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Bifurcation Analysis and Dimension Reduction of a Predator-Prey Model for the L-H Transition

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The L-H transition denotes a shift to an improved confinement state of a toroidal plasma in a fusion reactor. A model of the L-H transition is required to simulate the time dependence of tokamak discharges that include the L-H transition. A 3-ODE predator-prey type model of the L-H transition is investigated with bifurcation theory of dynamical systems. The model is recognized as a slow-fast system.

INTRODUCTION

The L- and H-modes are confinement states of a toroidal plasma, referring to states of low and high confinement, respectively. The transition from the L- to the H-mode is called the L-H transition. The T-mode is a transient, intermediate mode between the L- and H-mode, characterized by an oscillatory behavior. The L-H transition still lacks a first principle explanation. Modeling the L-H transition might contribute to a better understanding of the underlying mechanisms.

THE 3-ODE L-H TRANSITION MODEL

We consider the minimal 3-ODE predator-prey type L-H transition model suggested by Kim and Diamond [2, 3]. The model ignores spatial dependencies. The dependent variables in the model are:

- the drift wave turbulence level \( \xi \),
- the shear of the zonal flow \( \psi \), and
- the gradient of the ion pressure \( \nabla \).

The model can be formulated as

\[
\begin{align*}
\frac{d}{dt} \xi &= \xi \left( N - a_1 \xi - a_2 \psi V^1 - a_3 V^2 \right), \\
\frac{d}{dt} \psi &= \psi \left( b_1 \xi - 1 + b_2 \psi - b_3 \right), \\
\frac{d}{dt} V^1 &= V^1 \left( N - c_1 \xi - c_2 \right),
\end{align*}
\]

where \( a_1, b_1, c_1, i = 1, 2, 3 \) are parameters and \( Q \) is the heating power. Introducing new variables and time,

\[
\begin{align*}
\alpha &= a_1^{1/3} c_1^{2/3} \xi, \\
\nu &= a_2^{1/3} c_2^{2/3} \psi V^2, \\
\sigma &= a_3^{1/3} c_3^{2/3} V^1,
\end{align*}
\]

results in the non-dimensionalized system

\[
\begin{align*}
\dot{\alpha} &= \mu_1 \nu \left( \alpha - \mu_1 \alpha - \mu_2 \right), \\
\dot{\nu} &= \mu_2 \left( \sigma - \mu_2 \nu \right)
\end{align*}
\]

where, \( \mu_i, i = 1, \ldots, 2 \) are new parameters and \( \sigma \) is the rescaled heating power.

BIFURCATION ANALYSIS

The nullclines are

\[
\begin{align*}
N_\alpha &= \{ \alpha = 0 \} \cup \{ \alpha = \nu (1 - \omega - \nu) \}, \\
N_\nu &= \{ \nu = 0 \} \cup \{ \nu = \mu_2 (1 + \mu_1 \nu) \}, \\
N_{\sigma} &= \{ \sigma = 0 \} \cup \{ \sigma = \mu_2 (1 + \mu_1 \nu) \}.
\end{align*}
\]

Stability of the equilibrium points:

- \( L \) is a stable node when it is below \( N_\alpha \).
- \( H \) is always a saddle (unstable).
- \( T \) is a focus point. Stability depends on the value of \( \sigma \).
- \( QH \) is stable for \( \sigma > 1 \).

THREE TRANSITION TYPES

The bifurcation diagram structure depends on \( \mu_i, i = 1, \ldots, 2 \). By varying \( \mu_1 \) and \( \mu_2 \), three different transition types are observed.

DIMENSION REDUCTION WITH GSPT

Put \( \mu_1 = \frac{1}{\epsilon}, \) where \( 0 < \epsilon \ll 1 \).

- \( \nu \) and \( \sigma \) are slow variables,
- \( \nu \) is a fast variable.

For \( \epsilon > 0 \), but sufficiently small, solutions converge to the slow manifold,

\[
M = M_0 + \epsilon M_1 + \epsilon^2 M_2 + \cdots
\]

The reduced system of the flow is found by taking the limit \( \epsilon \to 0 \):

\[
\begin{align*}
\dot{\alpha} &= \frac{\mu_1 \nu \left( \alpha - \mu_1 \alpha - \mu_2 \right)}{1 + \mu_1 \nu}, \\
\dot{\nu} &= \frac{\mu_2 \left( \sigma - \mu_2 \nu \right)}{1 + \mu_1 \nu}
\end{align*}
\]

The reduced system contains the same dynamics as the full system.

CONCLUSION

Kim and Diamond’s 3-ODE L-H transition model was investigated with bifurcation theory.

- The model contains three types of transitions: an oscillating, a sharp with hysteresis, and a smooth transition.
- The system can be reduced to a 2-ODE system

A spatio-temporal L-H transition model has been proposed by Miki et al. [4].

References: