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GENERAL PREDICTIVE CONTROL USING THE DELTA OPERATOR

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Abstract: This paper deals with two discrete-time operators, the conventional forward shift-operator and the δ-operator. Both operators are treated in view of construction of suitable solutions to the Diophantine equation for the purpose of prediction. A general step-recursive scheme is presented. Finally a GPC is formulated and applied adaptively to a continuous-time plant.

Keywords: δ-operator, Diophantine equation, GPC, Prediction

1 Introduction

The increase in use of microprocessors and transputers within process automation during the last decade, has included the need for suitable discrete-time system descriptions at high sampling rates. Similar to the transform theory for continuous-time systems a discrete-time equivalent has been formulated usually involving the shift-operator, although descriptions based on the shift-operator tend to "loose" information at rapid sampling.

This paper deals with k-step predictors of ARMAX models for both the forward shift-operator and the δ-operator notation. This is a known issue for the conventional backward shift operator, see e.g. [3] and [5], however some preliminary results are derived for the forward shift-operator. This includes the advantages of getting results which are simpler to transform and interpret when deriving the expressions in terms of the δ-operator. Since prediction is closely related to the ubiquitous predictive control scheme, the paper describes a step-recursive k-step predictor for the purpose of making a receding-horizon general predictive controller (GPC). To make the GPC more precise the δ-description is included since the conventional GPC control based on the shift-operator often leads to numerical problems due to near common factors in the design polynomials.

The δ-operator is a discrete-time operator having properties similar to those of the continuous-time Laplace operator. In [2] a similar GPC scheme is derived, but based on the approximation $\delta \approx s$ and the results of [4]. In this paper it is believed that the δ-operator should be viewed more as a discrete-time operator and only in the limit process as a unifying operator with a continuous-time counterpart.

2 Discrete-time models

In the paper two discrete-time operators are considered, the forward shift-operator denoted by q and the δ-operator which is defined by

$$\delta = \frac{q-1}{T}, \qquad q = 1 + \delta T \tag{1}$$

T is a positive scalar, often chosen as the sampling period to obtain a unification between continuous and discrete-time results. Relative to the shift-operator, it contributes several advantages like a tighter relation between continuous and discrete-time systems theory and better numerical properties, see e.g. [6], [8], [9].

A unified notation is introduced since the two operators are linearly related and corresponding polynomials have identical polynomial degrees. Thus ξ denotes either of the two operators: q, δ.

2.1 The ARMAX model

Consider the discrete-time ARMAX model

$$A_\xi(\xi)y_t = B_\xi(\xi)u_t + C_\xi(\xi)e_t \tag{2}$$

where e_t is an uncorrelated zero-mean innovation sequence with constant variance and where the polynomial orders are $n = \rho(A_\xi(\xi)) = \rho(C_\xi(\xi))$ and $m = n - d = \rho(B_\xi(\xi))$. The polynomial $A_\xi(\xi)$ contains the open-loop poles. $C_\xi(\xi)$ is chosen apriori by the designer as a model of the disturbances and is required to have stable zeros. The case of $\xi = q$:

$$\begin{aligned} A_q(q) &= q^n + a_{n-1}q^{n-1} + \cdots + a_0 \\ B_q(q) &= b_m q^m + \cdots + b_0, \\ C_q(q) &= q^n + c_{n-1}q^{n-1} + \cdots + c_0 \end{aligned} \tag{3}$$

The case of $\xi = \delta$:

$$\begin{aligned} A_\delta(\delta) &= \frac{1}{T^n} A_q(q)|_{q=1+T\delta} = \delta^n + \cdots + \bar{a}_0 \\ B_\delta(\delta) &= \frac{1}{T^n} B_q(q)|_{q=1+T\delta} = \bar{b}_m\delta^m + \cdots + \bar{b}_0 \\ C_\delta(\delta) &= \frac{1}{T^n} C_q(q)|_{q=1+T\delta} = \delta^n + \cdots + \bar{c}_0 \end{aligned} \tag{4}$$

For an explicit relation between the corresponding polynomials of the two operators, see [10]. Note in the case of $C_q(q) = q^n$, that the corresponding δ-description does not have the same zero coefficients, i.e. $C_\delta(\delta) = (\delta + \frac{1}{T})^n$.

3 Prediction using q

Within the GPC two Diophantine equations are of special interest. The following shows that the structure allows an explicit recursive solution to the problem reducing the computational effort.

3.1 The two Diophantine equations

The linear model of (2) could be considered as the regulation about a particular operation point even for a nonlinear model. A novel method is the receding-horizon method dependent on predicting the plant's output over several steps based on assumptions about future control actions, e.g. [3]. To predict the output k steps ahead ($k = 1, 2, \cdots$) for the model in (2), one can translate the results of [5] and [3]. The Diophantine equation

$$A_q(q)E_q(q) + F_q(q) = q^{k-1}C_q(q) \tag{5}$$

is solved with respect to $E_q(q)$, $F_q(q)$ by use of the polynomial orders

$$\rho(E_q(q)) = k-1, \qquad \rho(F_q(q)) = n-1 \tag{6}$$

Now,

$$y_{t+k} = \frac{qB_q(q)E_q(q)}{C_q(q)}u_t + \frac{qF_q(q)}{C_q(q)}y_t + E_q(q)e_{t+1} \tag{7}$$

The first term consists of past, present and future control actions. In order to distinguish between known past and unknown present or future control actions an another Diophantine equation is stated,

$$C_q(q)G_q(q) + H_q(q) = qE_q(q)B_q(q) \tag{8}$$

where the polynomial orders are

$$\begin{aligned} \rho(G_q(q)) &= \max\{k-d, 0\} \\ \rho(H_q(q)) &= \min\{n-1, m+k\} \end{aligned} \tag{9}$$

The polynomial orders have twofold restrictions, those arisen from the *general* case where u_t affects the prediction and those arisen from the case of $m+k < n-1$ where $G_q(q)$ equals 0 and $H_q(q)$ assigns the right hand side of (8). (8) inserted in (7) gives,

$$y_{t+k} = G_q(q)u_t + \frac{H_q(q)}{C_q(q)}u_t + \frac{qF_q(q)}{C_q(q)}y_t + E_q(q)e_{t+1} \tag{10}$$

where $\rho(H_q(q))$ is strictly less than n, thus the four sources of (10) are

1. Present and future control actions $(u_t, \cdots, u_{t+k-d})$.
2. Past control actions $(u_{t-1}, u_{t-2}, \cdots)$.
3. Present and past output signals $(y_t, y_{t-1}, \cdots)$.
4. Innovations sequence $(e_{t+1}, \cdots, e_{t+k})$.

The optimal k-step prediction thus becomes

$$\hat{y}_{t+k} = G_q(q)u_t + \frac{H_q(q)}{C_q(q)}u_t + \frac{qF_q(q)}{C_q(q)}y_t \tag{11}$$

with the stationary error $\tilde{y}_{t+k} = E_q(q)e_{t+1}$. The variance of the stationary error equals,

$$\mathrm{V}\{\tilde{y}_t\} = (1 + (e_{k-2})^2 + \cdots + (e_0)^2)\mathrm{V}\{e_t\} \tag{12}$$

3.2 A recursive solution

Although (5) and (8) could be solved using the extended Euklidian algorithm, a recursive scheme is suggested based on the sparsity of the underlying Sylvester matrix, see e.g. [3].
The approach is, one seeks successively the Diophantine solutions for the prediction horizons $j = 1, 2, \cdots, k$, since the solution to the $j+1$-step prediction is closely related to that of the j-step prediction, so that a recursive scheme can be organized.
From (5):

$$\begin{aligned} j &: \; A_q(q)E_q^j(q) + F_q^j = q^{j-1}C_q(q) \\ j+1 &: \; A_q(q)E_q^{j+1}(q) + F_q^{j+1} = q^jC_q(q) \end{aligned} \tag{13}$$

Here a superscript denotes the prediction horizon which the polynomial is in correspondence with. Subtraction in (13) and introducing of

$$\begin{aligned} E_q^j(q) &= q^{j-1} + e_{j-2}^jq^{j-2} + \cdots + e_0^j \\ F_q^j(q) &= f_{n-1}^jq^{n-1} + \cdots + f_0^j \end{aligned} \tag{14}$$

results in the expression,

$$E_q^{j+1}(q) - qE_q^j(q) = e_0^{j+1} + qR_q(q) \tag{15}$$

where $\rho(R_q(q)) = j-1$. It is clear that $R_q(q) = 0$, so that

$$A_q(q)e_0^{j+1} + F_q^{j+1}(q) - qF_q^j(q) = 0 \tag{16}$$

A comparison of the coefficients to the powers of q^i, $i = 0, \cdots, n-1$ results in a recursion of $F_q^{j+1}(q)$ given $F_q^j(q)$:

$$\begin{aligned} e_0^{j+1} &= f_{n-1}^j \\ f_i^{j+1} &= f_{i-1}^j - a_ie_0^{j+1}, \qquad f_{-1}^j = 0 \\ & \quad i = 0, 1, \cdots, n-1 \end{aligned} \tag{17}$$

and from (15),

$$e_i^{j+1} = e_{i-1}^j, \qquad i = 1, \cdots, j \tag{18}$$

The recursion is initiated with $E_q^1 = 1$, $F_q^1 = C_q(q) - A_q(q)$. In case of an ARX-model the solution to the second Diophantine equation follows directly. In the general case a scheme similar to the preceding one must be generated. From (8) the expressions for the j and $j+1$ step predictions are derived, hence

$$\begin{aligned} & C_q(q)(G_q^{j+1}(q) - qG_q^j(q)) + H_q^{j+1}(q) \\ & -qH_q^j(q) = qf_{n-1}^jB_q(q) \end{aligned} \tag{19}$$

Denoting the coefficients in accordance with (14),

$$\begin{aligned} G_q^j(q) &= g_{k-d}^jq^{k-d} + \cdots + g_0^{k-d} \\ H_q^j(q) &= h_{n-1}^jq^{n-1} + \cdots + h_0^j \end{aligned} \tag{20}$$

then use of (19),

$$G_q^{j+1}(q) - qG_q^j(q) = g_0^{j+1} + qR_q(q) \tag{21}$$

where $\rho(R_q(q)) = j-d$ and $R_q(q) = 0$. Thus,

$$H_q^{j+1}(q) = qH_q^j(q) - C_q(q)g_0^{j+1} + f_{n-1}^jqB_q(q) \tag{22}$$

A comparison of the coefficients to the powers of q^i within (22) gives a recursion of $H_q^{j+1}(q)$ given $H_q^j(q)$. Reminding that $b_i = 0$ for $i > m$, then it follows that

$$g_0^{j+1} = \begin{cases} 0 & j+1 < d \\ h_{n-1}^j + f_{n-1}^j b_{n-1} & j+1 \geq d \end{cases} \qquad (23)$$

and keeping in mind that $h_i^j = 0$ whenever $0 < j < d-1, \quad m+j < i < n$ then:

$$h_0^{j+1} = -c_0 g_0^{j+1} \qquad (24)$$
$$h_i^{j+1} = h_{i-1}^j - g_0^{j+1} c_i + f_{n-1}^j b_{i-1}$$
$$i = 1, \cdots, \text{Min}\{n-1, m+j-1\} \qquad (25)$$

Finally from (21):

$$g_i^{j+1} = g_{i-1}^j, \quad i = 1, \cdots, \text{Max}\{j+1-d, 0\} \qquad (26)$$

The recursion is initiated with

$$m = n-1 \;:\; G_q^1 = b_m, \quad H_q^1 = qB_q(q) - b_m C_q(q)$$
$$m < n-1 \;:\; G_q^1 = 0, \quad H_q^1 = qB_q(q) \qquad (27)$$

4 Prediction using δ

As shown in e.g. [9] a discrete-time model formulated with the δ-operator perceives certain advantages regarding identification and the sensitivity to the accuracy of the transfer function coefficients at high sampling rates. Consequently, to take advantage of the δ-representation it is a possibility to use the δ-operator also in the resolution of the controller design. The following describes the predictive output form within a δ-operator formalism.

4.1 The two Diophantine equations

To predict the output of (2) k steps ahead, $k = 1, 2, \cdots$, two Diophantine equations are given. In order to keep the monicness and a close formalism between the Diophantine equations there is an inconvenience of some odd normalizing terms. Define

$$E_\delta(\delta) = \frac{1}{T^{k-1}} E_q(\delta T + 1) = \delta^{k-1} + \cdots + \bar{e}_0 \qquad (28)$$
$$F_\delta(\delta) = \frac{1}{T^{n+k-1}} F_q(\delta T + 1) = \bar{f}_{n-1}\delta^{n-1} + \cdots + \bar{f}_0$$

A mapping of (5) results in the Diophantine equation,

$$A_\delta(\delta)E_\delta(\delta) + F_\delta(\delta) = (\delta + \frac{1}{T})^{k-1} C_\delta(\delta) \qquad (29)$$

with the polynomial orders

$$\rho(E_\delta(\delta)) = k-1, \qquad \rho(F_\delta(\delta)) = n-1 \qquad (30)$$

Define also,

$$G_\delta(\delta) = \frac{1}{T^k} G_q(\delta T + 1), = \bar{g}_{k-d}\delta^{k-d} + \cdots + \bar{g}_0$$
$$H_\delta(\delta) = \frac{1}{T^{n+k}} H_q(\delta T + 1) = \bar{h}_{n-1}\delta^{n-1} + \cdots + \bar{h}_0 \qquad (31)$$

By mapping of (8),

$$G_\delta(\delta)C_\delta(\delta) + H_\delta(\delta) = (\delta + \frac{1}{T})E_\delta(\delta)B_\delta(\delta) \qquad (32)$$

where the polynomial orders are

$$\rho(G_\delta(\delta)) = \max\{k-d, 0\}$$
$$\rho(H_\delta(\delta)) = \min\{n-1, m+k\} \qquad (33)$$

In accordance with (10) the output is predicted as,

$$y_{t+k} = T^k G_\delta(\delta)u_t + \frac{T^k H_\delta(\delta)}{C_\delta(\delta)} u_t + \frac{(\delta + \frac{1}{T})T^k F_\delta(\delta)}{C_\delta(\delta)} y_t + T^{k-1} E_\delta(\delta) e_{t+1} \qquad (34)$$

where $\rho(H_\delta(\delta))$ is strictly less than n yielding the four different kind of sources. The optimal k-step prediction equals,

$$\hat{y}_{t+k} = T^k G_\delta(\delta)u_t + \frac{T^k H_\delta(\delta)}{C_\delta(\delta)} u_t + \frac{(\delta + \frac{1}{T})T^k F_\delta(\delta)}{C_\delta(\delta)} y_t \qquad (35)$$

with the stationary error

$$\tilde{y}_{t+k} = T^{k-1} E_\delta(\delta) e_{t+1} = E_q(q) e_{t+1} \qquad (36)$$

4.2 A recursive solution

Although (29) and (32) could be solved using the extended Euklidian algorithm, a recursive scheme similar to the one derived for the shift-operator description is presented. From (29),

$$j \;:\; A_\delta(\delta)E_\delta^j(\delta) + F_\delta^j = (\delta + \frac{1}{T})^{j-1} C_\delta(\delta)$$
$$j+1 \;:\; A_\delta(\delta)E_\delta^{j+1}(\delta) + F_\delta^{j+1} = (\delta + \frac{1}{T})^j C_\delta(\delta) \qquad (37)$$

using the notation from the preceding section. Subtraction in (37) and denoting the coefficients in accordance with (14)

$$E_\delta^j(\delta) = \delta^{j-1} + \bar{e}_{j-2}^j \delta^{j-2} + \cdots + \bar{e}_0^j$$
$$F_\delta^j(\delta) = \bar{f}_{n-1}^j \delta^{n-1} + \cdots + \bar{f}_0^j \qquad (38)$$

it follows that

$$E_\delta^{j+1}(\delta) - (\delta + \frac{1}{T})E_\delta^j(\delta) = \bar{e}_0^{j+1} - \frac{\bar{e}_0^j}{T} + \delta R_\delta(\delta) \qquad (39)$$

where $\rho(R_\delta(\delta)) = j-1$. It appears that $R_\delta(\delta) = 0$, so that

$$F_\delta^{j+1}(\delta) = (\delta + \frac{1}{T})F_\delta^j(\delta) - A_\delta(\delta)(\bar{e}_0^{j+1} - \frac{1}{T}\bar{e}_0^j) \qquad (40)$$

A comparison of the coefficients to the powers of δ^i, $i = 0, \cdots, n-1$ gives a recursion of $F_\delta^{j+1}(\delta)$ given $F_\delta^j(\delta)$:

$$\bar{f}_i^{j+1} = \bar{f}_{i-1}^j + \frac{1}{T}\bar{f}_i^j - \bar{a}_i \bar{f}_{n-1}^j, \quad \bar{f}_{-1}^j = 0$$
$$i = 0, 1, \cdots, n-1 \qquad (41)$$

since

$$\bar{e}_0^{j+1} = \frac{1}{T}\bar{e}_0^j + \bar{f}_{n-1}^j$$
$$\bar{e}_i^{j+1} = \bar{e}_{i-1}^j + \frac{1}{T}\bar{e}_i^j, \quad \bar{e}_j^j = 0, \quad i = 1, \cdots, j \qquad (42)$$

The recursion is started with $E_\delta^1 = 1$, $F_\delta^1 = C_\delta(\delta) - A_\delta(\delta)$. Concerning the second Diophantine equation some similar expressions for the j and $j+1$ step predictions are found. Subtraction and collecting terms of δ^i, $i = 0, \cdots, n+j+1$,

$$\begin{aligned} H_\delta^{j+1}(\delta) &= (\delta + \frac{1}{T})H_\delta^j(\delta) - C_\delta(\delta)(\overline{g}_0^{j+1} - \frac{1}{T}\overline{g}_0^j) \\ &\quad + (\delta + \frac{1}{T})\overline{f}_{n-1}^j B_\delta(\delta) \end{aligned} \tag{43}$$

The following recursion appears

$$\begin{aligned} \alpha_j &= \begin{cases} 0 & j+1 < d \\ \overline{b}_{n-1}\overline{f}_{n-1}^j + \overline{h}_{n-1}^j) & j+1 \geq d \end{cases} \\ \overline{h}_0^{j+1} &= \frac{1}{T}\overline{h}_0^j - \alpha_j c_0 + \frac{1}{T}\overline{b}_0\overline{f}_j^{n-1} \\ \overline{h}_i^{j+1} &= \overline{h}_{i-1}^j + \frac{\overline{h}_i^j}{T} - \alpha_j \overline{c}_i + \overline{f}_{n-1}^j \overline{b}_{i-1} + \frac{\overline{f}_{n-1}^j}{T}\overline{b}_i \\ & \quad i = 1, \cdots, \text{Min}\{n-1, m = j+1\} \end{aligned} \tag{44}$$

with the notational abuse of

$$\begin{aligned} \overline{b}_i &= 0, \quad i > m \\ \overline{h}_i^j &= 0, \quad 0 < j < d-1, \quad m+j < i < n \end{aligned} \tag{45}$$

Finally, keeping in mind that $G_\delta^{j+1}(\delta) = 0$ for $j+1 < d$, the recursion for $j+1 \geq d$ is given by:

$$\begin{aligned} \overline{g}_0^{j+1} &= \frac{1}{T}\overline{g}_0^j + \alpha_j \\ \overline{g}_i^{j+1} &= \overline{g}_{i-1}^j + \frac{1}{T}\overline{g}_i^j, \quad i = 1, \cdots, \text{Max}\{j+1-d, 0\} \end{aligned} \tag{46}$$

The recursion is initialized by

$$\begin{aligned} m = n-1 \quad &: \quad G_\delta^1 = b_m \\ & \quad\;\; H_\delta^1 = (\delta + \frac{1}{T})B_\delta(\delta) - b_m C_\delta(\delta) \\ m < n-1 \quad &: \quad G_\delta^1 = 0 \\ & \quad\;\; H_\delta^1 = (\delta + \frac{1}{T})B_\delta(\delta) \end{aligned} \tag{47}$$

5 The GPC controller

Let $\text{E}\{\cdot\}$ denote expectation. In order to guarantee the monicness of corresponding polynomials some odd normalizing terms have been introduced. Now, define

$$\begin{aligned} \overline{\xi} &= \begin{cases} q & \xi = q \\ \delta + \frac{1}{T} & \xi = \delta \end{cases} \\ \eta &= \begin{cases} 1 & \xi = q \\ T & \xi = \delta \end{cases} \end{aligned} \tag{48}$$

The optimal k-step prediction follows as:

$$\hat{y}_{t+k|k} = \eta^k G_\xi(\xi)u_t + \frac{\eta^k H_\xi(\xi)}{C_\xi(\xi)}u_t + \frac{\eta^k \overline{\xi} F_\xi(\xi)}{C_\xi(\xi)}y_t \tag{49}$$

Dependent of the preferred control criterion, the first term $T^k G_\delta(\delta)$ might be rewritten as $G_q(q)u_t$ by use of, [10]

$$\begin{aligned} g_i &= \sum_{j=i}^{k-d} T^{k-j} \begin{pmatrix} j \\ j-i \end{pmatrix} (-1)^{j-i}\overline{g}_j \\ & \quad i = 0, \cdots, k-d \end{aligned} \tag{50}$$

so that the first term of (49) corresponds to the first $k-d$ responses of the impulse response. Organized in a vector form the transformation of $[\overline{g}_0 \ \ldots \ \overline{g}_{k-d}]^\top$ into $[g_0 \ \ldots \ g_{k-d}]^\top$ forms an upper triangular matrix, which might be calculated prior to the design stage.

A partially constrained quadratic optimal control criterion is now imposed in terms of the control signal and the output error over a finite horizon, due to [1].

$$\underset{u_t}{\text{Min}} \ J = \text{E}\{\sum_{j=d}^{N_2}(y_{t+j} - w_{t+j-d})^2 + \lambda \sum_{j=0}^{N_u} u_{t+j}^2\} \tag{51}$$

subject to $N_2 \geq d$, $N_u \geq 0$, $N_2 \geq N_u$ and $u_{t+j} = u_{t+N_u}$ for $j \geq N_u$. Define the vector $\mathbf{f}$, composed of the free predictions,

$$\mathbf{f} = [\hat{y}_{t+d|t} \ \cdots \ \hat{y}_{t+N_2|t}]^\top \tag{52}$$

where

$$\hat{y}_{t+j|t} = \frac{\eta^j H_\xi^j(\xi)}{C_\xi(\xi)}u_t + \frac{\eta^j \overline{\xi} F_\xi^j(\xi)}{C_\xi(\xi)}y_t \tag{53}$$

Define the vectors $\hat{\mathbf{y}}$, $\mathbf{w}$ and $\mathbf{u}$, composed of predicted plant outputs, set-points and control inputs,

$$\begin{aligned} \hat{\mathbf{y}} &= [\hat{y}_{t+d} \ \cdots \ \hat{y}_{t+N_2}]^\top \\ \mathbf{w} &= [w_t \ \cdots \ w_{t+N_2-d}]^\top \\ \mathbf{u} &= [u_t \ \cdots \ u_{t+N_u}]^\top \end{aligned} \tag{54}$$

Now,

$$\hat{\mathbf{y}} = \mathbf{Gu} + \mathbf{f} \tag{55}$$

where the matrix $\mathbf{G}$, consisting of $N_u + 1$ columns and $N_2 - d + 1$ rows, is defined by:

$$\mathbf{G} = \begin{bmatrix} g_0^d & 0 & 0 \\ g_0^{d+1} & \ddots & 0 \\ g_0^{d+2} & & g_{N_u}^{d+N_u} \\ \vdots & & \vdots \\ g_0^{N_2} & \cdots & \sum_{j=N_u}^{N_2} g_j^{N_2} \end{bmatrix} \tag{56}$$

Note that $\mathbf{G}$ is composed of shift-operator impulse response parameters and forms a lower triangular Toeplitz matrix. This is due to the transformation in (50) that allows a conventional prediction error criterion rather than a δ-based derivative criterion that is the natural outcome of (49). By minimizing the criterion in (51),

$$\underset{u_t}{\text{Min}} \ J = [\mathbf{Gu} + \mathbf{f} - \mathbf{w}]^\top[\mathbf{Gu} + \mathbf{f} - \mathbf{w}] + \lambda \mathbf{u}^\top \mathbf{u}$$

yielding the optimal unrestricted control action

$$\mathbf{u} = [\mathbf{G}^\top\mathbf{G} + \lambda I]^{-1}\mathbf{G}^\top[\mathbf{w} - \mathbf{f}] \tag{57}$$

Here the matrix $\mathbf{G}^\top\mathbf{G}$ is of dimension $N_u + 1 \times N_u + 1$, thus N_u is closely related to the computational burden of the algorithm. Due to the receding horizon principle, u_t is found as the first element of $\mathbf{u}$. Extracting the main information needed,

$$[\alpha_d \ \cdots \ \alpha_{N_2}] = \text{First row of} \{[\mathbf{G}^\top\mathbf{G} + \lambda I]^{-1}\mathbf{G}^\top\} \tag{58}$$

Suppose now, that w_{t+j} remains constant for $j = d, \cdots, N_2$ which will simplify (57), so that the regulator can be written through the three polynomials,

$$Q_\xi(\xi) = C_\xi(\xi)\sum_{j=d}^{N_2}\alpha_j$$
$$R_\xi(\xi) = C_\xi(\xi) + \sum_{j=d}^{N_2}\eta^j\alpha_j H_\xi^j(\xi)$$
$$S_\xi(\xi) = \sum_{j=d}^{N_2}\eta^j\alpha_j\bar{\xi}F_\xi^j(\xi) \qquad (59)$$

defining the linear control law

$$R_\xi(\xi)u_t = Q_\xi(\xi)w_t - S_\xi(\xi)y_t \qquad (60)$$

Note in this case, that $R_\xi(\xi)$ becomes monic of order n. If the set-point supposition is not realistic, it is possible to formulate a similar regulator, unfortunately this has the effect of making $R_\xi(\xi)$ of order $N_2 + m$, introducing $N_2 - d$ dead-beat poles in the feed-forward action. Modification of the control law in (60) so that the reference signal appears as w_{t+N_2-d} rather than w_t, the controller equals:

$$Q_\xi(\xi) = C_\xi(\xi)\sum_{j=d}^{N_2}\alpha_j(\eta\bar{\xi})^{j-d}$$
$$R_\xi(\xi) = (\eta\bar{\xi})^{N_2-d}(C_\xi(\xi) + \sum_{j=d}^{N_2}\eta^j\alpha_j H_\xi^j(\xi))$$
$$S_\xi(\xi) = (\eta\bar{\xi})^{N_2-d}(\sum_{j=d}^{N_2}\eta^j\alpha_j\bar{\xi}F_\xi^j(\xi)) \qquad (61)$$

6 Numerical Example

Consider a single flexible beam which is modelled as in [7]. Only one mode is considered, so that a third order system describes the simplified motor dynamics and the flexibility of the beam. The voltage signal of the DC-motor is the input-signal and the output signal is the end-point position of the beam.

$$\dot{x} = \begin{bmatrix} 0 & 0.112 & 0 \\ 0 & 0 & 1 \\ 0 & -48.12 & -0.265 \end{bmatrix} x + \begin{bmatrix} 0.156 \\ -0.213 \\ 0.056 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0.7918 & 0 \end{bmatrix} x \qquad (62)$$

Sampled with a zero order hold every 0.02 sec a shift-operator model is generated. To improve the root sensitivity the δ-model is introduced, with $T = 1$ for simplicity.

6.1 The GPC design parameters

The observer polynomial is chosen as $C_\delta(\delta) = (\delta + \frac{1}{T})^3$. The GPC algorithm then leaves three design parameters to be adjusted. When the control cost horizon N_u is increased, the result is a better performance and even a reduction in the condition number of the matrix $\mathbf{G}^\mathsf{T}\mathbf{G} + \lambda I$.

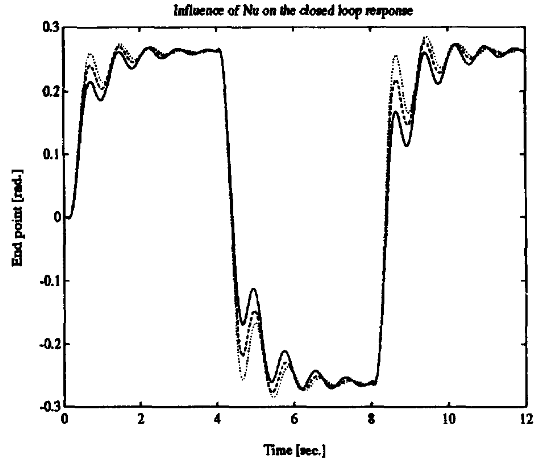

Figure 1: Effect of varying $N_u = 2, 6, 10$ for fixed $N_2 = 30$, $\lambda = 0.002$. The output response becomes faster and the poles move away from the Laplace imaginary axis as N_u increases.

However, N_u is deeply involved with the complexity of the algorithm due to the matrix inversion, see Fig. 1.
When the output horizon N_2 is increased, the result is a better performance, but with a drawback of an increase in the condition number of the matrix $\mathbf{G}^\mathsf{T}\mathbf{G} + \lambda I$. Since N_2 directly refers to the number of recursions in the algorithm, the computational burden is heavily affected by a large value of N_2. However, N_2 must be lower bounded in case of a non-minimum-phase plant, see Fig. 2.

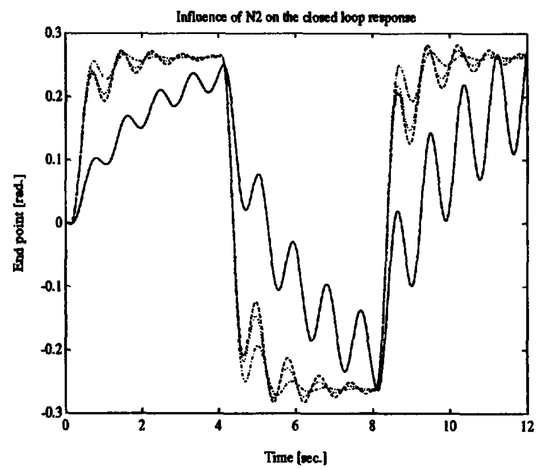

Figure 2: Effect of varying $N_2 = 10, 20, 30, 40$ for fixed $N_u = 6$, $\lambda = 0.002$. The output response becomes faster when N_2 is increased.

The control weighting λ plays a similar role to that of the control weighting of a LQG controller. It represents a tradeoff between better output performance and less effect used in the control action. When λ is lowered, more effect enters the regulation and the output reaches faster the set-point, see Fig. 3. However, the impact of decreasing λ affects the condition number of $\mathbf{G}^T\mathbf{G} + \lambda I$, which in general will increase.

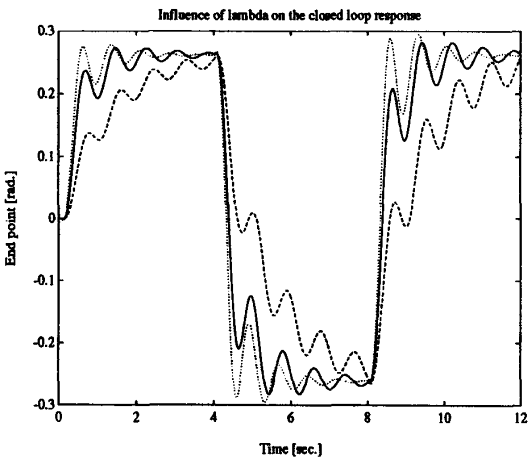

Figure 3: Effect of varying $\lambda = 0.0008, 0.002, 0.008$ for fixed $N_u = 6$, $N_2 = 30$. The output response becomes slower and poles move towards the Laplace origin as λ is increased.

The GPC has the properties, that for N_u, $N_2 \to \infty$ the algorithm coincides with the LQG-controller. In Fig. 4 the closed-loop poles of $A_\delta(\delta)R_\delta(\delta) + B_\delta(\delta)S_\delta(\delta)$ are shown for increasing values of the horizons.

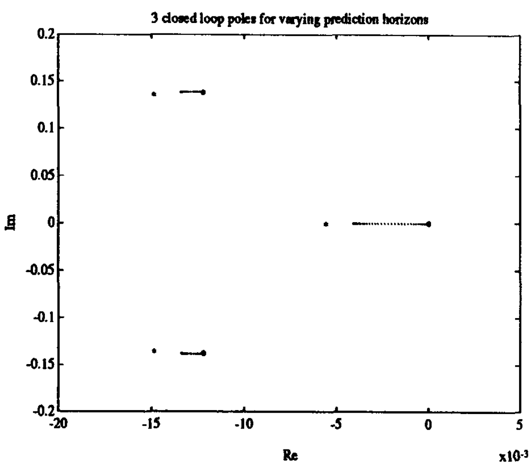

Figure 4: The closed-loop poles (omitting those of $C_\delta(\delta)$) for $\lambda = 0.1$ when N_2 and $1 + N_u$ grow from 1 (indicated by zeros) to 50.

7 Acknowledgement

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8 Conclusions

In this paper a new control strategy has been developed, i.e. the ubiquitous general predictive control scheme has been applied to the delta operator. Hence, a GPC for an ARMAX model has been formulated using step-recursions for the solution of the needed Diophantine equations.
A numerical example illustrates some of the properties. The algorithm can handle non-minimum-phase plants if the output horizon N_2 is chosen sufficiently large. Some guidelines for the selection of design parameters are discussed.
The algorithm has a close parallel to the LQG controller, but has built-in the capability of minimizing over a trajectory of future set-points. In the limit process N_u, $N_2 \to \infty$ the two controllers minimize the same criterion.

References

[1] R. R. Bitmead, M. Gevers, and V. Wertz. *Adaptive optimal control, - The thinking man's GPC.* Prentice Hall, 1990.

[2] P. Boucher, D. Dumur, and R. Neumann. Control axis using delta operator generalized predictive control. In *ECC '93*, volume 2, Groningen, The Netherlands, 1993.

[3] D. W. Clarke, C. Mohtadi, and P. S. Tuffs. Generalized predictive control - part 1,2. *Automatica*, 23(2):137–160, 1987.

[4] H. Demircioglu and P. J. Gawthrop. Continuous-time generalized predictive control. *Automatica*, 27(1):55–74, 1991.

[5] G. Favier and D. Dubois. A review of k-step-ahead predictors. *Automatica*, 26(1):75–84, 1990.

[6] G. Goodwin, R. L. Leal, D. Q. Mayne, and R. Middleton. A rapprochement between continuous and discrete model reference adaptive control. *Automatica*, 22(2):199–207, 1986.

[7] L. Kruise. *Modeling and control of a flexible beam and robot arm.* PhD thesis, Universiteit Twente, 1990.

[8] R. Middleton and G. Goodwin. Improved finite word length characteristics in digital control using delta operators. *IEEE Trans. on Auto. Control*, AC-31(11):1015–1020, 1986.

[9] R. Middleton and G. Goodwin. *Digital Control And Estimation, A unified Approach.* Prentice Hall, 1990.

[10] M. Rostgaard, N. K. Poulsen, and O. Ravn. A rapprochement between discrete-time operators. In *ECC '93*, volume 3, Groningen, The Netherlands, 1993. Also IMSOR TR-14/92.