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Published in:
Proc. American Control Conference, Albuquerque, New Mexico

Link to article, DOI:
10.1109/ACC.1997.611759

Publication date:
1997

Document Version
Publisher's PDF, also known as Version of record

Proceedings of the American Control Conference
Albuquerque, New Mexico June 1997
0-7803-3832-4/97/$10.00 © 1997 AACC

Computational Issues in Alternating Projection Algorithms for Fixed-Order Control Design

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Abstract

Alternating projection algorithms have been introduced recently to solve fixed-order controller design problems described by linear matrix inequalities and non-convex coupling rank constraints. In this work, extensive numerical experimentation using proposed benchmark fixed-order control design examples is used to indicate the computational efficiency of the method. These results indicate that the proposed alternating projections are effective in obtaining low-order controllers for small and medium order problems.

1. Introduction

In the last few year there is a significant interest in control design problems formulated in terms of Linear Matrix Inequalities (LMIs) [1], [5], [7], [12], [18]. Full-order stabilization, H_∞, μ-synthesis with constant scaling, gain-scheduling and other control design problems have been expressed as convex feasibility or optimization problems involving LMIs. Recently, efficient interior point algorithms have been proposed for computational solution of these type of problems [8], [17], [19] and an excellent introduction to the subject can be found in [5]. MATLAB packages have been developed based on these algorithms, see for example the LMI Control Toolbox for MATLAB [9] and the Semidefinite Programming Package with the user friendly MATLAB top LMITOOL [10]. These new approaches result in controllers of order equal to the order of the generalized plant. However, often the large order of the plant as well as control hardware implementation constraints dictate the need for low-order control design.

The low-order control design problems can be formulated in terms of LMIs and an additional coupling matrix rank constraint [7], [12], [18]. Unfortunately, because of this additional rank constraint, these problems are not convex and efficient convex programming cannot be used for a solution. Recently, algorithms that exploit the special structure of the fixed-order control design problems have been proposed for solution, but convergence of the algorithms is not guaranteed. In [15], alternating projection methods were used for fixed-order control design utilizing the projections onto the constraint sets. To this end, the control design problem was formulated as a feasibility problem of finding a matrix pair in the intersection of several constraint sets of simple geometry, and analytical expressions for the projection operators onto the constraint sets were obtained. In [3], a combined alternating projections and semidefinite programming algorithm was proposed that utilizes efficient convex programming techniques to compute the projections onto the combined LMI constraints and accelerate the convergence of the alternating projections algorithm.

In this work computational issues for the numerical implementation of the alternating projection algorithms are examined. Extensive numerical experimentation using proposed "benchmark" fixed-order control problems are used indicate the computational efficiency of the algorithm.

2. The Fixed-Order Control Design Problem

The reformulation of many control problems as feasibility problems of finding a solution to a set of coupled LMIs is now well examined and a number of control synthesis problems can be handled. This reformulation can be achieved using analysis tools in terms of LMIs (e.g., the Bounded Real Lemma for the case of H_∞ control) and deriving the corresponding LMI existence conditions for a solution, see for example [1], [5], [7], [12], [18].

Many control synthesis problems, such as stabilization, H_∞, μ-synthesis with constant scaling, gain-scheduling can be formulates as follows: Find matrices

[1]This work is sponsored by the Danish Research Counsil under grant no. 26-1830

$X > 0$ and $Y > 0$ to satisfy

$$E_xXF_x + (E_xXF_x)^T + Q_x < 0 \quad (1)$$
$$E_yYF_y + (E_yYF_y)^T + Q_y < 0 \quad (2)$$
$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \quad (3)$$

where $X, Y \in \mathcal{S}_n$ are the free variables and the other entries are affine functions of the open-loop matrices. The search for (X, Y) satisfying the LMIs (1)-(3) is a convex problem. Given (X, Y) satisfying the above conditions, a controller that solves the control synthesis problem can be computed via a new convex LMI problem or computed analytically from algebraic expressions, see [7], [12], [18]. We will denote the set of matrices (X, Y) satisfying (1)-(2) by Γ_{convex}, i.e.

$$\Gamma_{convex} = \{(X, Y) \mid X \text{ and } Y \text{ satisfy the LMIs (1)-(1)}\}$$

The order of the controller n_c can be restricted using the fact that

$$n_c = \operatorname{rank}(I - XY)\ . \quad (4)$$

We define the following set

$$\mathcal{Z}_{n_c} = \left\{(X, Y) \mid \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \text{ and } \operatorname{rank}(I - XY) \leq n_c\right\} \quad (5)$$

Hence, a controller of order at most n_c can be obtained by requiring that $(X, Y) \in \Gamma_{convex} \cap \mathcal{Z}_{n_c}$ However $\mathcal{Z}_{n_c}$ is a non-convex set and efficient convex programming cannot be used for low-order controller design.

2.1 Alternating Projection Methods

Alternating projection methods have been used in the past to solve statistical estimations and image restoration problems [20]. They provide iterative schemes for finding a feasible point in the intersection of a family of convex sets. The basic idea is that of a cyclic sequence of projections onto the constraint sets. Modifications of the standard algorithm provides accelerated convergence utilizing directional information about the constraint sets. Recently alternating projections have been used in covariance control design [14] and signal processing problems [13]. The fundamental result regarding alternating projection methods for convex sets is the following.

Theorem 1 *Let $\{C_i\}_1^k$ be a family of closed convex sets in a finite dimensional Hilbert space $\mathcal{H}$, and let $x_0 \in \mathcal{H}$. Then the sequence of alternating projections*

$$\begin{aligned} x_1 &= \mathcal{P}_{C_1} x_0 \\ x_2 &= \mathcal{P}_{C_2} x_1 \\ &\vdots \\ x_k &= \mathcal{P}_{C_k} x_{k-1} \\ x_{k+1} &= \mathcal{P}_{C_1} x_k \\ &\vdots \end{aligned} \quad (6)$$

converges to a vector $x \in \bigcap_1^k C_k$. If the intersection is empty then the sequence does not converge.

Unfortunately, due to the rank constraint set (5) the low-order control design is a non-convex problem and the sequence of alternating projections is not guaranteed to converge to a feasible solution. However, local convergence of the sequence is still guaranteed [6]. The following result provides the projection to the set of matrices of constraint rank, e.g. see [16].

Theorem 2 *Let R_k be the set of $n \times n$ matrices of rank less or equal than k, and let $X \in R^{n \times n}$. Suppose that the singular value decomposition of X is given $X = U\Sigma V^T$. A projection $X^* = \mathcal{P}_{R_k}(X)$ of X onto R_k is given by*

$$X_* = U\Sigma_k V^T$$

where Σ_k is the diagonal matrix obtained by replacing the $n - k$ smallest singular values of X by zero.

The above theorem is used to compute the projection onto the set $\mathcal{Z}_{n_c}$, see [15].

2.2 Semidefinite Programming

To find a solution to an LMI constraint is referred to as an LMI feasibility problem. This problem can been solved efficiently using interior point algorithms, see for instance [17]. The Semidefinite Programming (SP) Problem is to find a vector $x \in R^m$ to solve

$$\min c^T x$$

$$\text{subject to } F(x) \geq 0$$

where $F(x)$ is an LMI. Software, such as the MATLAB LMI Control Toolbox , and the Semidefinite Programming (SP) Software with a user-friendly top called 'LMITOOL' is available for numerical solution, see [9], [10]. In this paper, semidefinite programming is used to compute the projections onto the LMI constraints (1)-(2). LMITOOL and SP has been used for this purpose.

3. Mixed SP/AP Design for Low-Order Control

In [15], the alternating projection method is used for low-order control design. To this end, the LMI constraints (1)-(3) are expressed as intersections of sets of simpler geometry, and analytical expressions are provided for the projections onto these sets and the rank constraint set (4). However, the iterative projections onto these multiple sets require a large number of iterations for convergence to a feasible solution to be achieved. In the present work, the projections onto the LMI constraint sets described by (1)-(3) are computed solving SP problems. These projections, along with the analytical projections onto the rank constraint set ([?]), are utilized in alternating projection methods for low-order control design. Hence, the multiple projections required in onto the set Γ_{convex} are eliminated and faster convergence can be achieved. The following results, see [5], provides the projection onto the general LMI constraint set as a solution of an SP problem

Proposition 3 *Let Γ be the convex set described by an LMI. Then the projection $X^* = \mathcal{P}_\Gamma X$ can be computed as the unique solution Y to the SP problem*

$$\textit{minimize} \ \ \textit{trace}(S)$$

$$\textit{subject to} \ \begin{bmatrix} S & Y-X \\ Y-X & S \end{bmatrix} \geq 0 \ \textit{and} \ Y \in \Gamma, \ S \in \mathcal{S}_n .$$

The alternating projection algorithm can now be programed using SP. The proposed solution is the following: First find a solution that corresponds to a full-order controller. This is simply done by solving the LMI feasibility problem that corresponds to the set Γ_{convex}. Next we obtain a solution that corresponds to a controller of order at most $n_c - 1$. This can be done with the SP problem of minimizing trace$(X+Y)$ subject to $(X,Y) \in \Gamma_{convex}$. The given solution will be the starting point for our alternating projection algorithm.

Consider the following SP problem, where (X_0, Y_0) are fixed and $X, Y, S, T \in \mathcal{S}_n$ are the free variables

$$\text{minimize} \ \ \text{trace}(T+S)$$

$$\text{subject to} \ \begin{bmatrix} T & X-X_0 \\ X-X_0 & I \end{bmatrix} \geq 0$$

$$\text{and} \ \begin{bmatrix} S & Y-Y_0 \\ Y-Y_0 & I \end{bmatrix} \geq 0, \ (X,Y) \in \Gamma, \ T,S \in \mathcal{S}_n$$

Denote the minimizing solution by (X^*, Y^*) and write it in the short form

$$(X^*, Y^*) = \tilde{P}_\Gamma(X_0, Y_0)$$

The $\tilde{}$ indicates that this solution is found using semidefinite programming. Note that for simple set $\mathcal{S}$ the projection can be found explicitly, see [15]. However the feasible set Γ_{convex} presented in subsection contains the union of three different sets. These can be combined and solved in one single SP problem. The mixed AP/SP algorithm can now be written in the same way as in theorem 1 with two projecting operators as $\tilde{\mathcal{P}}_{\Gamma_{convex}}$ and $\mathcal{P}_{\mathcal{Z}_{n_c}}$. Note that in order to have fast convergence the final algorithm needs to use the directional alternating projection method as mentioned in [15].

4. Numerical Experiments

4.1 Randomly Generated Stabilizable Systems

The first numerical experiment considers randomly generated static-output-feedback stabilizable systems. A stabilizable system is obtained by reflecting the eigenvalues of randomly generated matrices via an eigenvalue-eigenvector decomposition where the positive eigenvalues are replaced with their negative values. Then a probuct of arbitrary input, feedback gain and output matrices is subtracted from this matrix to guarantee that the result is static-output-feedback stabilizable. By shifting all eigenvalues by an amount $-\alpha$, where α is a positive scalar, desired α-degree of stability can be introduced in the static output feedback problem.

Table 1 shows the cases we considered. Notice that the Kimura bound k_b that provides an upper bound for the lowest order stabilizing controller is equal to $k_b = 3$ for cases a), b) and c), and $k_b = 0$ for cases d), e) and f).

Case	System Order n	Number of inputs n_u	Number of outputs n_y
a	4	1	1
b	6	2	2
c	8	3	3
d	4	3	2
e	6	4	3
f	8	5	4

Table 1: Numerical Experiment Cases

For each one of the above six cases 200 hundred random experiments were carried out. A degree of stability $\alpha = 0.1$ has been introduced in the randomly generated systems and the objective is to obtain the lowest order stabilizing controller that places the closed-loop poles to the left of $-\alpha$. Table 2-7 show the results for each one of the above cases.

Number of experiments	Number of iterations	Lowest order achievable controller
176	0	0
6	7-38	0
5	0	1
4	4	0
3	5	0
3	3	0
3	2	0

Table 2: Computational results for case a)

Number of experiments	Number of iterations	Lowest order achievable controller
177	0	0
7	2	0
5	7-38	0
4	3	0
3	4	0
2	6	0

Table 3: Computational results for case b)

Number of experiments	Number of iterations	Lowest order achievable controller
184	0	0
7	2	0
3	1	0
2	3	0
1	4	0
1	6	0
1	7-38	0
1	0	1

Table 4: Computational results for case c)

Number of experiments	Number of iterations	Lowest order achievable controller
200	0	0

Table 5: Computational results for case d)

Number of experiments	Number of iterations	Lowest order achievable controller
199	0	0
1	3	0

Table 6: Computational results for case e)

Number of experiments	Number of iterations	Lowest order achievable controller
196	0	0
3	2	0
1	4	0

Table 7: Computational results for case f)

Table 8 shows the average CPU time that was needed in each one of the above cases to obtain the lowest order achievable controller.

Case	Average CPU time (sec)
a	18.71
b	15.93
c	26.15
d	1.17
e	3.19
f	11.18

Table 8: Average CPU time

From these results it is observed that in the majority of the experiments, the lowest order achievable controller is obtained in 0 iterations, that is, by solving the convex problem trace$(X+Y)$ subject to $(X,Y) \in \Gamma_{convex}$ as described in section 3. In all the above experiments, the lowest order achievable controller is of order lower or equal to the Kimura bound k_b. In 5 experiments for case a), the lowest order achievable controller was 1, instead of the zeroth order that is guaranteed by the construction of the experiments.

4.2 Helicopter Example

The following example is from [4]. The goal is to obtain a static state feedback controller for the following helicopter model such that the closed-loop poles are located to the left of $-\alpha = -0.1$ at the complex plane.

$$\dot{x} = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -0.4555 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x.$$

Zero iterations, that is the solution of the convex minimization problem: trace$(X+Y)$ subject to $(X,Y) \in \Gamma_{convex}$, provides a static controller

$$K = \begin{bmatrix} -0.2162 \\ 2.4942 \end{bmatrix}$$

to achieve the desired objective. The closed-loop poles are -20.8379, -0.1042 and $-0.2572 \pm 0.9738i$. The required CPU time to obtain this controller is 1.27 sec.

4.4 Spring-Mass Systems

In this numerical experiment interconnected spring-mass system models were generated. The order of the system is equal to twice the number of interconnected masses. The objective is to obtain the lowest order stabilizing controller that provides a desired degree of stability α. The following tables provides the lowest order achievable controller and the number of iterations needed for $\alpha = 0.001$ and $\alpha = 0.1$.

Number of masses	Number of iterations	Lowest order achievable controller
2	0	2
3	0	3
4	0	4
5	0	5

Table 9: Results for degree of stability $\alpha = 0.001$

Number of masses	Number of iterations	Lowest order achievable controller
2	0	2
3	0	3
4	1	5
5	1	6

Table 10: Results for degree of stability $\alpha = 0.1$

Either zero or one iterations of the algorithm is enough to provide a static output feedback controller depending on the desired degree of stability α.

4.5 Conclusions

Computational experiments have been used to demonstrate the applicability of alternation projection algorithms for low-order control design. Fixed and random benchmark examples have been developed for this objective. These results indicate that the proposed alternating projections combined with semidefinite programming are effective in obtaining low-order controllers for small and medium order problems.

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