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An Eigenvalue Study of a Double Integrator Oscillator

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Abstract—A tutorial study of an oscillator built from a loop of two active RC integrators and an ideal inverter. As a reference the linear harmonic oscillator is modeled as a LC circuit and as an ideal two integrator loop. The nonlinear amplifiers of the active RC integrator circuits are assumed to be linear with time varying gain and the eigenvalues are found as function of time. A design strategy based on the time-constants of the integrators is presented.

I. INTRODUCTION

In the recent 20 years the classic circuit theory with analysis and synthesis of linear electrical circuits has more or less disappeared from the curriculum of the electrical and electronic engineering students. The digital systems dominate over the analog systems. But in the future possibly nonlinear analog systems will be used instead of or as a supplement to digital systems due to lower power consumption and higher speed [1]. The aim of this tutorial is to demonstrate that the classic circuit theory with poles and zeros (eigenvalues) may be used to gain insight in the behavior of nonlinear circuits.

II. HARMONIC OSCILLATOR MODELS

The harmonic oscillator may be modeled as a capacitor coupled in parallel to a coil, Fig. 1(a). If the initial condition is a voltage across the capacitor and a current equal to zero through the coil the voltage becomes a cosine of time and the current becomes a sine of time. The capacitor may be modeled as a current source controlled by the time derivative of its voltage. The coil may be modeled as a voltage source controlled by the time derivative of its current. As variables we choose the voltage across the capacitor: x, and the current through the coil: y, (state variables). We choose voltage V and current I as variables because they are easy to measure (signals). We should use charge \( q = C \cdot V \) and flux \( \phi = L \cdot I \) as variables because they represent the energy in the system. Unfortunately they are difficult to measure. In the real world charge and flux are nonlinear functions of voltage and current. Note that current is time derivative of charge \( I = dq/dt \) and voltage is time derivative of flux \( V = d\phi/dt \). Charge provide the coupling between the chemical world and the electrical world (electrolysis). Flux provide the coupling between the mechanical world and the electrical world (generators, motors). Capacitors and coils are memory elements. The resistor

\[
\begin{align*}
\mathbf{L} & = -1 \cdot \mathbf{V}_2 \\
\mathbf{C} & = \frac{-1}{\sqrt{\mathbf{V}_2}} \\
\mathbf{L} & = \frac{-1}{\sqrt{\mathbf{V}_2}} \\
\mathbf{C} & = \frac{-1}{\sqrt{\mathbf{V}_2}} \\
\mathbf{I}_1 & = \frac{1}{\sqrt{\mathbf{C}_1 \cdot \mathbf{C}_2}} \\
\mathbf{I}_2 & = \frac{1}{\sqrt{\mathbf{C}_1 \cdot \mathbf{C}_2}}
\end{align*}
\]
with memory - the memristor \( \phi = M * q \) - has recently been implemented in the real world \[2\].

Now the harmonic oscillator may be modeled (defined) by means of two first order differential equations

\[
L \frac{dy}{dt} = +x \\
C \frac{dx}{dt} = -y
\]

The two equations may be combined into a second order differential equation.

\[
\frac{d^2x}{dt^2} + \frac{x}{LC} = 0
\]

The roots of the characteristic polynomial - the eigenvalues - are a complex pole pair \( s = \pm j * \omega \) on the imaginary axis where \( \omega = 2\pi f \). With \( \omega^2 = 1/(LC) \) the frequency becomes \( f = 1/[2\pi \sqrt{LC}] \). The amplitude is given by the initial condition.

An ideal integrator may be modeled as a capacitor loaded current source. The voltage of the capacitor is the time integral of the current source. By means of two ideal integrators the harmonic oscillator may be modeled as shown in Fig. 1(b).

The figures Fig. 2 and Fig. 3 show a PSpice comparison of two 100kHz oscillators with the following component values: \( L_3 = 1.591549431 \mu H, \ C_3 = 1.591549431 \mu F \) and \( C_1 = C_2 = 1.591549431 \mu F \). With an accuracy of \( RELTOL = 1 \mu \) and the capacitors \( C_1 \) and \( C_2 \) are chosen equal in PSpice a very close agreement between the models is observed.

Linear steady state oscillators are mathematical fiction. In the real world linear systems are always damped due to losses. Some kind of non-linearity must be introduced in order to obtain steady state oscillations. Oscillators are non-linear systems. It is to be expected that oscillators based on a double integrator will behave very close to sinusoidal so they have been reported frequently in the literature \[3\], \[4\], \[5\], \[6\], \[7\], \[8\], \[9\], \[10\], \[11\], \[12\]. In the following a double integrator oscillator based on operational amplifiers is investigated.

### III. DOUBLE INTEGRATOR OSCILLATOR

The circuit is a dedicated analogue computer circuit based on the definition of the sine and cosine functions: \( d(\sin(t))/dt = \cos(t) \) and \( d(\cos(t))/dt = -\sin(t) \). Input to the first integrator is \( -\sin(t) \) which give rise to input \( \cos(t) \) of the second integrator with output \( \sin(t) \) to be inverted for feed-back to the first integrator (see Fig. 4).

Assuming ideal operational amplifiers (\( A_1 = A_2 = \infty \)) the relation between the components and the frequency becomes \( \omega_0^2 = (2\pi f_0)^2 = 1/(\tau_1 * \tau_2) \) where \( \tau_1 = R_1 * C_1 \) and \( \tau_2 = R_2 * C_2 \) are the time constants of the integrators. If we choose \( R_1 = R_2 = R = 10k\Omega \) and \( C_1 = C_2 = C = 15.91549431 \mu F \) the nominal oscillating frequency becomes \( f_0 = 1 \) kHz, \( \tau_1 = \tau_2 = \tau = 159.1549431e^6 \), \( \omega_0 = 1/(RC) = 6.283185308e + 3 = 2\pi f_0 \).

![Double integrator oscillator](image)

**Fig. 4.** Double integrator oscillator. \( V(1) = -V(5) \)

### A. PSpice analysis

Now the two ideal operational amplifiers are replaced with PSpice op-amp macro-models, \( \mu A741 \) and an ideal PSpice inverter model (EVA3 1 0 5 0 -1) is introduced. The power supply is \( \pm 22 \) volts.

Figure 5 shows the frequency spectrum of the oscillator in steady state (FFT analysis over 800ms). Higher harmonics may be observed. In this case - where the resistors \( R_1 \) and \( R_2 \) are chosen equal and the capacitors \( C_1 \) and \( C_2 \) are chosen equal
clipping of both amplifier output voltages $V(3)$ and $V(5)$ is observed, Fig. 6.

Figure 6 shows that energy is transferred to the amplifiers as very narrow pulses at the maximums of the output voltages. This behavior is very similar to the behavior of the pendulum clock where the escape mechanism [13] delivers the energy as pulses when the angle from vertical is zero and the weights go down a step changing potential energy into a kinetic energy impulse. For small swing - i.e. when $x$ and $\sin(x)$ are almost equal - the pendulum clock is very close to a damped linear oscillator with very high quality factor and no harmonics. A strategy for design of electronic oscillators with minimum distortion could be optimization of the energy impulses observed.

A large number of PSpice simulations have been made with various combinations of the values of the two resistors and the two capacitors. Apparently it is possible to remove the energy impulses in connection with one of the amplifiers so that the harmonics in the output voltage becomes smaller. The following PSpice results demonstrates this assertion and gives more insight in the behavior of the oscillator.

By means of the following values a number of PSpice simulations have been made: $R_1 = 8.3333333333\, k\Omega$, $C_1 = 10.0\, nF$, $R_2 = 12.0\, k\Omega$, $C_2 = 25.33172448\, nF$, time constant amplifier $A_1$: $\tau_1 = R_1 \cdot C_1 = 83.333333336\, e^{-6}$, time constant amplifier $A_2$: $\tau_2 = R_2 \cdot C_2 = 303.9806938\, e^{-6}$, $\omega^2 = 1/(\tau_1 \cdot \tau_2) = 39.47619124\, e^{-6} = (2\pi f)$, $f = 0.9999718022\, kHz$.

A comparison of Fig. 7 with Fig. 5 shows that the harmonics of the output voltage $V(5)$ of amplifier $A_2$ have been reduced. Figure 8 shows that the power is supplied to amplifier $A_1$ with an almost constant current of $2.44\, mA$ and pulses of $15\, \mu A$ in the short clipping time intervals. The power is supplied to amplifier $A_2$ with an almost constant current of $2.438\, mA$. A comparison of Fig. 8 with Fig. 6 shows that the pulses in the power supply currents of amplifier $A_2$ have disappeared. This result depends of course on the op-amp macro-model used. Experiments with $LM741$ instead of $uA741$ did not show pulses but reduction of harmonics was obtained. Figure 9 shows the amplifier gains as functions of time over two periods. It is seen that the gain is varying slowly in the interval $\pm 5000$ in the whole period except at the maximums of the amplifier output voltages where it vary between very large and very small values in very short time intervals.

B. Eigenvalue calculation

A study of the eigenvalues of the time-varying Jacobian of the linearized differential equations may give some insight in the behavior of the oscillator [14]. The linear time-varying approach (LTV) is a method to calculate the time-varying
eigenvalues (dynamic eigenvalues) of a nonlinear circuit [15], [16], [17], [18], [19]. To calculate the dynamic eigenvalues, the Riccati equation must be solved. In the following it is assumed that a nonlinear circuit can be treated as a time-varying linear circuit.

If it is assumed that the amplifiers are perfect - i.e. the input impedance is infinite and the output voltage is equal to the input voltage times a time varying constant $A$ i.e. $V_{out} = (V_+ - V_-) * A(\text{time})$ - the characteristic polynomial of the circuit becomes

$$s^2 + 2\alpha s + \omega^2 = 0$$

where

$$2\alpha = \frac{C_1 R_1 (1 + A_1) + C_2 R_2 (1 + A_2)}{C_1 R_1 C_2 R_2 (1 + A_1)(1 + A_2)}$$

and

$$\omega^2 = \frac{1 + A_1 A_2}{C_1 R_1 C_2 R_2 (1 + A_1)(1 + A_2)}$$

The roots are

$$p_{1,2} = -\alpha \pm j\sqrt{\omega^2 - \alpha^2}$$

The roots of the characteristic polynomial (the poles) are found as function of time by means of a table of the gains as function of time found by means of PSpice. The result is shown in Fig. 10. It is seen that the imaginary part is close to $\omega_0$ over a period. The time constants of the integrators are chosen different in order to obtain minimum distortion of the output voltage of one of the amplifiers. The circuit is used mainly for low-frequency oscillators. The structure may be a candidate for IC implementation because it is based on resistors and capacitors only.

IV. CONCLUSION

It is demonstrated that the classic circuit theory with poles and zeros (eigenvalues) may be used in connection with the design of a double integrator oscillator. The non-linear circuit is treated as a linear time-varying circuit. The parameters are chosen in such a way that the imaginary part of the poles of the linear time-varying circuit is as close as possible to $\omega_0$ over a period. The time constants of the integrators are chosen different in order to obtain minimum distortion of the output voltage of one of the amplifiers. The circuit is used mainly for low-frequency oscillators. The structure may be a candidate for IC implementation because it is based on resistors and capacitors only.

REFERENCES

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Abstract

A tutorial study of an oscillator built from a loop of two active RC integrators and an ideal inverter. As a reference the linear harmonic oscillator is modeled as a LC circuit and as an ideal two integrator loop. The nonlinear amplifiers of the active RC integrator circuits are assumed to be linear with time varying gain and the eigenvalues are found as function of time. A design strategy based on the time-constants of the integrators is presented.

The aim of this tutorial is to demonstrate that the classic circuit theory with poles and zeros (eigenvalues) may be used to gain insight in the behavior of nonlinear circuits.

COMPUTER AIDED CIRCUIT ANALYSIS

- The kernel of analyzing nonlinear circuits is the solution of a linear circuit.
- All elements may be modeled as components of controlled sources.
- Op-amp circuits are generally approximated with either a dynamic value or a static value.

The CAPACITOR is a voltage controlled current source (VCCS) controlled by the time derivative of its voltage: \( \frac{dQ}{dt} = V \) or \( C = \frac{Q}{V} \).

The INDUCTOR is a current controlled voltage source (CCVS) controlled by the time derivative of its current: \( \frac{dI}{dt} = V \) or \( L = \frac{dI}{dt} \).

EXTENDED NODE EQUATIONS

- First order differential equations
- Variables: Node Potentials and Impedance Branch Currents
- Current sources: Admittance Branches
- Voltage sources: Impedance Branches

"FREEZE" TIME and replace nonlinearity with dynamic ratio (JACOBIAN) or with static ratio (LTV)

HARMONIC OSCILLATOR MODELS

Passive lossless LC circuit

Active ideal integrator circuit

NUMERICAL INTEGRATION with

and

and

Variable Integration Step

Newton-Raphson Iteration

Replace with dynamic value

Pixel Iteration

Replace with static value

Double Integrator Oscillator

TWO EXPERIMENTS

A

Same time constants of the integrators

B

Different time constants of the integrators

Two experiments

A

Same time constants of the integrators

B

Different time constants of the integrators

Two experiments

CONCLUSION

It is assumed that the amplifiers are perfect, i.e., the input impedance is infinite and the output voltage is equal to the input voltage times a time varying constant \( A(t) \). If the loop gain \( V(1)/V(2) \) is constant then the circuit becomes a time-constant simple oscillator with a constant frequency. The loop gain is the product of the amplifier gain and the feedback factor, which can be varied by the feedback factor. The feedback factor is the ratio of the output voltage to the input voltage and is usually the case for negative feedback.

The characteristic polynomial of the circuit becomes

\[ s^2 + 2\beta s + \gamma = 0 \]

where

\[ \beta = \frac{C_1 R_1 + C_2 R_2}{C_1 + C_2} \]

and

\[ \gamma = \frac{1}{C_1 + C_2} \]

The roots are

\[ s = \frac{-\beta \pm \sqrt{\beta^2 - \gamma}}{2} \]

The oscillations are damped exponential and the frequency is given by the time constant \( T = \frac{1}{\beta} \). The frequency is independent of the gain and it cannot be changed by changing the gain. The frequency is determined by the time constant which is independent of the gain.

The frequency is independent of the gain and it cannot be changed by changing the gain. The frequency is determined by the time constant which is independent of the gain.

Energy is transferred to the amplifier as a very narrow packet of the magnitude of the output voltage. This behavior is very similar to the behavior of the pendulum clock where the angular momentum of the energy is conserved as it passes through the pendulum. The energy is stored in the circuit and is released as a ramp which is a damped exponential. As the pendulum slows, the energy is released and the pendulum is released as an exponential. The pendulum clock is very close to a damped harmonic oscillator with very high quality factor and the energy is transferred at an exponential rate.

A strategy for design of electronic oscillators with non-linearities is to optimize the energy transfer rate.

In conclusion, the results of the characteristic polynomial show that the frequency of the circuit is independent of the gain and it cannot be changed by changing the gain. The frequency is determined by the time constant which is independent of the gain. The frequency is independent of the gain and it cannot be changed by changing the gain. The frequency is determined by the time constant which is independent of the gain.