p, q) \in \mathfrak{F}_i \tag{12}$$

$$W_{pq} \boldsymbol{c}_{pq}^T \boldsymbol{a} - \xi_i \le W_{pq} G_{pq} \;\forall (p, q) \in \mathfrak{F}_i \tag{13}$$

$$-\infty < \boldsymbol{a} < \infty \tag{14}$$

$$\xi_i \ge 0 \tag{15}$$

for all i. If we only have one $\mathfrak{F}_i = \mathfrak{F}$ we obtain the usual Chebyshev norm, and the corresponding LP problem.

The LP problems above are atypical in many respects. The matrix is fully dense, and the basis matrix tends to be ill-conditioned. The number of iterations used, which is much larger than for normal LP problems, is shown in Table I.

IV. An Analytical Least Squares Solution

Using the short-band notation D for $D(\boldsymbol{a})$ and $\Sigma_{p,q}$ instead of $\Sigma_{(\Omega_{1_p}, \Omega_{2_q}) \in \mathfrak{F}}$ the problem (7) can be rewritten as

$$\min D = \sum_{p,q} W_{pq}[G_{pq} - \boldsymbol{c}_{pq}^T \boldsymbol{a}]^2. \tag{16}$$

The gradient vector of D with respect to $\boldsymbol{a}$ is found by differentiating (16)

$$\begin{aligned} \nabla D &= \sum_{p,q} 2W_{pq}[G_{pq} - \boldsymbol{c}_{pq}^T \boldsymbol{a}](-\boldsymbol{c}_{pq}^T) \\ &= \boldsymbol{a}^T 2 \sum_{p,q} W_{pq} \boldsymbol{c}_{pq} \boldsymbol{c}_{pq}^T - 2 \sum_{p,q} W_{pq} G_{pq} \boldsymbol{c}_{pq}^T. \end{aligned} \tag{17}$$

A necessary condition for D being minimal is that the gradient vector (17) is zero.

$$\nabla D = \boldsymbol{0}^T \Leftrightarrow \boldsymbol{a}^T \boldsymbol{Q} = \boldsymbol{b}^T \tag{18}$$

where

$$\boldsymbol{Q} \equiv 2 \sum_{p,q} W_{pq} \boldsymbol{c}_{pq} \boldsymbol{c}_{pq}^T \tag{19}$$

is a $K \times K$ matrix and

$$\boldsymbol{b}^T \equiv 2 \sum_{p,q} W_{pq} G_{pq} \boldsymbol{c}_{pq}^T \tag{20}$$

is a vector of dimension K. The condition $\nabla D = \boldsymbol{0}^T$ is also sufficient since D is convex. Singularity of $\boldsymbol{Q}$ is avoided by increasing the number of frequency points in $\mathfrak{F}$. The *design procedure* is very simple: Given the desired frequency response G_{pq} and a nonnegative weighting function W_{pq} a weighted least squares solution is

TABLE I
MAXIMUM ERROR, NUMBER OF ITERATIONS NEEDED AND NUMBER OF ROWS IN THE LP-MODEL FOR DIFFERENT NUMBER OF ξ'S FOR SOLVING THE EVEN 13 × 13 FILTER PROBLEM

	Number of ξ's	Max. error	Number of Iterations	Rows in LP-model
Chebyshev crit.	1	0.0905	5688	86
Partitioned Chebyshev crit.	2	0.1049	4728	87
	8	0.1383	2559	93
	18	0.1523	1972	103
	32	0.2954	1611	117
	50	0.3063	1851	135

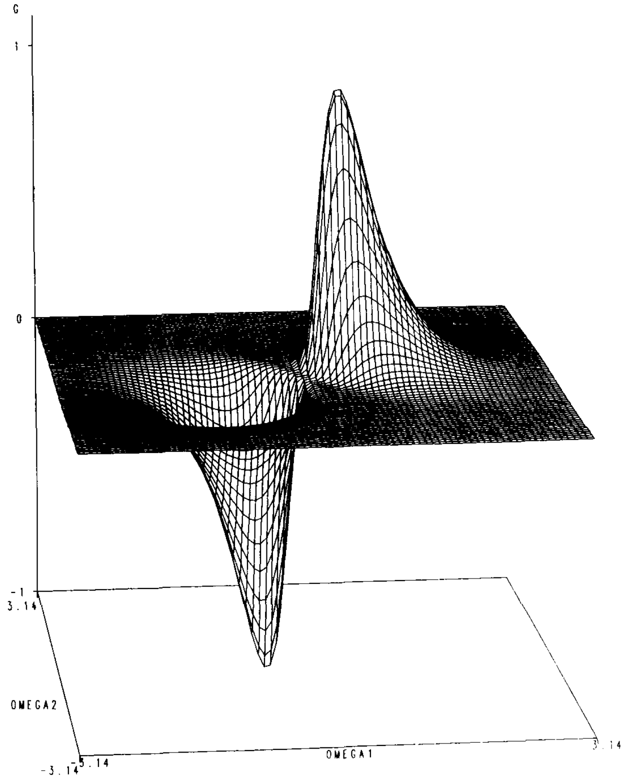

Fig. 1. Perspective plot of the desired frequency response of the filter (odd part).

obtained by the following steps: 1) calculate the Hessian matrix $\boldsymbol{Q}$ using (19); 2) calculate the right-hand side $\boldsymbol{b}$ using (20); and 3) solve $\boldsymbol{Qa} = \boldsymbol{b}$ for the filter coefficients $\boldsymbol{a}$.

Practical experience with this approach has been encouraging, no numerical problems were encountered for filters of size up to 33 × 33, and the $\boldsymbol{Q}$ matrix tends to be well conditioned.

V. DESIGN EXAMPLE

We have designed a number of filter pairs approximating the specifications of Knutsson and Granlund [6] for detection of local orientation using all three criteria. The designed frequency response of the odd part of the filter is shown in Fig. 1. All weights were equal to one in the examples.

Fig. 2 shows the approximation (odd filter) using the Chebyshev error criterion. Fig. 3 shows the corresponding approximation using the partitioned Chebyshev error criterion with 50 ξ's and Fig. 4 shows the approximation using the WLS error criterion. The filter size is 13 × 13 for these filters which corresponds to 84 free filter

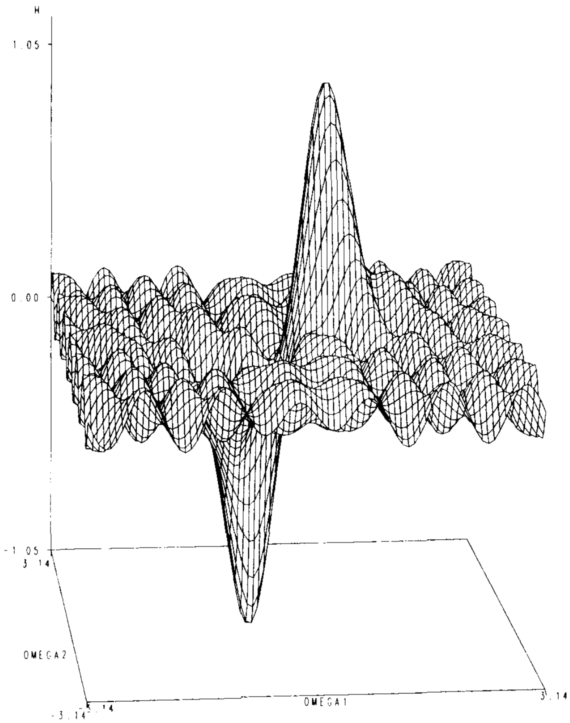

Fig. 2. Perspective plot of the frequency response of the designed odd 13 × 13 filter using the Chebyshev criterion.

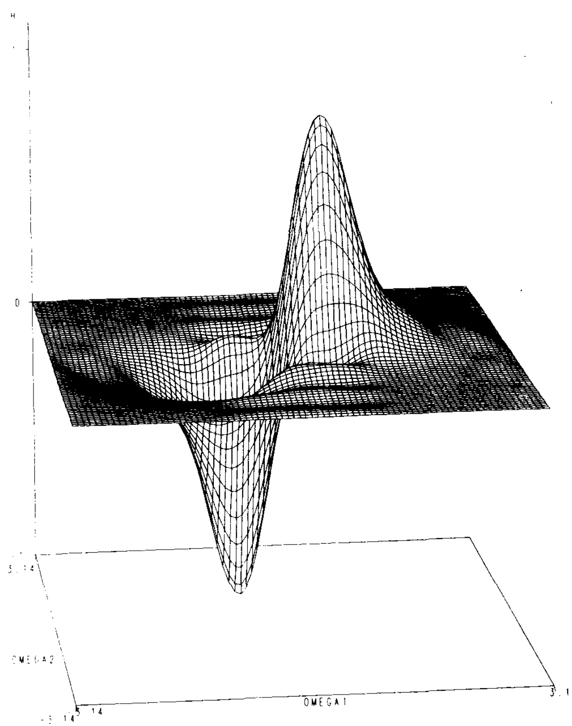

Fig. 3. Perspective plot of the frequency response of the designed odd 13 × 13 filter using the partitioned Chebyshev criterion where the number of subsets is 50.

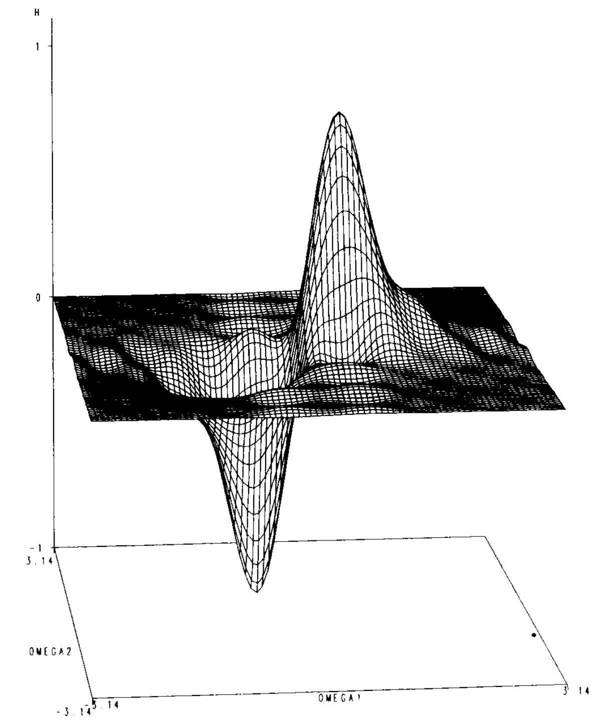

Fig. 4. Perspective plot of the frequency response of the designed odd 13 × 13 filter using the WLS criterion.

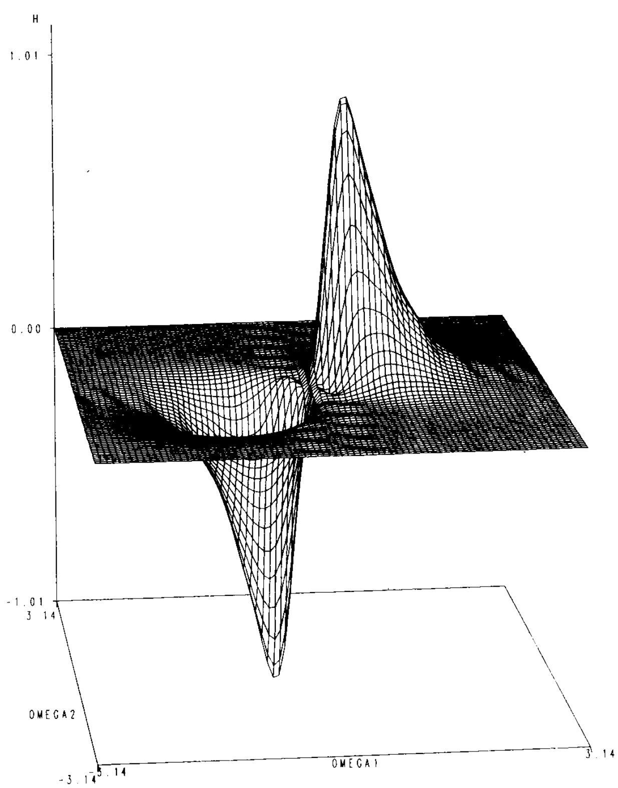

Fig. 5. Perspective plot of the frequency response of the designed odd 25 × 25 filter using the WLS criterion.

TABLE 2
PERFORMANCES FOR THE APPROXIMATED FILTERS USING THE THREE DESIGN CRITERIA. THE MINIMUM DEVIATION FOR EACH FILTER SIZE AND NOISE COMBINATION IS MARKED IN BOLD

	Noise		Deviation in degrees						
Filter	in	in	Chebyshev	Partitioned Chebyshev crit.					WLS
Size	%	dB	crit.	2ξ	8ξ	18ξ	32ξ	50ξ	crit.
11 x 11	0	∞	11.28	11.28	**4.59**	9.96	15.72	13.27	14.93
	5	20	14.15	14.15	**8.70**	13.21	17.54	15.17	15.78
	9	15	16.19	16.19	**13.70**	17.04	20.94	17.85	19.38
	16	10	19.26	19.26	**19.19**	21.31	23.70	22.94	23.47
	28	5	**24.80**	**24.80**	27.08	27.87	29.95	25.88	28.20
	50	0	30.30	**30.28**	41.06	40.08	41.50	39.55	40.73
13 x 13	0	∞	4.28	5.78	3.82	2.56	2.46	**1.54**	1.86
	5	20	4.82	6.08	4.05	8.42	4.95	**1.98**	2.23
	9	15	5.27	6.74	7.67	12.29	10.40	**3.00**	3.94
	16	10	7.04	7.81	12.18	19.25	15.37	7.15	**6.86**
	28	5	10.51	12.33	19.61	26.53	21.57	**10.44**	11.68
	50	0	13.78	18.55	24.65	36.53	29.58	15.37	17.29
25 x 25	0	∞	-	-	-	-	-	-	0.63
	5	20	-	-	-	-	-	-	0.99
	9	15	-	-	-	-	-	-	1.53
	16	10	-	-	-	-	-	-	2.62
	28	5	-	-	-	-	-	-	6.00
	50	0	-	-	-	-	-	-	8.59

coefficients. Fig. 5 shows the approximation of the reference function using the WLS criterion with filter size 25 × 25 corresponding to 312 free filter coefficients.

The approximated filters are tested on the same test image as in Knutsson and Granlund [6] with various amounts of Gaussian noise added. The results are shown in Table II. The deviation in degrees is a comparable performance measure for the filters with respect to the application of local orientation estimation. The best filter, i.e., the filter showing a minimum deviation is marked with boldface characters. The design times varied a lot. For the two 13 × 13 filters the computing times were 9.6 h (5688 LP-iterations) for the Chebyshev design, a third of this for the partitioned Chebyshev design (Table I), and 1.5 min for the WLS design. The memory requirements were 4.4 Mb for the Chebyshev-design and 60 kb for the WLS design.

Increasing the filter size to 15 × 15 the Chebyshev design did not converge after 116 h (41000 LP-iterations) whereas the WLS filter was designed in 2 min. Increasing the filter size to 25 × 25 results in an WLS design time of 17 min, and for a 33 × 33 filter the execution time was 1 h.

It is noteworthy that in this application the best filters up to size 13 × 13 were obtained with the partitioned Chebyshev criterion with one tié and one exception where the WLS was the best. For the larger kernels solutions were only obtained for the WLS model.

VI. CONCLUSION

We have presented a new error criterion, the partitioned Chebyshev criterion for the "numerical" design of FIR filters. Furthermore a WLS technique for "analytical" design was presented.

The partitioned criterion leads to a considerable improvement in computing time when compared to the ordinary Chebyshev setup. In the examples considered the performance of the resulting filters is better than the performance of the "ordinary" Chebyshev designed filters. Even for large filters the WLS design works very fast and produces (in the case considered) reasonably good filters.

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Regularized Image Reconstruction Using SVD and a Neural Network Method for Matrix Inversion

Ronald J. Steriti and Michael A. Fiddy

***Abstract*—We compare two methods of matrix inversion when used in an image reconstruction algorithm. The first is based on energy minimization using a Hopfield neural network; this is compared with the inverse obtained using singular value decomposition (SVD). We show with a practical example that the neural network provides a more useful and robust matrix inverse.**

I. Introduction

Many image reconstruction problems require the calculation of a matrix inverse which can be ill-conditioned and time consuming to calculate reliably. Regularization techniques are commonly used to overcome these difficulties but determining an optimal value for the regularization parameter is not straightforward. Singular value decomposition is the most systematic approach for calculating a matrix inverse and it suffers from this problem. To overcome these difficulties, the matrix inversion problem was mapped onto a Hopfield neural network. We wished to determine whether this approach would provide a useful and robust inverse for our purposes.

The relationship between the matrix inversion calculated by a Hopfield neural network and that calculated using a regularized SVD algorithm was investigated in two ways. One was to find the relationship between the singular values of the inverses calculated. The second was to evaluate their use in an image reconstruction algorithm. The image reconstruction method used was the PDFT (Priorized Discrete Fourier Transform) algorithm [1], [2].

Manuscript received December 24, 1991; revised December 9, 1992. The associate editor coordinating the review of this paper and approving it for publication was Dr. B. H. Juang.

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IEEE Log Number 9210948.

II. The PDFT Algorithm

We wish to recover an estimate of the original image f from the available data g where

$$g(x) = \int A(x, y)\mathrm{f}(y)\, dy + \text{noise} \tag{1}$$

where A is the transformation kernel. The data may be image or spectral data, the latter being used in the PDFT algorithm. In discrete form we have

$$g = A\mathrm{f} + \text{noise} \tag{2}$$

where g and f are vectors and A is the transformation matrix.

The PDFT estimate is defined by finding a data consistent minimizer of $E = \|\mathrm{f} - \hat{f}\|^2$ in a weighted Hilbert space. The solution vector is expressed as a linear combination of basis functions that span the chosen Hilbert space, i.e.

$$\hat{f}(n) = \sum_{m=1}^{M} c_m \varphi_m(n) \tag{3}$$

where φ_m are given by

$$\phi_m(n) = p(n)\mathrm{e}^{[mn\ 2\pi\ i/N]} \tag{4}$$

$n = 1,2, \cdots N\ m = 1,2, \cdots M\ i = \sqrt{-1}$. The function $p(n)$ is a prior estimate of the function f. The coefficients $\{c_m\}$ minimizing E can be found using the orthogonality principle:

$$\langle f, \varphi_m \rangle - \sum_{n=1}^{M} c_n \langle \varphi_n, \varphi_m \rangle = 0 \tag{5}$$

for all $m = 1, 2, \cdots N$. A regularized form of this expression can be written as $G = (P + \beta I)c$ where we assume that G is the vectorized Fourier data and c the vectorized coefficients. The matrix P necessarily has the elements of the DFT of the prior, p.

Using the PDFT estimator with band-limited Fourier data, (i.e., low pass filtered images) we assume the prior equals one within the image support and zero elsewhere; hence P takes the form of a sinc matrix, $P_{n,m} = \Omega$ sinc Ω $(n - \mathrm{m})$, where Ω is the ratio of the cut-off to the maximum frequency. It is well known that this matrix is highly ill-conditioned. The singular values are one and decrease suddenly (exponentially) to a minimum value of zero. Some form of regularization is necessary to calculate a useful (pseudo) inverse and estimate. For computational purposes, the two dimensional PDFT estimates used in the examples shown in the later sections employ a separable prior, that is $p(x_1, x_2) = p_1(x_1)p_2(x_2)$, with $p_1 = p_2$ in the simplest case.

III. Matrix Inversion

SVD allows direct control over the inversion of a matrix [3]. If $A = U\Sigma W^*$ then $A^{-1} = W\Sigma^{-1}U^*$ where Σ^{-1} is a diagonal matrix with elements $\{\alpha_n^{-1}\}$. When the singular values of A, α_n, are very small the matrix is ill-conditioned. Substituting $\alpha_n/(|\alpha_n|^2 + \beta)$ for α_n^{-1} calculates a regularized inverse of A. Simply removing the zero singular values results in the calculation of the Moore-Penrose inverse; this inverse is unique but may not be optimal for use in an image reconstruction algorithm because small nonzero α_n remain.

The Hopfield neural network has been shown to minimize [4], [5]

$$E = \|Tv - b\|^2 \tag{6}$$