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Geometry of Hamiltonian Chaos

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The characterization of chaotic Hamiltonian systems in terms of the curvature associated with a Riemannian space associated with the metric \( g_{ij} \) where

\[
\frac{\partial}{\partial x^i} \propto \frac{\partial}{\partial x^j}
\]

is equivalent to (1). For a metric of conformal form \( g_{ij} \), the curvature associated with this metric on the manifold, in this case, the kinetic energy, thus corresponding to the Hamilton action. Transformation to parametrization by exponents of a locally linearized system.

The Eisenhardt metric, leading to geodesic motion in \( t \), involves the addition of an extra dimension. As noted by Caini et al. [5], this metric leads to the tangent dynamics commonly used to measure Lyapunov exponents in standard Hamiltonian systems. The method that we use, associated with a curvature that is explicitly energy dependent, appears to be a more sensitive diagnostic than the computation of exponents of a locally linearized system.

The formulation of Hamiltonian dynamics of the type of Eq. (6) in the form (1) is carried out by requiring that (6) be equivalent to (1). For a metric of conformal form

\[
g_{ij} = \varphi \delta_{ij},
\]

on the hypersurface defined by \( H = E = \text{const} \), the requirement of equivalence implies that

\[
\frac{\partial}{\partial x^i} \propto \frac{\partial}{\partial x^j}
\]
\[ \varphi = \frac{E}{E - V(x)} . \]  

Substituting this result in the geodesic Eqs. (2), one obtains an equation that does not coincide in form with the Hamilton equations obtained from (6).

To see that the Hamilton equations obtained from (1) can, however, be put into correspondence with those obtained from the Hamiltonian of the potential model (6), we first note, from the Hamilton equations for (1), that

\[ \dot{x}_i = \frac{\partial H}{\partial p^i} = \frac{1}{M} g_{ij} p^j . \]  

We then use the geometrical property that \( \dot{x}_i \) is a first rank tensor (as is \( p^i \)), under local diffeomorphisms that preserve the constraint that \( H \) be constant, to define the \textit{velocity field}

\[ \dot{x}^j \equiv g^{ij} x_i = \frac{1}{M} p^j , \]  

coinciding formally with one of the Hamilton equations implied by (6). From this definition, we recognize that we are dealing with two manifolds, each characterized, as we shall see, by a different connection form, but related by

\[ dx^j = g^{ij} dx_i \]  

on a common tangent space at each point (for which \( g^{ij} \) is nonsingular).

To complete our correspondence with the dynamics induced by (6), consider the Hamilton equation for \( \dot{p}^\ell \),

\[ \dot{p}^\ell = - \frac{\partial H}{\partial x_\ell} = - \frac{1}{2M} \frac{\partial g_{ij}}{\partial x_\ell} p^i p^j . \]  

With the form (7) for \( g^{ij} \), we obtain in the particular coordinate system in which (6) is defined,

\[ \dot{p}^\ell = - \frac{E}{E - V} \frac{\partial V}{\partial x_\ell} . \]  

Considering (11) as a change of variables, (13) becomes

\[ \dot{p}^\ell = - \frac{\partial V}{\partial x_\ell} , \]  

the second Hamilton equation in the usual form, where \( V \) is considered a function of the \( \{ x^\ell \} \), now considered as independent variables.

As a coordinate space, the \( \{ x^\ell \} \), which we shall call the \textit{Hamilton manifold}, is not uniquely defined in terms of the original manifold \( \{ x_\ell \} \), which we shall call the \textit{Gutzwiller manifold}, since (11) is not an exact differential. As we have remarked, we shall be working with two manifolds (characterized by the connection forms (3) and (21)). It is the local relation (11) which induces, from the geometry of the Gutzwiller manifold, a corresponding geometry on the Hamilton manifold. We shall discuss applications and interpretation of the physics of the Gutzwiller manifold elsewhere, but turn now to a further examination of the consequences of the relations (10) and (11).

The geodesic Eq. (2) can be transformed directly from an equation for \( \dot{x}_i \) to an equation for \( \dot{x}^j \), the motion defined in the Hamilton manifold. From (10), it follows that

\[ \dot{x}_\ell = g_{\ell j} \dot{x}^j = \frac{1}{M} \frac{\partial g_{ij}}{\partial x_\ell} \delta_{j \ell} , \]

\[ = \frac{1}{2M} \frac{\partial g_{ij}}{\partial x_\ell} \left[ \frac{\partial g_{nm}}{\partial x_n} + \frac{\partial g_{kn}}{\partial x_m} - \frac{\partial g_{mn}}{\partial x_k} \right] \dot{x}_m \dot{x}_n . \]  

Now, using the identity

\[ \frac{\partial g_{ij}}{\partial x_\ell} = - g_{\ell k} \frac{\partial g_{km}}{\partial x_n} g_{mj} , \]  

it follows that, with the symmetry of \( \dot{x}_m \dot{x}_n \),

\[ \frac{\partial g_{ij}}{\partial x_\ell} \dot{x}_m \dot{x}_n = - \frac{1}{2M} \frac{\partial g_{ij}}{\partial x_\ell} \left[ \frac{\partial g_{km}}{\partial x_n} + \frac{\partial g_{kn}}{\partial x_m} \right] \delta_{j \ell} \delta_{i \ell} . \]  

Thus, the term on the left side of (15) containing the derivative of \( g_{ij} \) cancels the first two terms of the connection form; multiplying the result by the inverse of \( g_{ij} \), and applying the identity (16) to lower the indices of \( g_{ij} \) in the remaining term on the right side of (15), one obtains

\[ \dot{x}_\ell = -M_{mn} \dot{x}_m \dot{x}_n , \]  

where

\[ M_{mn} = \frac{1}{2M} \frac{\partial g_{ij}}{\partial x_\ell} \frac{\partial g_{ij}}{\partial x_\ell} . \]  

Equation (18) has the form of a geodesic equation, with a truncated connection form. In fact, it can be shown (a full proof will be given elsewhere) that the form (19) is indeed a connection form, transforming as

\[ M_{mn} = \frac{\partial x^{\ell}}{\partial x^m} \frac{\partial x^{\ell}}{\partial x^n} M_{\ell q} = \frac{\partial x^{\ell}}{\partial x^m} \frac{\partial x^{\ell}}{\partial x^n} M_{\ell q} . \]  

consistent with the covariance of (18) under local diffeomorphisms of the Hamilton manifold.

Substituting (7) and (8) into (18) and (19), the Kronecker deltas identify the indices of \( \dot{x}^m \) and \( \dot{x}^n \); the resulting square of the velocity cancels a factor of \( (E - V)^{-1} \), leaving the Hamilton-Newton law (14). Equation (18) is therefore a covariant form of the Hamilton-Newton law, exhibiting what can be considered an underlying geometry of standard Hamiltonian motion.

The geometrical structure of the Hamilton manifold can be understood as follows. Let us write the covariant derivative for a (rank one) covariant tensor on the Gutzwiller manifold (defined as transforming in the same way as \( \partial/\partial x_\ell \)), using the full connection form (3),

\[ A^m_{n q} = \frac{\partial A^m}{\partial x_q} - \Gamma^m_{k q} A^k . \]  

Lowering the index \( q \) with \( g_{\ell q} \), we obtain the covariant
curvature of that manifold, determined by fold, as we shall see below, is not the same as the intrinsic curvature, associated with the geodesic deviation in the Hamilton manifold. We note here, however, that the curvature associated with the geodesics themselves. This expression, as remarked above, is not the curvature of the Hamilton manifold (given by this formula with $\Gamma^I_{Iglm}$ in place of $M_{glm}$), but a dynamical curvature which is appropriate for geodesic motion.

We give in the following a general formula for the geodesic deviation in the Hamilton manifold in two dimensions (the resulting conditions can be easily generalized to three dimensions, for which they are also effective), and then show results of computer simulation for Poincaré plots showing a correspondence with the prediction of instability from the geodesic deviation.

With the conformal metric in noncovariant form (7) and (8), the dynamical curvature (25) can be written in terms of derivatives of the potential $V$, and the geodesic deviation Eq. (24) becomes

$$\frac{D^2M_\xi}{D^2t^2} = -\mathcal{V}P_\xi, \quad (26)$$

where the matrix $\mathcal{V}$ is given by

$$\mathcal{V}_{\xi} = \left[ \frac{3}{M^2} \frac{\partial V}{\partial x^i} \frac{\partial V}{\partial x^j} + \frac{1}{M} \frac{\partial^2 V}{\partial x^i \partial x^j} \right] \quad (27)$$

and

$$P_{ij} = \delta_{ij} - \nu_i \nu_j \nu^2, \quad (28)$$

with $\nu^i = \dot{x}^i$, defining a projection into a direction orthogonal to $\nu^i$. We then find for the component orthogonal to the motion

$$\frac{D^2M_\xi}{D^2t^2} (v_\perp \cdot \xi) = -[\lambda_1 \cos^2 \phi + \lambda_2 \sin^2 \phi] (v_\perp \cdot \xi). \quad (29)$$

where $\lambda_1$ and $\lambda_2$ are eigenvalues of the matrix $\mathcal{V}$, and $\phi$ is the angle between $v_\perp$ and the eigenvector for $\lambda_1$.

Instability should occur if at least one of the eigenvalues of $\mathcal{V}$ is negative, in terms of the second covariant

![Figure 1](image-url)

(a) The dark area shows the region of negative eigenvalues for the matrix $\mathcal{V}$ for a Hamiltonian with potential (30). The light area corresponds to physically allowable motion for $E = 1/6$. The region of negative eigenvalues does not penetrate the physically accessible region in this case. (b) The dark area of negative eigenvalues for the matrix $\mathcal{V}$ is seen to penetrate deeply into the light region of physically allowable motion for $E = 3$. 

FIG. 1.
derivatives of the transverse component of the geodesic deviation.

One may easily verify that the oscillator potential is predicted to be stable. Our criteria imply that the Duffing oscillator (without perturbation, not a chaotic system) clearly indicates instability in a neighborhood of the unstable fixed point. The potentials discussed by Oloumi and Teychenne [6] also demonstrate the effectiveness of our procedure; our results in these cases are in agreement with theirs. The relation (29) provides a clear indication of the local regions of instability giving rise to chaotic motion in the Hénon-Heiles model (this result will be discussed in detail elsewhere).

We take for a simple illustration here a slight modification of the fifth order expansion of a two body Toda lattice Hamiltonian (for which the fourth order expansion coincides with the Hénon-Heiles model)

\[
V(x, y) = \frac{1}{2}(x^2 + y^2) + x^3y - \frac{1}{3}x^3 + \frac{3}{2}x^4 + \frac{1}{2}y^4.
\]

This provides a new Hamilton chaotic system for which our criterion gives a clear local signal for the presence of instability. Figure 1(a) shows that the region of negative eigenvalues does not penetrate the physically accessible region for \(E = 1/6\); Fig. 2(a) shows a Poincaré plot in the \(y, p_y\) plane for this case, indicating completely regular orbits. In Fig. 1(b), the distribution of negative eigenvalues for \(E = 3\) is shown to penetrate deeply into the physical region, and Fig. 2(b) shows the corresponding Poincaré plot displaying a high degree of chaotic behavior. The criterion for instability we have given depends sensitively on the energy of the system. The critical energy for which the negative eigenvalues begin to penetrate the physically accessible region, in this example, is \(E \approx 1/5\), also the energy of transition for the Poincaré plot.

The condition implied by the geodesic deviation Eq. (26), in terms of covariant derivatives, in which the orbits are viewed geometrically as geodesic motion, is a new condition for instability, based on the underlying geometry, for a Hamiltonian system of the form (6). This geometrical picture of Hamiltonian dynamics provides, moreover, new insight into the structure of the unstable and chaotic behavior of Hamiltonian dynamical systems.

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