Optimal Design of Composite Structures Under Manufacturing Constraints

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Optimal Design of Composite Structures Under Manufacturing Constraints

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In loving memory of
Professor George Athanasiadis
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Preface

This thesis is submitted in partial fulfillment of the requirements for obtaining the degree of Ph.D. at the Technical University of Denmark (DTU). The Ph.D. project was funded by The Danish Council for Independent Research | Technology and Production Sciences (FTP) under the grant "Optimal Design of Composite Structures under Manufacturing Constraints" and has been carried out at the Wind Energy Department and the Department of Applied Mathematics and Computer Science at DTU during the period August 1st 2011 – July 31st 2014. Senior Scientist Dr. Techn. Mathias Stolpe, Professor Erik Lund and Associate Professor Lars Pilgaard Mikkelsen were the supervisors on the project. I would like to thank all my supervisors for supporting my research with their directions and valuable experience throughout the course of the studies. I am very grateful to Dr. Ming Zhou, Boris Lauber, Dr. Stefan Schwarz, Dr. Yosuke Ueki and Dr. Peter Kopalidis for their continuous support during my PhD studies. Finally, I would like to express my deep gratitude to my family: my parents and my brothers for all their love and support.

Roskilde, July 2014

Konstantinos Marmaras
Summary

This thesis considers discrete multi material and thickness optimization of laminated composite structures including local failure criteria and manufacturing constraints. Our models closely follow an immediate extension of the Discrete Material Optimization scheme, which allows simultaneous determination of the appropriate laminate thickness and the material choice in the structure. The optimal design problems that arise are stated as nonconvex mixed integer programming problems. We resort to different reformulation techniques to state the optimization problems as either linear or nonlinear convex mixed integer 0–1 programming problems. The manufacturing constraints have been treated by developing explicit models with favorable properties.

In this thesis we have developed and implemented special purpose global optimization methods and heuristic techniques for solving this class of problems. The continuous relaxation of the mixed integer programming problems is being solved by an implementation of a primal–dual interior point method for nonlinear programming that updates the barrier parameter adaptively. The method is chosen for its excellent convergence properties and the ability of the method to react swiftly to changes of scale in the problem. As opposed to the original Discrete Material Optimization methodology, we obtain discrete feasible solutions to the stated mixed 0–1 convex problems by the application of advanced heuristic techniques. Our heuristics are based on solving a finite sequence of well–posed optimization problems. They provide us with a discrete feasible solution or correctly determine problem infeasibility. Our aim is to solve the considered problems to proven global optimality. We propose a combination of the convergent Outer Approximation and Local Branching algorithms to perform the global optimization. The efficiency of the proposed models is examined on a set of well–defined discrete multi material and thickness optimization problems originating from the literature. The inclusion of manufacturing limitations along with structural considerations in the early design phase results in structures with better structural performance reducing the need of manually post–processing the found designs.
Resumé (in Danish)


Chapter 1

Introduction
In many cases, using composite materials is more advantageous than using conventional materials such as steel and aluminum. These advantages may include higher specific strength and stiffness, improved fatigue and impact resistance, improved thermal conductivity and corrosion resistance. Applications of composites range from aerospace, medical, automotive applications, etc. It is the ability of composite structures to adapt to local design requirements that designates them as a compelling alternative to metallic structures. However, this highly attractive feature is accompanied by the complexity of the resulting design problem, making design optimization an appropriate tool for the design of laminated composite structures.

The main project contributions involve the development and implementation of new special purpose numerical methods and design parameterizations of composite structures which can be used to conveniently model manufacturing constraints and failure criteria. The theoretical implications introduced by the additional constraints are examined on a set of well-defined benchmark examples originating from the literature. Our implementation proves to be competitive with existing methods for optimal design of laminated composite structures.

This thesis consists of six chapters. In Chapter 1 we present an extensive literature study of optimal design of laminated composite structures. An overview study of existing methods and heuristics for mixed integer nonlinear optimization is presented in Chapter 2. The main theoretical and numerical results, the main research contributions, potential scientific impact and future research developments are reported in Chapter 3. Chapter 4 describes our implementation of global optimization methods and heuristic techniques. In Chapter 5 we extend our models in order to perform simultaneous multi-material and thickness optimization of laminated composites. The problem formulations include additional constraints which conveniently model manufacturing limitations. Finally, Chapter 6 investigates the addition of local failure criteria into our models.

1.1 Laminated Composite Structures

A composite material consists of two or more constituent materials, not soluble in each other, which are combined at a macroscopic level. There are two main categories of constituent materials, the reinforcing phase and the matrix. The reinforcing phase may exist in the form of fibers, particles or flakes and it is embedded into the matrix. The matrix functions include binding the reinforcing together, distributing the load to the reinforcement phase material, and protecting the reinforcement from the environment or from damage due to handling.

Composites are classified by the geometry of the reinforcement (fibers, particles or flakes) and the type of matrix. The most common composites are polymer matrix composites due to their low cost, high strength and simple manufacturing principles. Techniques of manufacturing a polymer matrix composite include filaments winding, autoclave forming, and resin transfer molding. Polymers are further classified as thermosets and thermoplastics. Thermoset polymers (e.g. epoxies, polyesters, phenolics and polyamide), are insoluble and infusible after cure, while thermoplastics (e.g. polyethylene, polyesterene and polyphenylene sulfide), are formable at high temperatures and pressure.

The fiber factors that contribute the most to the mechanical performance of a composite are the length, the shape, the orientation and the material of the fibers. The fibers may be oriented in specific directions in order to obtain high stiffness in the loading direction and low stiffness in other directions. Although matrices by themselves generally have low mechanical properties compared to those of fibers, the matrix influences many mechanical
properties of the composites. These properties include transverse modulus and strength, shear modulus and strength, compressive strength, interlaminar shear strength, thermal expansion coefficient, thermal resistance, and fatigue strength.

Figure 1.1: Composite Materials.

1.2. Optimal Design of Laminated Composite Structures

Laminate design is typically accompanied by optimization of the thickness and orientation of the anisotropic or orthotropic layers and materials. Formulating practical composite design problems as mathematical programming problems introduces several complexities such as the nonconvexity of the objective and constraint functions, the associated large number of design variables and the inherent difficulties involved with solving large-scale optimization problems.

This section reviews the main optimization methods used for the design of laminated composite structures. We have categorized the considered optimization approaches into three groups; Gradient based methods, Direct search methods and Specialized Techniques. An exhaustive review regarding the application and classification of existing optimization techniques and design parameterizations for composite structures can be found in [14, 13].

1.2.1 Gradient based methods

These algorithms are based on calculating the gradient of the objective and constraint functions in order to find the direction towards the optimal solution. Gradient based methods provide a faster convergence rate compared to direct search methods. Moreover, the application of gradient based methods can be utilized as a warm start technique in a more general global optimization framework. However, the extensive computational effort spend on calculating the gradients of the objective and constraints functions together with the fact that the global optimum cannot be guaranteed with these methods, are preventing their wider application in optimal design of laminated composites structures.
CHAPTER 1. INTRODUCTION

Closed-form expressions of the objectives and constraints often do not exist for the problems under study and therefore gradient based methods mainly resort to the application of approximation schemes. The original problem is replaced by a sequence of well-posed approximate subproblems generated through the first or second order Taylor series expansions of these functions in terms of the design variables. There exist many general robust and efficient methods developed for solving the resulting subproblems. Two methods that have received large acceptance in this field are Conlin’s method, see e.g. [15], and the Method of Moving Asymptotes (or MMA), see e.g. [63]. In Conlin’s method the approximate function is obtained by a combination of a linear approximation and an inverse approximation according to the sign of the derivatives of the structural responses. The Method of Moving Asymptotes consists of a linearization of the response functions with respect to variables of the type $1/(x_i - L_i)$ or $1/(U_i - x_i)$, depending on the sign of the derivatives of the corresponding function, where $x_i$ are the design variables and $U_i$ and $L_i$ represent the corresponding limits (or asymptotes) of the variables. The values of the limits $U_i$ and $L_i$ are updated at each iteration of the optimization run.

1.2.2 Direct search methods

Direct search methods constitute the most popular approach for optimizing laminated composite structures. They are global optimization methods that have the ability to work directly with integer variables. In contrast to gradient based methods, these methods do not require any gradient information. This attribute is a theoretical and computational significant advantage in composite laminate design where sensitivities of structural responses are difficult to calculate.

Genetic algorithms have seen the widest application among the direct search methods. They are evolutionary optimization techniques using Darwin’s principal of survival of the fittest to improve a population of solutions. They have been successfully applied to design problems with structural criteria such as strength, stiffness, buckling loads and fundamental frequencies, see e.g. [51] and [49]. However, the use of evolutionary techniques has been limited to small scale problem instances because of the exhaustive computing cost. Furthermore, if the initial population is not appropriately selected, the application of a genetic algorithm might suffer from premature convergence.

1.2.3 Specialized techniques

In this category fall methods that can efficiently exploit the structure of the optimal design problems in order to simplify the optimization process. These techniques are tailored to the characteristics of the considered problems and thus may lack robustness when applied to a general optimization problem.

Multi-phase topology optimization

The multi-material selection problem in a fixed reference design domain, was first considered in [55] and [26] using a three-phase topology optimization method. This approach was later extended to handle any number of phases in the parameterization scheme commonly referred to as Discrete Material Optimization (or DMO), see e.g. [26] and [22]. The DMO methodology has been applied successfully to optimal design problems with structural criteria such as compliance in [26], [22], [55] and [31], eigenfrequencies in [22], and buckling load factors
1.2. OPTIMAL DESIGN OF LAMINATED COMPOSITE STRUCTURES

in [34]. The design variables $x_{ijk} \in \{0,1\}$ are in this case binary and associate candidate materials $i$ from a given set to each layer $k$ and every design domain $j$. The choice among the candidate materials supposes the selection of a single material in each design domain. This condition is enforced by employing the following linear equality constraints, also called generalized upper bound equality constraints

\[ \sum_{i=1}^{I} x_{ijk} = 1, \quad \forall (j,k). \]  

(1.1)

Together with the integrality conditions $x_{ijk} \in \{0,1\}$ the generalized upper bound equality constraints (1.1) ensure that only one material can be chosen in each design domain. The material selection can be generalized to the case where the set of candidate materials is not necessarily the same for each design domain.

The basic approach followed in [26] and [22] involves relaxing the integer constraints on the binary design variables, thus allowing values between 0 and 1 during the optimization. The relaxed problem can then be solved by the application of a gradient based method. The continuous design variables are eventually penalized to 0 or 1 through the use of a material interpolation scheme, see e.g. [11], [13], [52] and [59].

An interesting alternative parameterization scheme named shape function with penalization (or SFP) was recently introduced in [7]. The design variables in this case are based on the shape functions of a quadrilateral first order finite element, i.e. each vertex of the reference quadrangle represents a candidate material. SFP is restricted to the design problem of four candidate orientations ($0^\circ, 90^\circ$ and $\pm 45^\circ$). The approach has been later generalized with a bi–value parameterization (or BCP) scheme [16] to deal with any finite number of materials. The main advantage of BCP compared to DMO is a reduction in the total number of design variables. Note that multi–phase topology optimization methods cannot theoretically guarantee the global optimum and the results depend on the chosen material interpolation scheme.

### Design with lamination parameters

In this case, the design variables (also called lamination parameters, see e.g. [66]) are based on integrated trigonometric functions through the thickness of a laminate. A total of at most twelve lamination parameters suffices to fully describe the laminate properties, considerably reducing the total number of design variables appearing in the resulting optimization problem. Moreover, the mathematical structure of the parameterization allows the optimal design problems to be stated as convex problems which offers significant theoretical and computational advantages. However, the lamination parameters are not independent and therefore do not provide a direct description of the laminate construction. Moreover, the solution process involves solving the inverse problem in order to obtain the stacking sequence corresponding to the optimum lamination parameters, which complicates the optimization process significantly. Therefore the application of this methodology to general laminate composite optimization problems is rather limited.

### Optimality criteria

The optimality criteria methods are based on the derivation of an appropriate mechanical criterion such as the energy density of a uniform strain field, see e.g. [45], or the co–alignment
of the principal stress directions with the fiber orientations, see e.g. [45]. They refer to specialized design conditions, where the optimal design is obtained by following an iterative solution process.

1.3 Manufacturing constraints

The final quality of a composite component is dependent on a number of manufacturing related factors such as the occurrence of voids, dry spots and high temperature peaks. For this reason a good understanding of the manufacturing process is crucial in achieving high quality components. We present a number of manufacturing related constraints appearing to a varying extent in the literature. Most of these constraints correspond to well established design rules that prevent laminate failure initiation, such as matrix cracking and delamination.

1.3.1 Blending

In the design of laminated composite structures, it is common practice to divide the structure into panels that may be designed independently. In general such an approach results in designs which lack continuity of laminate layers across the individual panels. The resulting discontinuities can cause stress concentrations and increase manufacturing difficulty and cost. Designing composite laminates which exhibit continuity of the ply orientation angles (or materials) across adjacent panels is commonly referred to as blending.

Optimization approaches to such problems have attracted considerable attention, typically using genetic algorithms and heuristic techniques. A blending methodology was presented in [31] based on the concept of key regions (or heaviest loaded regions) and a "greater–than–or–equal–to" rule to ensure manufacturability of the obtained designs. Assessing manufacturing complexity was achieved in [35] by introducing appropriate measures of material composition and stacking sequence continuity between adjacent laminates. A guide based design methodology within a genetic algorithm optimization scheme was employed in [54] to satisfy the blending requirements. A heuristic technique, the so–called shared–layer blending process, was applied in [65] within a bi–level optimization strategy.

![Figure 1.2: Stacking sequence continuity between adjacent laminates.](image)

1.3.2 Ply–drop constraints

Tapering of laminate thickness is often necessary for reasons of material saving (in low stress areas) and shape conformity requirements. Laminate thickness variation throughout the entire structure results in the termination of plies at different locations, known as ply–drops. The resulting stress concentrations appearing at the drop locations can lead to the development
of delamination and premature buckling of the laminate. We can reduce the risk of such a failure by designing composite structures consisting of large zones (or patches) of uniform thickness and smooth plydrops between these zones.

Ply–drop constraints expressed as the ratio of two thicknesses were employed in [63] in order to limit the thickness variation rate between adjacent zones. The sublaminates (group of plies) approach was followed in [39] to yield the plydrop tapering. A genetic algorithm implementation within an expert system shell was developed in [34] to minimize the weight of a composite laminate with ply–drop.

![Figure 1.3: Schematic illustration of laminate cross section with ply drops (source: Collier Research at www.compositesworld.com).](image)

1.3.3 Ply blocking

The lay–up design of laminated composite structures has a strong influence on the formation of interlaminar normal and shear stresses near the laminate edge zones, see e.g. [42]. It has been demonstrated in [48, 44] that the existence of interlaminar stresses initiates two different failure modes at a free edge: (1) crack initiation through the thickness of the interior plies and (2) free–edge delamination along the midsurface of the laminate. The main purpose of the ply blocking (or contiguity) constraints is to reduce the risk of such a failure by limiting the allowable number of identical contiguous uni–directional (UD) fiber material candidates through the thickness of the laminate.

The performance of a genetic algorithm for buckling load maximization of a composite laminate subject to constraints on ply contiguity was investigated in [51]. A permutation genetic algorithm using a repair strategy for permutations violating the contiguity constraint was developed in [33]. An approximation of the ply blocking constraints as functions of the lamination parameters was provided in [24] to optimize anisotropic composite panels with T–shaped stiffeners.

1.3.4 Balanced condition

Another common guideline in the design of composite structures is the requirement for a balanced and symmetric laminate. The balanced condition requires that for every angle ply (those at any angle θ other than 0° and 90°) with a positive fiber orientation angle θ, there is a corresponding ply with the negative fiber orientation angle –θ. A balanced and symmetric lay–up eliminates the occurrence of any couplings between the structural responses of the laminate. Moreover, such a configuration limits the occurrence of manufacturing induced
shape distortions such as warping of the composite structure during the cool-down from the cure temperature, see e.g. [60]. For an application of the balanced condition on optimization problems of laminated composite structures, see e.g. [34], [65] and [64].

1.3.5 Available amount of plies

The manufacturing process of composite materials such as resin infusion moulding may dictate a minimum number of plies in order to avoid the creation of dry spots in the manufactured state. In addition, a minimum number of plies of a given orientation may be specified to increase the strength in directions that are not primary load paths. For composite design, it was proposed in [42] that at least 10% of each ply orientation should be provided. The inclusion of constraints on the available amount of plies has been considered in problems with buckling, strength, and tip twist constraints, see e.g. [63], [65] and [24].

1.3.6 Placement of the angle plies

Based on the fact that under normal buckling the $\pm 45^0$ plies will tend to group towards the outside of the laminate surface in order to maximize the in-plane buckling resistance, the outer plies for the skin should always contain at least one pair of $\pm 45^0$ plies, see e.g. [34] and [65].

1.3.7 Draping

The forming (draping) stage of the manufacturing process in case of doubly curved components results in significant shear deformation due to inter-fibre angle change and inter-fibre
1.3. MANUFACTURING CONSTRAINTS

sliding. The resulting localized buckling and wrinkling which may occur due to this excessive shear deformation can be eliminated by localized stitching. In such a case a correction of the fiber orientation needs to be considered in the analysis models by performing a draping analysis, see e.g. [69].

Figure 1.6: Ply orientation draping on a doubly curved surface. Adapted from Optimization of Composites – Recent Advances and Applications, by M. Zhou, R. Fleury and M. Kemp, 2011, The 7th Altair CAE Technology Conference.

1.3.8 Tow–placement

Laminated composite structures are traditionally manufactured by hand lay–up techniques. Among the automated production processes, placing fibers using an advanced tow–placement machine is becoming an increasingly popular alternative to traditional hand lay–up techniques. By combining features of automated tape laying and filament winding, the tow–placement technology enables the mass production of a wide range of high quality composite products. In this case stiffness tailoring of the laminated composite structures is achieved by allowing the fibers to follow curvilinear paths within the plane of the laminate. This novel design concept is particular applicable in structures with highly non–uniform stress states.

In order to take advantage of these advanced capabilities, the designer must carefully consider the limitations imposed by the fibre–placement. A minimum turning radius for the fiber path must be defined, see e.g. [41], in order to avoid any issues with out of plane wrinkling of the inner tows. The minimum tow length is also of concern to the designer. The occurrence of too many gaps and overlaps between tows of adjacent courses, leads to the creation of resin–rich regions which might initiate laminate damage and failure. Manufacturing constraints that apply for tow–placement were used in [4] to optimize fibre–reinforced composite conical shells for maximum fundamental frequency.
Manufacturing of laminated composite structures involves various sources of uncertainty resulting in a variation of the material properties, see e.g. [36]. Variations in the fibre architecture associated with tow waviness, distribution of fibers inside the tows, resin content variations, misalignment of fibers or imperfect bonding between fibers and matrix can directly affect the filling and curing processing steps which in turn influence the quality of the final product. These effects can intensify the formation of residual stresses following the cure process and lead to delamination or warping of the composite during the cool-down from the cure temperature. Therefore, the development and implementation of stochastic simulation tools, which can appropriately quantify the process output variability as a function of material selection and process parameter definitions is crucial.

A two-step optimization process was developed and implemented in [56] for laminated plate stacking sequence and thickness optimization taking the uncertainties of material properties into account. A Robust Design Optimization (or RDO) procedure was presented in [67] for minimum weight optimization of a symmetrically laminated plate, with manufacturing uncertainty in the layup angle and thickness accounted for. A coupled evolutionary and heuristic algorithm was utilized in [17] for stacking sequence and thickness optimization of dimensionally stable composites. The reliability of the in-plane designs was estimated by performing a Monte Carlo analysis based on material and geometry property distributions.
Chapter 2

Mixed Integer Nonlinear Optimization Techniques
Throughout this Thesis our models will closely follow the problem formulations presented in the Discrete Material Optimization parameterization scheme, see e.g. [26] and [22]. The design problem in this case becomes a combinatorial problem of choosing the fiber direction from a permissible set for each ply. By using this design parameterization it is possible to model several of the manufacturing constraints mentioned in Section 1.3. Moreover, if also the ply thicknesses are included as variables then additional manufacturing constraints can be modeled. The optimal design problems that arise are classified as mixed integer optimization problems. In the following we will present several methods and heuristic techniques for solving mixed integer optimization programming problems.

2.1 Global Optimization

We consider the nonlinear (convex) programming problem of the form

\[
\begin{align*}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad g(x, y) \leq 0.
\end{align*}
\]  

We consider the objective function \( f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R} \) and the (nonlinear) constraints \( g : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R} \) to be convex and continuous differentiable. In this section we will present a number of (convergent) cutting plane methods in order to solve the optimization problem (2.1) to proven global optimality. Cutting plane methods poses several advantages such as simplicity in the implementation and robustness of the solution. These methods are suitable for solving convex mixed integer nonlinear problems with a moderate degree of nonlinearity.

2.1.1 Outer approximation

In the outer approximation algorithm, see e.g. [21] and [23], we solve a finite sequence of nonlinear continuous subproblems and relaxations of a linear mixed integer master program. The main idea behind outer approximation relies on the representation of the nonlinear convex objective and constraint functions by a collection of supporting planes. The relaxation of the outer approximation master problem of (2.1) is given by the linear mixed 0–1 problem

\[
\begin{align*}
\text{minimize} & \quad \eta \\
\text{subject to} & \quad \eta \leq \min\{f(x^k, y^k)\}, & k = 0, \ldots, P, \\
& \quad f(x^k, y^k) + (\nabla f(x^k, y^k))^T \left( \begin{array}{c} x \\ y \end{array} \right) - \left( \begin{array}{c} x^k \\ y^k \end{array} \right) \leq \eta, & k = 0, \ldots, P, \\
& \quad g(x^k, y^k) + (\nabla g(x^k, y^k))^T \left( \begin{array}{c} x \\ y \end{array} \right) - \left( \begin{array}{c} x^k \\ y^k \end{array} \right) \leq 0, & k = 0, \ldots, P, \\
& \quad g(x^r, y^r) + (\nabla g(x^r, y^r))^T \left( \begin{array}{c} x \\ y \end{array} \right) - \left( \begin{array}{c} x^r \\ y^r \end{array} \right) \leq 0, & r = 0, \ldots, T.
\end{align*}
\]  

Note that an upper bound on variable \( \eta \) has been set in order to ensure that a previously found feasible design \((x^k, y^k)\) with \( k = 1, \ldots, P \) is not replicated by the algorithm, see e.g. [23]. The projection of (2.1) onto the space of the integer variables, at iteration \( i \) of the solution process, gives rise to the following nonlinear subproblem
2.1. GLOBAL OPTIMIZATION

\[
\begin{align*}
\text{minimize} & \quad f(x^i, y) \\
\text{subject to} & \quad g(x^i, y) \leq 0.
\end{align*}
\]

(2.3)

Problem (2.3) is parameterized by the integer solution \( x^i \in \mathbb{Z}^n \) and therefore includes only the continuous variables \( y \in \mathbb{R}^{n_2} \). Problem (2.2) includes additional constraints derived from infeasible subproblems of (2.3), i.e. the constraints

\[
g(x^r, y^r) + (\nabla g(x^r, y^r))^T \left( \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^r \\ y^r \end{pmatrix} \right) \leq 0, \ r = 0, \ldots, T.
\]

These (feasibility) constraints (or cuts) ensure that integer assignments which produce infeasible subproblems are also infeasible in the master program (2.2). Each iteration of the linear outer approximation algorithm chooses a new integer assignment \( x^i \) and attempts to solve (2.3). Either a feasible solution \( y^i \) is obtained or infeasibility is detected and \( y^i \) is the solution of a feasibility problem, see e.g. [23]. Note, that in the particular case of pure integer nonlinear programming problems there is no advantage obtained by solving (2.3).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2_1.png}
\caption{Outer approximation of a convex function in \( \mathbb{R}^1 \).}
\end{figure}

2.1.2 Extended cutting plane method

The extended cutting plane method, see e.g. [31], [58, 59], is closely related to the outer approximation methodology [21] and [23]. The main difference between the two approaches relies on the fact that while outer approximation solves problem (2.1) by decomposing it into a sequence of mixed integer linear programming problems and nonlinear continuous problems, the extended cutting plane method does not solve any nonlinear continuous problems separately. In each extended cutting plane method iteration, one or several cutting planes are constructed based on the most violated constraint(s) on the current iterate, i.e. \( \max\{g(x^i, y^i)\} \), whose linearization is added at the subsequent mixed integer problem. For
strongly nonlinear mixed integer programming problems which mainly consist of continuous variables, the outer approximation method is proven to be more efficient than the proposed extended cutting plane method, see e.g. [59].

2.1.3 Generalized Benders Decomposition

Like outer approximation, see e.g. [21] and [23], Generalized Benders Decomposition [22] solves (2.1) by alternating finitely between a nonlinear continuous subproblem and a linear mixed integer master problem. However, in the Generalized Benders Decomposition method [22] nonlinear duality theory rather than outer approximation is being utilized to obtain the mixed integer master program.

2.2 Interior Point Methods

Cutting plane methods are often criticized because their convergence speed is quite slow and can only handle small scale problem instances. In a practical design situation of composite structures the number of design variables can easily exceed hundreds of thousands which is a practical limit on the problems that can be considered. The basic approach followed in the Discrete Material Optimization parameterization scheme, see e.g. [26] and [22], is to relax the integer constraints on the binary design variables thus allowing values between 0 and 1 during the optimization. The resulting continuous (or relaxed) problem can then be solved by the application of a gradient based method. In this section we will discuss interior point methods for solving nonlinear continuous convex programming problems which include inequality constraints.

2.2.1 The Barrier Method

We consider the nonlinear continuous (convex) programming problem of the form

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_L \leq g(x) \leq g_U \\
& \quad x_L \leq x \leq x_U,
\end{align*}
\]

(2.4)

where \(x \in \mathbb{R}^n\) are the primal variables, with lower and upper bounds \(x_L, x_U \in \mathbb{R}^n\), and \(g : \mathbb{R}^n \to \mathbb{R}^m\) are the constraints, with lower and upper bounds \(g_L, g_U \in \mathbb{R}^m\). We consider the objective function \(f : \mathbb{R}^n \to \mathbb{R}\) and the constraints to be convex and twice continuously differentiable. First we make an explicit distinction between the equality (i.e. \(g_L = g_U\)) and inequality constraints

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad b(x) = 0 \\
& \quad d_L \leq d(x) \leq d_U \\
& \quad x_L \leq x \leq x_U,
\end{align*}
\]

(2.5)

where \(b : \mathbb{R}^n \to \mathbb{R}^{mb}\) are the equality constraints, and \(d : \mathbb{R}^n \to \mathbb{R}^{md}\) are the inequality constraints, with lower and upper bounds \(d_L, d_U \in \mathbb{R}^{md}\). Next we make a simple transformation of the inequality constraints by adding slack variables \(s \in \mathbb{R}^{md}\) with their corresponding bounds into the problem.
2.2. INTERIOR POINT METHODS

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad b(x) = 0 \\
& \quad d(x) - s = 0 \\
& \quad d_L - s \leq 0, \quad s - d_U \leq 0 \\
& \quad x_L - x \leq 0, \quad x - x_U \leq 0.
\end{align*}
\] (2.6)

If a variable bound does not exist, we set the respective value into a large number \((-\infty, +\infty\)). We allow only valid bounds to appear in the problem. We achieve this by introducing permutation matrices between the variables \(x\) and \(s\) and their respective bounds.

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad b(x) = 0 \\
& \quad d(x) - s = 0 \\
& \quad d_L - (P^L_s)^T s \leq 0, \quad (P^U_s)^T s - d_U \leq 0 \\
& \quad x_L - (P^L_x)^T x \leq 0, \quad (P^U_x)^T x - x_U \leq 0,
\end{align*}
\] (2.7)

where \(P^L_x \in \mathbb{R}^{n \times n_x_L}, P^U_x \in \mathbb{R}^{n \times n_x_U}, P^L_s \in \mathbb{R}^{n \times n_s_L}, P^U_s \in \mathbb{R}^{n \times n_s_U}\) are the permutation matrices, and \(n_x_L, n_x_U, n_s_L, n_s_U\) are the number of valid lower and upper bounds of the variables. Our main goal is to approximately formulate the inequality constrained problem as an equality constrained problem to which Newton’s method can be applied, see e.g. [17].

We reformulate problem (2.7) by implicitly introducing the inequality constraints into the objective function of the problem

\[
\begin{align*}
\text{minimize} & \quad f(x) + I_-(x, s) \\
\text{subject to} & \quad b(x) = 0 \\
& \quad d(x) - s = 0,
\end{align*}
\] (2.8)

where

\[
I_-(x, s) = \sum_{i=1}^{m_d} I_-(d_L - (P^L_s)^T s) + \sum_{i=1}^{m_d} I_-(d_U - d_L) + \sum_{i=1}^{n} I_-(x_L - (P^L_x)^T x) + \sum_{i=1}^{n} I_-(x_U - x_L).
\] (2.9)

The function \(I_-(u) : \mathbb{R} \to \mathbb{R}\) is called the indicator function of the nonpositive reals, with the following interpretation

\[
I_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0 \end{cases}.
\] (2.10)

Problem (2.8) is not explicitly subjected to inequality constraints. However, the objective function of the equality constrained problem (2.8) is not differentiable and therefore Newton’s method cannot be directly applied. We further approximate the indicator function with a differential function called the logarithmic barrier function

\[
\hat{I}_-(u) = -(1/t)\ln(-u),
\] (2.11)

where \(t > 0\) is the parameter controlling the approximation. Problem (2.8) now becomes
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\[
\begin{align*}
\text{minimize} & \quad f(x) + (1/t)\phi(x, s) \\
\text{subject to} & \quad b(x) = 0 \\
& \quad d(x) - s = 0,
\end{align*}
\]

(2.12)

where \(\phi(x, s)\) is the logarithmic barrier of the associated problem

\[
\phi(x, s) = -\left(\sum_{i=1}^{m_d} \ln((P^L_s)^T s - d_L) + \sum_{i=1}^{m_d} \ln((d_U - (P^U_s)^T s)) + \sum_{i=1}^{n} \ln((P^L_x)^T x - x_L) + \sum_{i=1}^{n} \ln((x_U - (P^U_x)^T x))\right).
\]

(2.13)

Problem (2.12) has a convex and differentiable objective function, where Newton’s method can be directly applied. The quality of the approximation of problem (2.12) with respect to the original nonlinear program (2.7) increases with increasing values of the barrier parameter \(t\). However, as the parameter \(t\) grows the Hessian of the objective varies rapidly near the boundary of the feasible set, see e.g. [17], which constitutes problem (2.12) very difficult to solve. A common approach to circumvent the difficulties related with the solution of problem (2.12) is to instead solve a sequence of unconstrained (or linearly constrained) minimization problems with increasing value of the barrier parameter \(t\). This approach is called the barrier method where the solution found at each step (also called centering steps or outer iterations) of the solution process, denoted as \(x(t)\), is used as the starting point for the next minimization problem. The barrier method algorithm is described in Algorithm 1.

![Figure 2.2: Central path for a linear program with total number of constraints \(m = 6\). The dashed curves show the contour lines of the logarithmic barrier function \(\phi\). Adapted from Convex Optimization (p. 580), by S. Boyd and L. Vandenberghe, 2009, Cambridge University Press.](image)
2.2. INTERIOR POINT METHODS

Algorithm 1: Barrier Method

Given \((x^0, s^0, y_c^0, y_d^0, z_L^0, v_L^0, t^0, \mu, \epsilon)\) that satisfies

\[
(z_L^0, v_L^0) > 0, \quad d_L < s^0 < d_U, \quad x_L < x^0 < x_U, \quad t^0 > 0, \quad \mu > 1, \quad \epsilon > 0
\]

Set \(t = t^0\) and \(x = x^0\).

while \(m/t < \epsilon\) do
  Compute \(x(t)\) by solving the barrier problem (2.12) by Newton’s method.
  Update \(x \leftarrow x(t)\).
  Update \(t \leftarrow \mu t\).
end

where \(m\) is the total number of constraints in problem (2.4). Note, that the barrier method requires a strictly feasible starting point \((x^0, s^0, y_c^0, y_d^0, z_L^0, v_L^0, t^0, \mu, \epsilon)\), see e.g. [17]. The choice of the parameter \(\mu\) is a trade–off between the number of centering steps and the number of Newton iterations (also called inner iterations) performed at each centering step. A small value of \(\mu\) results in a small increase of the barrier parameter \(t\) and therefore a good starting point is provided for the next minimization problem. This is interpreted in a small number of Newton iterations required to compute the next iterate. However, a small value of \(\mu\) reduces the duality gap \(m/t\) by only a small amount which results in a large number of centering steps. On the other hand, a large value of \(\mu\) produces a larger reduction of the gap \(m/t\), but the iterate \(x(t)\) will not be a good starting point of the next minimization problem. Therefore an aggressive updating strategy of the barrier parameter \(t\) requires a larger number of inner iterations.

2.2.2 Primal–Dual Interior Point Method

Another class of interior point methods used to solve the nonlinear inequality constrained problem (2.7) are the primal–dual interior point methods. Numerical evidence, see e.g. [17], indicate that for basic problem classes (linear, quadratic, second–order cone and semidefinite programming) primal–dual interior point methods can exhibit better than linear convergence. In the following we will describe the solution steps involved in the implementation of the primal–dual interior point method.

Search Direction

We first derive the Lagrange function of the nonlinear programming problem (2.12)

\[
L = f(x) + y_b^T b(x) + y_d^T (d(x) - s) + z_L^T (x_L - (P^L_x)^T x) \\
+ z_L^T (x_L - (P^L_x)^T x) + z_U^T ((P^U_x)^T x - x_U) \\
+ v_L^T (d_L - (P^L_s)^T s) + v_U^T ((P^U_s)^T s - d_U),
\]

(2.14)

where \(y_b \in \mathbb{R}^{m_b}\) and \(y_d \in \mathbb{R}^{m_d}\) are the Lagrange multipliers for the equality and inequality constraints, \(z_L \in \mathbb{R}^n\) and \(z_U \in \mathbb{R}^n\) are the Lagrange multipliers for the lower and upper bounds of the primal variables, \(v_L \in \mathbb{R}^m\) and \(v_U \in \mathbb{R}^m\) are the Lagrange multipliers for the lower and upper bounds of the slack variables. We state the modified KKT optimality conditions of the logarithmic barrier centering problem (2.12).
\[ \nabla_x L = \nabla_x f(x) + J_b(x)^T y_b + J_d(x)^T y_d - P^L_z z_L + P^U_z z_U = 0 \]
\[ \nabla_s L = -y_d - P^L_s v_s + P^U_s v_U = 0 \]
\[ S^L_z z_L - (1/t)e = 0 \]
\[ S^U_z z_U - (1/t)e = 0 \]
\[ S^L_s v_s - (1/t)e = 0 \]
\[ b(x) = 0 \]
\[ d(x) - s = 0 \]

where \( J_b \in \mathbb{R}^{n \times m_b} \) and \( J_d \in \mathbb{R}^{n \times m_d} \) are the Jacobian of the equality and inequality constraints respectively and we have defined

\[ S^L_x = \text{diag}((P^L_x)^T x - x_L), \quad S^U_x = \text{diag}(x_U - (P^U_x)^T x), \]
\[ S^L_s = \text{diag}((P^L_s)^T s - d_L), \quad S^U_s = \text{diag}(d_U - (P^U_s)^T s). \]  

The primal–dual search directions are computed by solving the KKT system

\[
\begin{bmatrix}
H & 0 & J_b^T & J_d^T & -P^L_x & P^U_x & 0 & 0 \\
0 & 0 & 0 & -I & 0 & 0 & -P^L_s & P^U_s \\
J_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
J_d & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
Z_L(P^L_x)^T & 0 & 0 & S^L_x & 0 & 0 & 0 & 0 \\
-Z_U(P^U_x)^T & 0 & 0 & 0 & S^U_x & 0 & 0 & 0 \\
0 & V_L(P^L_s)^T & 0 & 0 & 0 & S^L_s & 0 & 0 \\
0 & -V_U(P^U_s)^T & 0 & 0 & 0 & 0 & S^U_s & 0 \\
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta s \\
\Delta y_b \\
\Delta y_d \\
\Delta z_L \\
\Delta z_U \\
\Delta v_L \\
\Delta v_U \\
\end{bmatrix} =
\begin{bmatrix}
\nabla_x L \\
\nabla_s L \\
b(x) \\
d(x) - s \\
S^L_x z_L - (1/t)e \\
S^U_x z_U - (1/t)e \\
S^L_s v_s - (1/t)e \\
S^U_s v_U - (1/t)e \\
\end{bmatrix}
\]  

where \( H \) is the Hessian of the Lagrangian (2.14), and we have defined

\[ Z_L = \text{diag}(z_L), \quad Z_U = \text{diag}(z_U), \]
\[ V_L = \text{diag}(v_L), \quad V_U = \text{diag}(v_U). \]  

**Line Search**

The line search in the primal–dual interior point method is a standard backtracking line search, see e.g. [17], based on the KKT error, and modified to ensure that

\[ (z_L, z_U, v_L, v_U) > 0, \quad d_L < s < d_U, \quad x_L < x < x_U. \]  

We denote the current iterate as \( (x, s, y_b, y_d, z_L, z_U, v_L, v_U) \) and the next iterate as \( (x^+, s^+, y^+_b, y^+_d, z^+_L, z^+_U, v^+_L, v^+_U) \), i.e.

\[ x^+ = x + \alpha_x \Delta x, \quad s^+ = s + \alpha_s \Delta s, \]
\[ y^+_b = y_b + \alpha_z \Delta y_b, \quad y^+_d = y_d + \alpha_z \Delta y_d, \]
\[ z^+_L = z_L + \alpha_z \Delta z_L, \quad z^+_U = z_U + \alpha_z \Delta z_U, \]
\[ v^+_L = v_L + \alpha_z \Delta v_L, \quad v^+_U = v_U + \alpha_z \Delta v_U. \]
The residual (KKT error) evaluated at the next iterate will be denoted $r^+$. We first compute the steplengths $\alpha_x, \alpha_s, \alpha_z$, that satisfy the fraction to the boundary rule, i.e.

$$\alpha^\text{max}_x = \sup\{ \alpha_x \in [0, 1] \mid \lambda + a_x \Delta \lambda \geq 0 \}$$
$$= \min\{ 1, 0.995 \min \{ -\lambda_i / \Delta \lambda_i \mid \lambda + \Delta \lambda < 0 \} \},$$

$$\alpha^\text{max}_z = \sup\{ \alpha_z \in [0, 1] \mid x_L \leq x + a_z \Delta x \leq x_U \}$$
$$= \min\{ 1, 0.995 \min \{ (x_i - x_{L_i}) / \Delta x_i \mid x + \Delta x < x_L, (x_{U_i} - x_i) / \Delta x_i \mid x + \Delta x > x_U \} \},$$

$$\alpha^\text{max}_s = \sup\{ \alpha_s \in [0, 1] \mid s_L \leq s + a_s \Delta s \leq s_U \}$$
$$= \min\{ 1, 0.995 \min \{ (s_i - s_{L_i}) / \Delta s_i \mid s + \Delta s < s_L, (s_{U_i} - s_i) / \Delta s_i \mid s + \Delta s > s_U \} \},$$

(2.21)

where $\lambda$ are the dual variables and $\Delta \lambda$ the associated search directions. We continue by multiplying the step lengths by $\beta$ until $\|r^+\|_2 \leq (1 - \alpha_z \alpha) \|r\|_2$. $\alpha$ is typically chosen in the range 0.01 to 0.1, and $\beta$ is typically chosen in the range 0.3 to 0.8.

**Barrier Parameter Update**

In practice, two basic approaches for updating the barrier parameter $t$ are being followed, the monotone and the adaptive strategies. The most widely used monotone strategy is the Fiacco–McCormick algorithm, see e.g. [12], where the barrier parameter is held fixed until an approximate solution of the barrier parameter is computed. The adaptive strategies allow the barrier parameter to increase or decrease at every iteration of the solution process in order to correct overly aggressive increases in the barrier parameter. The most important adaptive strategies are the LOQO and the Mehrotra probing, see e.g. [13].

We describe here an efficient adaptive barrier parameter strategy first presented in [43]. In the proposed scheme it is assumed that $(1/t)$ is proportional to the current complementarity value $\delta$, i.e. $(1/t) = \sigma \frac{\delta}{m}$, where $m$ denotes the number of constraints, and $\sigma \geq 0$ is the centering parameter to be determined. We choose $\sigma$ in order to minimize the following nonlinear quality function which represents the KKT error

$$q_N(\sigma) = \|r_d(\sigma)\|_2^2 + \|r_p(\sigma)\|_2^2 + \|\delta(\sigma)\|_2^2$$

(2.22)

where

$$r_d(\sigma) = \begin{bmatrix} \nabla_x S_L(\sigma) \\ \nabla_s S_L(\sigma) \end{bmatrix}, \quad r_p(\sigma) = \begin{bmatrix} b(x(\sigma)) \\ d(x(\sigma)) - s(\sigma) \end{bmatrix}, \quad \delta(\sigma) = \begin{bmatrix} S^L_L(\sigma) z_L(\sigma) \\ S^U_L(\sigma) z_U(\sigma) \\ S^L_s(\sigma) v_L(\sigma) \\ S^U_s(\sigma) v_U(\sigma) \end{bmatrix}.$$  

(2.23)

The evaluation of $q_N$ requires the evaluation of the problem functions and derivatives for every value of $\sigma$. In order to avoid these computations we will instead use a linear quality function. We define $\Delta(\sigma)$, to be the solution of the KKT system (2.17) as a function of $\sigma$. Once the KKT system (2.17) has been solved twice to obtain $\Delta(0)$ and $\Delta(1)$, $\Delta(\sigma)$ can be easily computed as

$$\Delta(\sigma) = \Delta(0) + \sigma (\Delta(1) - \Delta(0)).$$

(2.24)

By letting $\alpha^\text{max}_x(\sigma), \alpha^\text{max}_z(\sigma)$ denote the step lengths that satisfy the fraction to the boundary rule (2.21) for the step $\Delta = \Delta(\sigma)$, and assuming the problem functions to be linear, we define the linear quality function as
\[ q_L(\sigma) = (1 - \alpha^\text{max}_x(\sigma))^2 \| r_d \|^2 + (1 - \alpha^\text{max}_z(\sigma))^2 \| r_p \|^2 + \| \delta(\sigma) \|^2. \quad (2.25) \]

**Implementation**

We can now describe the implementation of the primal–dual interior point method

**Algorithm 2: Primal–Dual Interior Point Method**

Given \((x^0, s^0, y^0_c, y^0_d, z^0_L, v^0_U, \epsilon, \epsilon_{\text{feas}})\) that satisfies

\[ (z^0_L, v^0_U) > 0, \ d_L < s^0 < d_U, \ x_L < x^0 < x_U \]

while \((\| r_p \|_2 \leq \epsilon_{\text{feas}}, \| r_d \|_2 \leq \epsilon_{\text{feas}}, \text{ and } \hat{\eta} \leq \epsilon)\) do

Choose a target value of the barrier parameter \(t\).

Compute the search directions by solving the primal–dual system (2.17).

Perform the line search and update the design variables.

end

where \(\hat{\eta}\) is the surrogate duality gap, see e.g. [17]. Compared with the barrier method, in the primal–dual interior point method there is only one loop or iteration, i.e. there is no distinction between inner and outer iterations. At each iteration, both the primal and dual variables are updated. Moreover, in a primal–dual interior point method, the primal and dual iterates are not necessarily feasible.

### 2.3 Heuristic Techniques

The optimal solution to the continuous relaxation of (2.1) is rarely binary feasible, hence it is not a feasible solution to the original mixed integer optimization problem (2.1). In the Discrete Material Optimization methodology a feasible solution is obtained by the application of an implicit material interpolation scheme, see e.g. [11], [13], [52], [59] and [72]. Several alternative approaches have been proposed in the literature which explicitly address the integrality conditions on the design variables, see e.g. [31] and [33]. In this section we will elaborate on existing heuristic techniques that can provide us with a discrete feasible solution to the mixed integer programming problem (2.1).

#### 2.3.1 Feasibility Pump

In the feasibility pump heuristic, see e.g. [17] and [15], we generate a sequence of points \((\bar{x}^0, \bar{y}^0), \ldots, (\bar{x}^k, \bar{y}^k)\) that satisfy \(g(\bar{x}^i, \bar{y}^i) \leq 0\) but \(\bar{x}^i \notin \mathbb{Z}^n\). Associated with the sequence of integer infeasible points is a sequence of points \((\hat{x}^1, \hat{y}^1), \ldots, (\hat{x}^{k+1}, \hat{y}^{k+1})\) which satisfy \(\hat{x}^i \in \mathbb{Z}^n\) but do not necessarily satisfy \(g(\hat{x}^i, \hat{y}^i) \leq 0\). The two sequences exhibit the property that at each iteration of the solution process the distance between \(\bar{x}^i\) and \(\hat{x}^{i+1}\) is nonincreasing.

Given a number of points \((\bar{x}^k, \bar{y}^k)\), with \(k = 0, \ldots, P\) and \((\bar{x}^0, \bar{y}^0)\) being an optimal solution to the continuous relaxation of the mixed integer programming problem (2.1), we generate the sequence \((\hat{x}^1, \hat{y}^1), \ldots, (\hat{x}^{k+1}, \hat{y}^{k+1})\), by performing an outer approximation of the (convex) region \(g(x, y) \leq 0\), see e.g. [21] and [23].
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\[
\begin{align*}
\text{minimize} & \quad \|x - \overline{x}^i - 1\|_1 \\
\text{subject to} & \quad g(\overline{x}^k, \overline{y}^k) + \left(\nabla g(\overline{x}^k, \overline{y}^k)\right)^T \left(\begin{array}{c} x \\ y \end{array}\right) - \left(\begin{array}{c} \overline{x}^k \\ \overline{y}^k \end{array}\right) \leq 0, \quad k = 0, \ldots, P, \\
& \quad (\overline{x}^k - \hat{x}^k)(x - \overline{x}^k) \geq 0, \quad k = 1, \ldots, P. \\
\end{align*}
\] (2.26)

The enhanced version of the feasibility pump \[15\] used here includes the inequalities

\[(\overline{x}^k - \hat{x}^k)(x - \overline{x}^k) \geq 0, \quad k = 1, \ldots, P.\]

Each of these inequalities represents a supporting hyperplane that separates \(\hat{x}^k\) from the convex region defined by the nonlinear constraints. For a proof on the validity of these inequalities see e.g. \[15\]. We then compute \((\overline{x}^i, \overline{y}^i)\) by solving the nonlinear program

\[
\begin{align*}
\text{minimize} & \quad \|x - \hat{x}^i\|_2 \\
\text{subject to} & \quad g(x, y) \leq 0.
\end{align*}
\] (2.27)

The enhanced feasibility pump is an exact algorithm: it iterates between solving (2.26) and (2.27) until either a feasible solution of (2.1) is found or (2.26) becomes infeasible.

2.3.2 Relaxation enforced neighborhood search

Large neighborhood search strategies are an alternative approach to rounding heuristics such as the feasibility pump. In this case, we first define the neighborhood around the incumbent solution to be a subproblem of the original mixed integer problem (2.1). The neighborhood is then completely or partially explored by solving the resulting mixed integer subproblem.

The relaxation enforced neighborhood search heuristic (or RENS) \[3\] cleverly uses the information provided by the solution \((\overline{x}^0, \overline{y}^0)\) on the continuous relaxation of (2.1) to define the solution neighborhoods. It focuses attention on those variables that attain integer values in \(\overline{x}^0 \in [0, 1]^{n_1}\). The main idea in RENS is based on the intuition that these variables form a partial solution that can be extended towards a complete solution that achieves both integrality and a good objective value. We first fix the variables that attain integer values in \(\overline{x}^0 \in [0, 1]^{n_1}\). We then solve a mixed integer programming subproblem on the remaining variables. This subproblem actually represents the space of all feasible roundings of \((\overline{x}^0, \overline{y}^0)\). RENS either finds a discrete feasible solution to problem (2.1), which is in fact the best rounding possible for the particular solution \((\overline{x}^0, \overline{y}^0)\), or correctly proves that the original problem (2.1) is indeed infeasible.
Chapter 3

Conclusion
CHAPTER 3. CONCLUSION

3.1 Summary of results

In this section we present a summary of the main theoretical and numerical results reported in the appended articles. Our primary aim in all three articles is to solve the problems under study to proven global optimality. Special purpose numerical methods that can conveniently model manufacturing constraints of laminated composite structures have been developed and implemented in the first two articles. The modeling of local failure criteria is investigated in the final article.

**Article 1: Optimal Design of Laminated Composite Structures by Mixed 0–1 Nonlinear Optimization Techniques (Chapter 4 of the thesis)**

In the first article we present new global optimization methods and heuristics for optimal lay–up design of laminated composite structures by discrete material optimization. In the problems the objective is compliance (or mass) which is minimized with a mass (or compliance) limitation. The design variables are, in our case binary and represent locally presence or absence of a material in the structure from a list of pre–defined candidate materials. The optimal design problems that arise are formulated as nonconvex mixed 0–1 optimization problems. The mathematical structure of the problems allows them to be reformulated as convex 0–1 problems. This gives significant theoretical and computational advantages when developing global optimization methods as well as efficient heuristics. The continuous relaxation of the mixed integer problems is being solved by an implementation of a modern primal–dual interior point method for nonlinear programming. Several efficient heuristic techniques are designed to obtain discrete feasible solutions, based on the optimal solution to the continuous relaxations of the considered problems. The heuristics provide a measure of closeness to a global minimizer and provide feasible 0–1 designs without the use of advanced material penalization schemes or elaborate continuation approaches. These heuristics have advantageous theoretical properties and can be successfully applied to large–scale problems. Our primary aim is to solve the considered problems to proven global optimality. We propose a combination of the convergent Outer Approximation and Local Branching algorithms to perform the global optimization.

The numerical experiences of our methods and heuristics are reported on a set of discrete material optimization problems. The obtained results showcase the excellent convergence properties and the ability of the primal–dual interior point method to react swiftly to changes of scale of our problems. The method managed to converge in 13 to 29 iterations in all the examined cases. Our heuristics are highly dependent on the considered design domain, loading and boundary conditions. The relative optimality gap varies by as low as 2% to as high as 22.8% depending on the considered case. Several of the problem instances were solved to global optimality. The numerical results indicate the importance of including manufacturing constraints in the design problem. The article includes ideas on how manufacturing constraints can be included as linear constraints in the problem formulations and the methods. Some promising preliminary results have been demonstrated.


3.1. SUMMARY OF RESULTS

Article 2: Optimal Design of Laminated Composite Structures Under Manufacturing Constraints by Mixed 0–1 Nonlinear Optimization Techniques (Chapter 5 of the thesis)

In this work we have extended the global optimization methods and heuristic techniques proposed in the first article to perform simultaneous material and thickness optimization of laminated composite structures including manufacturing considerations. The objective is either a weighted sum of the individual static compliances subject to a mass constraint or the total mass of the structure which is minimized with constraints on the individual static compliances. The problem formulations are appropriately extended to handle the addition of manufacturing considerations as explicit linear constraints. A material deposition constraint has been implemented together with restrictions on the laminate thickness variation rate resembling the manufacturing technology of laminated composites. Furthermore, we impose limitations on the fiber angles variations through an implementation that is based on the perimeter method for variable topology shape optimization of elastic structures.

The ability of our global optimization methods and heuristics to perform simultaneous multi–material and thickness optimization is examined on a set of discrete material optimization problems of laminated composite plates. The obtained results showcase the excellent convergence properties of the global optimization methods when manufacturing considerations are included in the design problem. Problem instances of up to 19456 design variables, 68269 constraints and 21780 degrees of freedom were solved to global optimality. The activation of the manufacturing constraints results in designs with a uniform lamination sequence and smooth ply–drops throughout the entire laminate structure.

Article 3: Optimal Design of Laminated Composite Structures Including Local Failure Criteria by Mixed 0–1 Nonlinear Optimization Techniques (Chapter 6 of the thesis)

In the third and final article we consider the multi–material optimization of laminated composite structures subject to constraints on local failure criteria. The objective is a weighted sum of the individual static compliances subject to a mass constraint. The optimal design problems are stated as nonconvex mixed integer problems. We resort to different reformulation techniques and state the original nonconvex mixed–integer problems as either linear or nonlinear convex mixed 0–1 programs. By performing the reformulations, the optimal design problems are made accessible to general robust and efficient branch–and–cut methods developed for solving this class of problems.

The chosen parameterization offers significant advantages in modeling failure criteria. In this manuscript we consider the maximum strain and maximum stress failure criteria which are introduced into the problem formulation as a set of linear inequalities. The addition of local failure criteria in our models preserves the favorable mathematical properties that have been achieved by performing the reformulations. Nevertheless, with the suggested reformulation the size of the problem is increased substantially, both in terms of variables and constraints, which constitutes the problems difficult to solve.

We examine the performance of our models in the case of multi–material optimization under different loading and boundary conditions. In the numerical experiments it is shown that the quadratic formulations clearly outperform the linear formulations. Several of the problem instances were solved to global optimality.
3.2 Contributions and impact

The objective of this work was to develop new special purpose global optimization methods and heuristic techniques for optimal design of laminated composite structures under manufacturing constraints. In the following we summarize the main research contributions.

Design parameterization

The modelling of the considered problems is strongly based on the design parameterization proposed in [30]. The methodology constitutes an immediate extension of the original Discrete Material Optimization method introduced in [26] and [22], and allows the simultaneous determination of the appropriate laminate thickness and the material choice in the structure. The potential of the chosen parameterization in performing simultaneous multi-material and thickness optimization has never before been demonstrated in case of layered composite structures.

Interior point methods

The continuous relaxation of the mixed integer programming problems is being solved by our own implementation of a primal–dual interior point method for nonlinear programming. The barrier parameter is updated adaptively as the iteration progresses based on the minimization of a quality function which represents the KKT error, following the ideas presented in [43]. The primal–dual interior point method is for the first time utilized for discrete material optimization of laminated composite structures.

Heuristic techniques

In this thesis we address the issue of feasibility in mixed integer nonlinear programming. As opposed to the original Discrete Material Optimization methodology, we obtain feasible 0–1 designs to the considered problems, by the application of advanced heuristic techniques, without the use of material penalization schemes or elaborate continuation approaches. Our heuristics are based on solving sequences of well-posed optimization problems for which efficient methods and robust implementations exist. The developed heuristics either guarantee feasible 0–1 designs or correctly determine problem infeasibility.

Global optimization methods

Multi-phase topology optimization methods such as DMO, cannot theoretically guarantee the global optimum and the results depend on the chosen material interpolation scheme. Our primary aim in this thesis was to develop special purpose optimization methods which are capable of solving the mixed 0–1 problems to global optimality. We developed and implemented a novel framework based on the convergent outer approximation and local branching algorithms that provides us with guarantee globally optimal solutions to the considered problems. The design examples presented in the numerical experiments have never before been solved to global optimality.
3.3. **FUTURE RESEARCH**

Manufacturing constraints

In this thesis we propose a novel approach of assessing manufacturing complexity in composite structures with an implementation that is based on the perimeter method for variable topology shape optimization of elastic structures, see e.g. [27]. Our models include additional structural considerations which correspond to well-established design rules that prevent laminate failure initiation such as matrix cracking and delamination. The additional constraints are sparse and there exist specialized algorithms and data structures that can efficiently handle the sparse structure of these equations. In the numerical experiments it is proven that the application of the perimeter constraints constitutes a regularization technique for this class of problems.

3.3 Future research

Considering the promising results presented in this thesis we propose some interesting topics for future research developments.

Numerical experiments

In the numerical experiments we study maximum stiffness optimization problems of laminated composites assuming quasi-static loading and linear elastic material behavior. In most practical applications laminated plates are thin and subjected to in-plane loading, so that elastic instabilities (known as buckling) can occur. The effect of thermal stresses on the optimal design of panels can also be significant. The problem arises from the fact that the coefficients of thermal expansion in the longitudinal and transverse direction can assume widely different values. In order to capture these effects multicriteria optimization techniques should be employed in our models. Such techniques possess several theoretical and computational challenges that have to be overcome before they can be included within the global optimization framework presented in this thesis.

Our numerical methods are validated on simple design examples composed of plane geometries discretized by 9 node Mindlin type plate elements. Our optimization models are general enough to be coupled with a commercial FE package. The main advantage of this approach is that all the pre- and post-processing capabilities, meshing and finite element implementations existing in the FE package become accessible to us and would allow us to interface with more complex structures and applications of industrial relevance.

Design parameterization

So far the major drawback of the chosen parameterization is the large number of design variables introduced. In a practical design situation the number of design variables can easily exceed hundreds of thousands which is a practical limit on the problems that can be considered. Some focus should therefore be directed in experimenting with alternative design parameterizations that can lead to reduced problem sizes.

Throughout this thesis the structural behavior of the laminate is described using an equivalent single-layer (or ESL) theory, see e.g. [23]. The layers are assumed to be perfectly bonded together and thus displacements and strains vary continuous across the thickness of the structure. Although the ESL theory provides a good approximation of the structural
stiffness, it cannot predict interlaminar effects such as delamination due to the absence of normal stress and strain components in the thickness direction. In order to capture these effects, a 3D–solid finite element model should be applied. However, the excessive computing cost associated with this approach is prohibiting its use in our applications. An alternative approach could be to include stress recovery in our models as discussed by e.g. [10], or to introduce a modified kinematic assumption, also known as the zig–zag approach, see e.g. [8], which allows piecewise continuous displacements.

Heuristic techniques

The heuristics presented in this thesis either provide us with a feasible solution to the mixed–integer programming problems or correctly detect problem infeasibility. However, for none of these heuristics there are any guarantees on the quality of the obtained feasible designs. Finding good feasible designs to the considered mixed 0–1 problems is important for the practical convergence rate of most modern global optimization methods. Some research could therefore be conducted in developing heuristic techniques which can provide us with improved feasible designs in a relative fast way.

Manufacturing constraints

The manufacturing constraints considered in this thesis correspond to thumb rules and design guidelines based on experience. They are not directly linked to a specific manufacturing process such as compression moulding or vacuum infusion. Numerical simulations investigating the influence of the processing conditions of compression moulding in thickness optimization of composite plates have been recently performed in [9]. Some more attention could therefore be directed towards defining the processing conditions as design variables for structural optimization of laminated composite structures.

During manufacturing of composites, one of the common challenges met is the development of process induced shape distortions and residual stress build–up. It is a highly nonlinear thermo–mechanical process that results in dramatic changes of the mechanical properties of the manufactured specimen. Using existing thermo–mechanical process models, see e.g. [60], it is possible to predict the process outcome and final part degree of cure. A proper integration of these models into a general optimization framework could therefore be an important step towards more optimized manufacturing of composites.

Failure criteria

A comparison between the available failure theories of composite materials has taken place as part of a co–ordinated international study, referred to as the "World–Wide Failure Exercise", see e.g. [25], [29], [58], [49], [35], [23]. The selected test cases included a wide range of varying parameters, such as fibre type, matrix type, lay–up configuration and loading states. The results of the exercise indicated a large spread in the final failure strengths predicted by the different failure theories. In many instances, the theories differed in the mode (and occasionally the location) of failure each predicted. This divergence in behavior was attributed primarily to the following factors:
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- The different ways in which curing residual stresses are introduced in the predictions, especially in the case of first ply failure.
- The concept of in situ behavior of a lamina within the laminate.
- The different methods of modeling the progressive failure process and the definition of ultimate laminate failure.
- The nonlinear behavior of matrix-dominated laminates.

Although it is not possible currently to relate the complex mechanisms and modes of failure to all existing macroscopic failure criteria, it is safe to say that interactions exist among the various mechanisms and modes. The maximum stress and maximum strain failure criteria studied in this thesis are non-interactive criteria and therefore constitute gross simplifications. We could easily extend our problem formulations to model interactive (or partially interactive) failure criteria such as the Tsai–Hill and Tsai–Wu criteria in the form of quadratic constraints, see e.g. [40].
Bibliography


Optimal Design of Laminated Composite Structures by Mixed 0–1 Nonlinear Optimization Techniques

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Abstract

We present new global optimization methods and heuristics for optimal design of laminated composite structures by discrete material optimization. In the problems the objective is compliance (or mass) which is minimized with a mass (or compliance) limitation and constraints modeling manufacturing and design rules. The heuristics provide a measure of closeness to a global minimizer and provide feasible 0–1 designs or correctly determine infeasibility. Numerical experiences are reported on a set of discrete material optimization problems. The results indicate that only small–scale problems can be solved to global optimality. The heuristics are competitive to existing heuristics for discrete material optimization.

Mathematical Subject Classification (2000): 90C59, 90C90, 74P15, 90C26, 74E30

Keywords: Structural Optimization, Laminated Composites, Discrete Material Optimization, Heuristics, Global Optimization

4.1 Introduction

Optimal design of laminated composite structures requires determination of the appropriate number of layers, the material choice, and thickness of each individual layer. Throughout this article the number and the thickness of the individual layers are considered to be fixed over the entire design domain. We are thus considering the laminate layup design problem of choosing among a set of pre–defined candidate materials. The materials may for example be oriented in specific directions in order to obtain high stiffness in the loading directions and lower stiffness in other directions.

The optimal design problems are formulated as nonconvex mixed 0–1 optimization problems. The design variables are, in our case binary and represent locally presence or absence of a material in the structure from a list of pre–defined candidate materials. In our numerical experiments the list consists of isotropic and orthotropic materials, but other materials (e.g. quasi–isotropic and anisotropic) can be included without any further modifications to the

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methods and heuristics. This parameterization scheme has been proposed in [26] and [22] and is commonly referred to as Discrete Material Optimization (DMO). The parameterization is a generalization of the modeling techniques used in structural topology optimization, see e.g. [6]. The mathematical structure of the problems allows them to be reformulated as convex 0–1 problems. This gives significant theoretical and computational advantages when developing global optimization methods as well as efficient heuristics.

The discrete material optimization approach has been applied successfully to design problems with structural criteria like compliance in [26], [22], [55] and [31], eigenfrequencies in [22], and buckling load factors in [34]. The parameterization was extended in [56] and [57] to consider combined topology and thickness optimization of laminated composite structures including manufacturing constraints. The main idea in [26] and [22] is to relax the integrality constraints on the binary design variables, thus allowing values between 0 and 1 during the optimization. The material properties in the continuous relaxation of the design variables are obtained as weighted averages of the constituent properties. The continuous design variables are eventually penalized to 0 or 1. Normally, an almost feasible point to the optimization problem is obtained. Mixtures of materials are penalized either through a constitutive interpolation scheme, see e.g. [11], [13], [52] and [59], or by explicitly adding a penalty term to the objective function, see e.g. [16]. The material interpolation scheme approach has gained large acceptance in the community of structural topology optimization, see e.g. [6].

Our primary aim is to provide modern optimization methods which are capable of solving the mixed 0–1 problems to global optimality. The numerical results presented in [2], [43], [40], and [53] for truss topology optimization problems indicate that only small– to medium–size problems can be solved to global optimality. Our secondary aim is therefore to provide advanced heuristics which are guaranteed to provide feasible 0–1 designs or correctly determine that the problem is infeasible. The heuristics we present are based on solving sequences of well–posed optimization problems and they can be applied to large–scale problems. Our heuristics provide feasible 0–1 designs to the considered problem, without the use of advanced material penalization schemes or elaborate continuation approaches.

Several methods exist in the literature for solving general classes of mixed integer nonlinear optimization problems. An extension of the branch and bound method [39], that solves nonlinear relaxed problems at each node of the branch and bound tree, was presented in [25]. The extended cutting plane method proposed in [58, 59], is a direct extension of the cutting plane method presented in [31]. In [21] an outer approximation algorithm for solving a particular class of mixed integer convex programming problems was presented. The outer approximation idea was generalized in [23] to treat a wider class of convex problems. The generalized Benders’ decomposition approach presented in [22], is another method developed for mixed integer nonlinear optimization.

In this manuscript we develop a framework based on outer approximation, that provides us with globally optimal solutions to the considered problems. Outer approximation was used in [51] for global optimization of laminated composites. A variant of outer approximation has been applied for structural synthesis by means of simultaneous topology, parameter and standard dimension optimization in [32, 33]. A method based on generalized Benders’ decomposition has been developed and implemented for maximum stiffness truss topology optimization in [40].

We also present another approach for solving our problems to global optimality by developing a framework based on the concept of local branching. Local branching is a strategy presented in [22] for linear mixed integer programs. Local branching has been applied success-
fully, both as a method and as a heuristic, see e.g. [15, 46, 20, 47, 19] and [15]. It is today implemented in commercial packages for mixed integer optimization, such as IBM CPLEX [19]. Local branching was used in [54] to solve a set of challenging topology optimization benchmark problems.

This manuscript is organized as follows. In Section 4.2 we state the considered minimum compliance and minimum mass problems, and the basic assumptions on the analysis models and problem data. The heuristic techniques are presented in Section 4.3 while the global optimization methods are described in Section 4.4. The implementation of the algorithms are described in Section 4.5. In Section 4.6 we report the numerical experiences of our methods and heuristics on a set of discrete material optimization problems. Possible topics for future research in this area and our concluding remarks are finally presented in Section 4.7.

4.2 Problem statements and assumptions

We consider discrete multi–material minimum compliance and minimum mass problems for optimal lay–up design of laminated composite structures. Our models closely follow the problem formulations proposed in [26] and [22]. We provide several different problem formulations which can prove useful in different design situations. The problems are in general not equivalent and have very different mathematical properties. Due to this they require development of different heuristics and methods. The layered design domain is partitioned into finite elements and the design variables associate candidate materials from a given set to each layer and every finite element (or patch). The candidate materials are represented by their constitutive matrices. The binary design variables $x_{ijk} \in \{0, 1\}$ indicate presence or absence of material $i$ in layer $k$ of element (or patch) $j$. All materials behave linearly elastic and the structural behavior of the laminate is described using an equivalent single–layer (ESL) theory, see e.g. [23]. The layers are assumed to be perfectly bonded together and thus displacements and strains vary continuous across the thickness of the structure. The finite element formulations are based on the first–order shear deformation theory (FSDT), see e.g. [23]. In the first multiple load minimum compliance problem a weighted sum of the individual static compliances $f_l^T u_l$ is minimized, where $f_l$ are the static design independent loads and $u_l$ is the corresponding displacement vector satisfying the linear elasticity equilibrium equations $K(x) u_l = f_l$. The considered minimum compliance problem is

\[
\begin{align*}
  z^* = \min_{x \in \mathbb{R}^{n_d \times n_d}, u_1, \ldots, u_L \in \mathbb{R}^{n_d}} & \sum_{l=1}^L w_l f_l^T u_l \\
\text{subject to} & \quad K(x) u_l - f_l = 0, \quad \forall l \\
& \quad m(x) \leq m_{\text{max}}, \\
& \quad \sum_{i=1}^L x_{ijk} = 1, \quad \forall (j,k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall (i,j,k),
\end{align*}
\]

where $K(x) \in \mathbb{R}^{n_d \times n_d}$ is the stiffness matrix, $f_1, \ldots, f_L \in \mathbb{R}^{n_d}$ are vectors of work–equivalent nodal forces, and $u_1, \ldots, u_L \in \mathbb{R}^{n_d}$ are the continuous nodal displacements. The integer $L$ represents the number of load cases while $n_d$ denotes the number of free finite element degrees of freedom. The relative importance of each load case is given by a weighting factor $w_l \geq 0$. The function $m(x)$ represents the mass of the structure and $m_{\text{max}} > 0$ is the maximum allowable mass. The constraint on the total mass is redundant when all candidate materials
4.2. PROBLEM STATEMENTS AND ASSUMPTIONS

have the same mass density, e.g. in the case of pure fibre angle optimization. The mass of
the structure is computed by

\[ m(x) = \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{i=1}^{I} x_{ijk} t_k a_j \rho_i, \]

where \( t_k \) is the thickness of layer \( k \), \( a_j \) is the area of element \( j \), \( \rho_i \) is the density of the given
material \( i \). Furthermore, \( J \) is the total number of elements, \( K \) is the number of layers, and \( I \)
is the number of candidate materials. The number of binary variables \( n \) is thus \( n = J \cdot K \cdot I \).

An alternative formulation to the minimum compliance problem \((P^c_D)\), is by minimizing
the maximum compliance among all the load cases, i.e. the worst–case formulation,

\[
\begin{align*}
\min_{x \in \mathbb{R}^n, u_L \in \mathbb{R}^{nL}} & \max_{1 \leq l \leq L} \{ f_l^T u_l \} \\
\text{subject to} & \quad K(x) u_l - f_l = 0, \quad \forall l \\
& \quad m(x) \leq m_{\text{max}}, \\
& \quad I \sum_{i=1}^I x_{ijk} = 1, \quad \forall (j, k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k).
\end{align*}
\]

A related minimum mass problem formulation is

\[
\begin{align*}
\min_{x \in \mathbb{R}^n, u_L \in \mathbb{R}^{nL}} & \quad m(x) \\
\text{subject to} & \quad K(x) u_l - f_l = 0, \quad \forall l \\
& \quad f_l^T u_l \leq c_{l, \text{max}}, \quad \forall l \\
& \quad I \sum_{i=1}^I x_{ijk} = 1, \quad \forall (j, k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k),
\end{align*}
\]

where \( c_{l, \text{max}} > 0 \) is the maximum allowable compliance for the \( l \)–th load condition.

The feasible sets of the problems \((P^c_D)\) – \((P^m_D)\) can be additionally restricted. The formu-
lations are general enough to cover also the case that the design variables are coupled
into patches (larger areas). Alternatively, the design variables can be coupled with the ma-
terial properties of an entire lamina. The latter possibility is investigated in the numerical
experiments presented in Section 4.6.

4.2.1 Assumptions

Throughout this paper we make a number of technical assumptions on the analysis models
and the problem data. Similar assumptions were stated in [2] and [6] for continuous minimum
compliance problems and in [1], [40] and [43] for truss topology optimization problems with
discrete design variables. Most of these assumptions are natural and generally satisfied in
practical design applications. Some of them are technical but at the same time necessary for
showing the advantageous theoretical properties of the developed methods and heuristics.

\textbf{(A1)} The topology of the structure does not change and the stiffness matrix \( K(x) \) is sym-
metric and positive definite for all \( x \in [0,1]^n \). The stiffness \( K(x) \) matrix is linear (or
possibly affine) in the design variables and

\[
K(x) = K_0 + \sum_{i,j,k} x_{ijk} K_{ijk} = K_0 + \sum_{i,j,k} x_{ijk} B_j^T C_{ik} B_j = K_0 + \sum_j B_j^T (\sum_{i,k} x_{ijk} C_{ik}) B_j
\]
where $C_{ik} = C_{ik}^T \succeq 0$ is the constitutive matrix for the $i$–th given material for the $k$–th layer and $B_j$ is the strain–displacement matrix for the $j$–th element. $K_{ijk}$ is the stiffness matrix for the $k$–th layer of the $j$–th element and the $i$–th material and $K_0$ is a given symmetric and positive semidefinite matrix.

(A2) The external loads $f_l \in \mathbb{R}^{n_d \setminus \{0\}}$ for $l = 1, \ldots, L$. Furthermore, we assume that the load vectors are independent of the design variables.

(A3) The mass limit, $m_{\text{max}}$ satisfies

$$\sum_{k=1}^J \sum_{i=1}^K t_i a_j \min \{\rho_i\} \leq m_{\text{max}} \leq \sum_{k=1}^J \sum_{i=1}^K t_i a_j \max \{\rho_i\}.$$  

(A4) The compliance limits $c_{l,\text{max}} > 0$ are chosen such that there exists a design vector $x$ with the properties that

$$0 \leq x_{ijk} \leq 1 \forall (i,j,k), \sum_{i=1}^I x_{ijk} = 1 \forall (j,k)$$

and

$$f_l^T K^{-1}(x) f_l < c_{l,\text{max}} \forall l.$$  

(A5) The weighting factors $w_l \geq 0$ for all load cases $l$.

The summation in (A1) should be considered as a standard finite element stiffness matrix assembly process. The matrix $K_0$ can be used to model the situation that the structure also contains domains which are not part of the design domain. Assumptions (A1) – (A3) guarantee that the feasible sets of the minimum compliance problems $(P'_{c,D})$ and $(P'_{w,D})$ are non–empty. Note also that the objective functions in all problems $(P'_{c,D})$ – $(P'_{m,D})$ are bounded from below by zero. Note that assumption (A4) is not sufficiently strong to guarantee that the feasible set of the minimum mass problem $(P'_{m,D})$ is non–empty. It will later be used to assure existence of solutions of the continuous relaxation of $(P'_{m,D})$. Assumption (A1) guarantees that for any design vector $x \in \{0, 1\}^n$, there is a unique displacement solution of the equilibrium equations. This assumption is extensively used for stating the problem reformulations and in the development of methods and heuristics in later sections. Assumption (A2) is stated to avoid the trivial situation in which the design domain is not subjected to any load.

### 4.2.2 Nested problem formulations

All problem formulations $(P'_{c,D})$ – $(P'_{m,D})$ can be classified as nonconvex mixed 0–1 optimization problems due to the bilinear terms in the equilibrium equations. Assumption (A1) allows us to eliminate the nodal displacement variables by $u_l = K(x)^{-1} f_l \forall l$. This way the original nonconvex 0–1 problem $(P'_{c,D})$ can be reformulated as the 0–1 program with a nonlinear objective function

\[
\min_{x \in \mathbb{R}^n} c(x) \\
\text{subject to } m(x) \leq m_{\text{max}}, \\
\sum_{i=1}^I x_{ijk} = 1, \forall (j,k) \\
x_{ijk} \in \{0, 1\}, \forall (i,j,k)
\]  

\[(P'_{c,N})\]
where \( c(x) \) is the weighted sum of the individual static compliances and is computed by

\[
c(x) = \sum_{l=1}^{L} w_l f_l^T K(x)^{-1} f_l.
\]

Assumption (A5) stating that the weights \( w_l \geq 0 \) and assumption (A1) stating that the stiffness matrix is linear (or affine) as a function in the design variables ensure convexity of the objective function in \((P^c_N)\), see e.g. [61]. Problem \((P^c_N)\) includes only the binary design variables. The equilibrium equations are solved as part of computing the objective function. This problem is often referred to as nested formulation and is commonly used in (structural) topology optimization, see e.g. [6]. Similarly, the corresponding nested formulation of the minimum mass problem \((P^m_D)\) is

\[
m^* = \min_{x \in \mathbb{R}^n} \quad m(x)
\text{subject to} \quad c_l(x) \leq c_{l, \text{max}}, \quad \forall l
\sum_{i=1}^{l} x_{ijk} = 1, \quad \forall (j, k)
x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k)
\]

where \( c_l(x) = f_l^T K(x)^{-1} f_l \) is the individual static compliance for the \( l \)-th load case.

**Continuous relaxations**

By relaxing the integer constraints on the design variables we get the continuous relaxation of the minimum compliance problem \((P^c_N)\)

\[
z^*_R = \min_{x \in \mathbb{R}^n} \quad c(x)
\text{subject to} \quad m(x) \leq m^\text{max},
\sum_{i=1}^{l} x_{ijk} = 1, \quad \forall (j, k)
x_{ijk} \geq 0, \quad \forall (i, j, k).
\]

The upper bounds on the design variables \( x_{ijk} \leq 1 \) are not included in \((P^c_R)\), since they are implicitly satisfied due to the generalized upper bound constraints.

Assumptions (A1) – (A3) guarantee that the continuous problem \((P^c_R)\) has a non–empty feasible set. Furthermore, the objective function is bounded from below by zero and unboundedness of the objective function is thus not an issue. Since the feasible set is also compact (closed and bounded) and the objective function is continuously differentiable, it follows (by Weierstrass theorem) that there is at least one optimal solution of \((P^c_R)\). Since the constraints are linear and the objective function is convex in \((P^c_R)\) finding a KKT–point to \((P^c_R)\) assures global optimality.

Since the feasible set of \((P^c_R)\) is larger than the feasible set of \((P^c_N)\), it follows that \((P^c_R)\) is a relaxation of \((P^c_N)\). Hence, it is a lower bounding problem of \((P^c_N)\) and subsequently also of \((P^m_D)\), i.e. \( z^*_R \leq z^* \). If an optimal solution to the relaxed problem \((P^c_R)\) is binary, then it is also a global optimal solution to the original discrete problems \((P^c_D)\) and \((P^m_N)\).
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In the case of the minimum mass problem \((P_{m}^{R})\), the continuous relaxation is

\[
m_{R} = \min_{x \in \mathbb{R}^{n}} \quad m(x) \\
\text{subject to} \quad c_{l}(x) \leq c_{l}^{\max}, \quad l = 1, \ldots, L \\
\sum_{i=1}^{L} x_{ijk} = 1, \quad \forall (j, k) \\
x_{ijk} \geq 0, \quad \forall (i, j, k).
\]

Assumptions \((A1), (A2), \) and \((A4)\) guarantee that the continuous problem \((P_{m}^{R})\) has a non-empty feasible set. Furthermore, the objective function in \((P_{m}^{R})\) is bounded from below and unboundedness is therefore not an issue. Due to assumption \((A4)\) the feasible set of problem \((P_{m}^{R})\) satisfies some constraint qualifications. Combined with the convexity of the feasible set a KKT-point of \((P_{m}^{R})\) guarantees both local and global optimality. If \(\bar{x}\) is an optimal solution of \((P_{m}^{R})\) then \(m(\bar{x}) = m_R \leq m^\star\).

4.3 Heuristics

Finding good feasible designs to the considered mixed 0–1 problems is important for the practical convergence rate of most modern global optimization methods. Furthermore, in a practical design situation it is important to have efficient and reliable heuristics. We present one heuristic for the minimum compliance problems \((P_{c}^{D})\) and \((P_{w}^{D})\) and one heuristic for the minimum mass problem \((P_{m}^{D})\). The first heuristic rounds the optimal solution of the continuous relaxation of the minimum compliance problem by solving a linear 0–1 optimization problem. The method is guaranteed to find a feasible design to \((P_{c}^{D})\) or \((P_{w}^{D})\), respectively. This heuristic is easy to implement given a solver for mixed 0–1 linear programs. The heuristic for the minimum mass problem \((P_{m}^{D})\) is more advanced due to the nonlinearity of the constraints and it is based on the feasibility pump, see e.g. [17] and [15]. For neither of these heuristics, there are any guarantees on the quality of the obtained feasible design. We therefore attempt to find improved feasible designs after performing the heuristics for the minimum compliance problems \((P_{c}^{D})\) and \((P_{w}^{D})\) and minimum mass problem \((P_{m}^{D})\) by applying a gap improvement method/heuristic as described in Section 4.3.3.

4.3.1 Rounding heuristic for minimum compliance problems

We first present an efficient heuristic for obtaining a feasible 0–1 solution to the minimum compliance problems \((P_{c}^{D})\) and \((P_{w}^{D})\). The heuristic is based on adjusting the solution of the continuous relaxation to satisfy all the constraints, including the 0–1 requirements in \((P_{c}^{D})\) and \((P_{w}^{D})\). Given a non-discrete solution \(\bar{x}\) of the continuous relaxation problem \((P_{c}^{D})\), i.e. the continuous relaxation of \((P_{c}^{D})\), a feasible design to \((P_{c}^{D})\) is obtained by solving the 0–1 program

\[
\text{minimize} \quad \|\bar{x} - x\|_1 \\
\text{subject to} \quad m(x) \leq m^{\max}, \\
\sum_{i=1}^{L} x_{ijk} = 1, \quad \forall (j, k) \\
x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k).
\]

A similar heuristic is proposed for truss topology optimization problems in [30]. Problem \((4.1)\) is in the implementation and the numerical experiments reformulated as a mixed linear
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0–1 problem by introducing additional continuous variables $z_{ij}$ with the interpretation

$$|\overline{x}_{ijk} - x_{ijk}| = z_{ijk}.$$ 

The resulting problem is

$$\begin{align*}
\text{minimize} & \quad \sum_{ijk} z_{ijk} \\
\text{subject to} & \quad m(x) \leq m^{\text{max}}, \\
& \quad \sum_{i=1}^{I} x_{ijk} = 1, \quad \forall (j,k), \\
& \quad -z_{ijk} \leq \overline{x}_{ijk} - x_{ijk} \leq z_{ijk}, \quad \forall (i,j,k), \\
& \quad x_{ijk} \in \{0,1\}, \quad \forall (i,j,k), \\
& \quad z_{ijk} \geq 0, \quad \forall (i,j,k). \\
\end{align*} \tag{4.2}$$

The feasible set of problem (4.2) is non–empty due to assumptions (A2) and (A3). Furthermore since the objective function is bounded from below and the problem is linear there exists at least one optimal solution of (4.2). Problem (4.2) is a linear mixed 0–1 problem and for this class of problems there is a wide selection of heuristics and global optimization methods. There are also several robust and efficient implementations of these methods such as the branch–and–cut solvers in IBM CPLEX [19] and GUROBI [26].

In the numerical experiments we first solve the continuous relaxation of the minimum compliance problem (P$^c_N$) and then round the obtained (generally) non–discrete solution by solving (4.2). In fact, any non–discrete design $\overline{x}$ can be rounded to a feasible design of (P$^c_D$) or (P$^w_D$) by solving (4.2).

**Algorithm 3:** Heuristic for the minimum compliance problems (P$^c_D$) and (P$^w_D$).

1. Solve the continuous relaxation of the relevant minimum compliance problem.
2. Denote the optimal solution $\overline{x}$ with objective function value $z_R$.
3. Solve the mixed 0–1 problem (4.2). Denote the optimal solution $\hat{x}$.
4. Set the lower bound $lb = z_R$.
5. Compute displacement vectors $u_l(\hat{x}) = K(\hat{x})^{-1} f_l \forall l$ and the upper bound

$$ub = \sum_{l=1}^{L} u_l f_l^T u_l(\hat{x}) \text{ or } ub = \max_{1 \leq l \leq L} \{ f_l^T u_l(\hat{x}) \}.$$ 

$(\hat{x}, u_1(\hat{x}), \ldots, u_L(\hat{x}))$ is feasible to the minimum compliance problems (P$^c_D$) and (P$^w_D$).
6. Compute the relative optimality gap

$$\delta = ((ub - lb)/ub) \times 100.$$ 

It is not necessary to solve (4.2) to global optimality. In fact, any point $\hat{x}$ which is feasible to (4.2) is feasible to (P$^c_N$).

The constraint on the total mass is redundant in (P$^c_D$), (P$^w_D$), and (P$^c_N$) when all candidate materials have the same mass density, e.g. in the case of pure fibre angle optimization. In this case solving (4.2) becomes trivial.

4.3.2 A feasibility pump for minimum mass problems

Due to the nonlinear inequality constraints in the minimum mass problem (P$^m_D$) the above mentioned heuristic cannot be applied. For the minimum mass problem (P$^m_D$) we instead
address the issue of feasibility by applying a special purpose version of the enhanced feasibility pump developed in \[15\]. The feasibility pump was originally proposed in \[17\] for finding a feasible solution to mixed integer linear programs. The feasibility pump has been proven to be very successful in finding feasible solutions even for very hard mixed integer problem instances, see \[1\]. It was further improved in \[10\] and \[1\] to solve mixed integer programming problems with binary and general integer variables. An algorithm for finding a feasible solution to convex mixed integer nonlinear problems was later on presented in \[15\].

In the feasibility pump heuristic two sequences of points are generated. They exhibit the property that at each iteration the distance between them is non increasing. The first sequence \(\{x^r\}\), with \(x^r \in [0, 1]\) consists of points which satisfy the constraints in the continuous relaxation \(\text{P}_m^R\) of the minimum mass problem \(\text{P}_m\). Associated with these (generally) 0–1 infeasible points is a sequence of points \(\{\hat{x}^r\}\), with \(\hat{x}^r \in \{0, 1\}\) which are all 0–1 but do not necessarily satisfy the nonlinear constraints of the problem. The feasibility pump heuristic starts with a solution \(x^0\) of the continuous relaxation \(\text{P}_m^R\) of the mixed 0–1 nonlinear programs \(\text{P}_m^D\) or \(\text{P}_m^N\). At iteration \(p\) of the feasibility pump we attempt to find a point \(\hat{x}^p\) that solves

\[
\begin{align*}
\text{minimize} & \quad \|x - x^{p-1}\|_1 \\
\text{subject to} & \quad c_l(x^r) + (\nabla c_l(x^r))^T(x - x^r) \leq c^\text{max}_l, \quad \forall r = 0, \ldots, p - 1, \quad \forall l \\
& \quad \sum_{i=1}^I x_{ijk} = 1, \quad \forall (j, k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k).
\end{align*}
\]

Since the function \(c_l(x)\) is convex the linearization constraints represent supporting hyperplanes (an outer approximation). Thus, if the original minimum mass problem \(\text{P}_m^D\) has a non–empty feasible set, so does \(4.3\). If on the other hand, \(4.3\) is infeasible, so is \(\text{P}_m^D\). If \(4.3\) has an optimal solution it is denoted \(\hat{x}^p\).

Adding valid inequalities to problem \(4.3\) may improve the outer approximation. The enhanced version of the feasibility pump \[15\] includes the inequality

\[
(\hat{x} - \bar{x})^T x \leq (\hat{x} - \bar{x})^T \bar{x}.
\]

It represents a hyperplane separating \(\hat{x}\) from the convex region defined by the nonlinear constraints. Since \(\bar{x}\) is on the boundary of the convex region defined by the nonlinear constraints and minimizes \(\|x - \bar{x}\|_2^2\), it follows (by the Minimum Principle, see e.g. Theorem 9.3.3 in \[36\]) that \(4.4\) is a valid inequality for \(4.3\).

Problem \(4.3\) can be rewritten as a linear mixed 0–1 problem by introducing additional continuous variables \(z_{ijk}\) with the interpretation

\[
\left| \bar{x}_{ijk}^{p-1} - x_{ijk} \right| = z_{ijk}.
\]
The linear reformulation of (4.3) is

\[
\begin{align*}
\text{minimize} & \quad \sum_{ijk} z_{ijk} \\
\text{subject to} & \quad c_l(x) + (\nabla c_l(x))^T(x - \bar{x}) \leq c_l^{\text{max}}, \quad \forall r = 0, \ldots, p - 1, \quad \forall l \\
& \quad (\bar{x} - \hat{x})^T(x - \bar{x}) \geq 0, \quad \forall r = 1, \ldots, p - 1 \\
& \quad \sum_{i=1}^I x_{ijk} = 1, \quad \forall (j, k) \\
& \quad -z_{ijk} \leq x_{ijk} - \bar{x}_{ijk}^{p-1} \leq z_{ijk}, \quad \forall (i, j, k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k) \\
& \quad z_{ijk} \geq 0, \quad \forall (i, j, k).
\end{align*}
\]

(4.5)

Note that problem (4.5) has at least one optimal solution if (P_{mD}) has a non-empty feasible set. Then we compute \(\bar{x}^p\) by solving the nonlinear continuous problem

\[
\begin{align*}
\text{minimize} & \quad \|x - \hat{x}^p\|_2^2 \\
\text{subject to} & \quad c_l(x) \leq c_l^{\text{max}}, \quad \forall l \\
& \quad \sum_{i=1}^I x_{ijk} = 1, \quad \forall (j, k) \\
& \quad x_{ijk} \geq 0, \quad \forall (i, j, k).
\end{align*}
\]

(4.6)

Problem (4.6) is a convex and continuous problem and can therefore be solved to global optimality. Assumptions (A1) – (A4) also guarantee that problem (4.6) has a non-empty feasible set. Assumption (A4) states that the feasible set of (4.6) has a (relative) interior point which implies that Slater’s constraint qualifications are satisfied. This regularity property is sufficient to guarantee that every KKT-point of (4.6) is both a local minimizer and a global minimizer.

The feasibility pump alternates between solving (4.5) and (4.6) until either a feasible solution to the original mixed 0–1 problem (P_{mD}) is found or (4.5) becomes infeasible. In the latter situation it is concluded that (P_{mD}) is also infeasible. The feasibility pump algorithm is
presented in Algorithm 4.

**Algorithm 4:** Feasibility pump for the minimum mass problem \((P^m_N)\).

Solve the continuous relaxation \((P^m_R)\) of the minimum mass problem \((P^m_N)\).
Denote the optimal solution \(\mathbf{x}^0\) with objective \(m_R\).
Generate compliance inequalities on \(\mathbf{x}^0\) for each load case, i.e.
\[
c_l(\mathbf{x}^0) + (\nabla c_l(\mathbf{x}^0))^T (\mathbf{x} - \mathbf{x}^0) - c_l^{\text{max}} \leq 0, \quad \forall l
\]
and add them to (4.5).
Set the lower bound \(lb = m_R\).
Set \(p = 1\).

**while** \(c(\hat{x}_p^l) > c_l^{\text{max}}\) **for some** \(l\) **do**

 Attempt to solve the linear mixed 0–1 problem (4.5).

 **if** (4.5) **is infeasible** **then**
  | Problem \((P^m_P)\) is infeasible. Stop and exit.
 **else**
  | Denote the optimal solution \(\hat{x}^p\).
 **end**

 Generate compliance inequalities on the solution of problem (4.5) for each load case.
 Solve the convex problem (4.6). Denote the optimal solution \(\mathbf{x}^p\).
 Generate compliance inequalities on \(\mathbf{x}^p\) for each load case, i.e.
\[
c_l(\mathbf{x}^p) + (\nabla c_l(\mathbf{x}^p))^T (\mathbf{x} - \mathbf{x}^p) - c_l^{\text{max}} \leq 0, \quad \forall l
\]
 and add them to (4.5).
 Let \(p \leftarrow p + 1\).

 **end**

 Compute displacement vectors \(u_l(\hat{x}) = K(\hat{x})^{-1} f_l\) \(\forall l\) and the upper bound
\[
ub = \sum_{l=1}^{L} w_l f_l^T u_l(\hat{x}).
\]

 Compute the relative optimality gap \(\delta = ((ub - lb)/ub) \times 100\).

The procedure defined by the enhanced feasibility pump cannot cycle, see [15] for a proof. Furthermore, the region defined by our nonlinear constraints is convex, making the enhanced feasibility pump an exact algorithm. Either it finds a feasible solution or it proves that the problem is infeasible.

### 4.3.3 Gap improvement method/heuristic

If the relative optimality gap achieved after performing the heuristics for the minimum compliance problems \((P^c_D)\) and \((P^w_D)\) and minimum mass problem \((P^m_D)\) is too large we attempt to find improved feasible designs. In case of the minimum compliance problems \((P^c_D)\) and \((P^w_D)\) we apply the feasibility pump heuristic with an artificial upper bound \(c^{\text{max}}\) on the weighted compliance. The artificial bound is then adjusted by a bisection procedure. This combined
heuristic solves a sequence of problems given by

\[
\begin{align*}
\text{minimize} & \quad \|x - \bar{x}^{p-1}\|_1 \\
\text{subject to} & \quad c(\bar{x}^r) + (\nabla c(\bar{x}^r))^T(x - \bar{x}^r) \leq c^{\max}, \quad \forall r = 0, \ldots, p - 1 \\
& \quad (\bar{x}^{r'} - \hat{x}^r)^T(x - \bar{x}^r) \geq 0, \quad \forall r = 1, \ldots, p - 1 \\
& \quad m(x) \leq m^{\max}, \\
& \quad \sum_{i=1}^{l} x_{ijk} = 1, \quad \forall (j, k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k),
\end{align*}
\]  

(4.7)

and the convex problems

\[
\begin{align*}
\text{minimize} & \quad \|x - \hat{x}^p\|_2^2 \\
\text{subject to} & \quad c(x) \leq c^{\max}, \\
& \quad m(x) \leq m^{\max}, \\
& \quad \sum_{i=1}^{l} x_{ijk} = 1, \quad \forall (j, k) \\
& \quad x_{ijk} \geq 0, \quad \forall (i, j, k).
\end{align*}
\]  

(4.8)
The gap improvement method/heuristic is described in detail in Algorithm 5 for the minimum compliance problem \( \text{P}_c^N \).

**Algorithm 5: Gap improvement method/heuristic for the minimum compliance problem \( \text{P}_c^N \).**

1. Solve the continuous relaxation \( \text{P}_c^R \) of the minimum compliance problem \( \text{P}_c^N \).
2. Denote the optimal solution \( \bar{x}^0 \) with objective \( z_R \).
3. Generate a compliance inequality \( c(\bar{x}^0) + (\nabla c(\bar{x}^0))^T (x - \bar{x}^0) - c_{\text{max}} \leq 0 \) and add to (4.7).
4. Solve the rounding heuristic problem (4.2) and denote the optimal solution \( \hat{x}^0 \).
5. Generate a compliance inequality on the solution of problem (4.2) and add to (4.7).
6. Set the lower bound \( lb = z_R \).
7. Compute displacement vectors \( u_l(\hat{x}^0) = K(\hat{x}^0)^{-1}f_l \forall l \) and the upper bound
   \[
   ub = \sum_{l=1}^{L} w_l f_l^T u_l(\hat{x}^0). 
   \]
8. Set the target value \( c_{\text{max}} = \frac{ub + lb}{2} \).
9. **while** \( (ub - lb)/ub > \epsilon_0 \) **do**
   1. Set \( p = 0 \).
   2. **while** \( c(\hat{x}^p) > c_{\text{max}} \) **do**
      1. Let \( p \leftarrow p + 1 \).
      2. Attempt to solve the 0–1 problem (4.7).
      3. **if** \( (4.7) \) **is infeasible** **then**
         1. Update the lower bound \( lb \leftarrow c_{\text{max}} \).
         2. Update the target value \( c_{\text{max}} \leftarrow (ub + lb)/2 \).
      **else**
         1. Denote the optimal solution \( \hat{x}^p \).
         2. Generate a compliance inequality on the solution of problem (4.7).
         3. Solve the convex problem (4.8). Denote the optimal solution \( x^p \).
         4. Generate a compliance inequality on \( x^p \), i.e.
            \[
            c(\bar{x}^p) + (\nabla c(\bar{x}^p))^T (x - \bar{x}^p) - c_{\text{max}} \leq 0
            \]
         and add to (4.7).
      **end**
   3. Update the upper bound \( ub \leftarrow c(\hat{x}^p) \).
   4. Update the target value \( c_{\text{max}} \leftarrow (ub + lb)/2 \).
 **end**

If the problem (4.7) is solved to optimality at each iteration then Algorithm 5 terminates with a feasible solution to the minimum compliance problem \( \text{P}_c^D \) which has a relative optimality gap no larger than \( \epsilon_0 \). In this situation the algorithm constitutes a global optimization method. If on the other hand we limit the maximum number of nodes solved or the time limit in the branch–and–cut solver for solving (4.7), it might happen that the mixed 0–1 solver terminates without reaching optimality. In this case Algorithm 5 becomes a heuristic.
4.4 Global optimization methods

The models and methods used for the optimal design of laminated composite structures, in general, only provide locally optimal designs and one has to resort to direct search methods in order to achieve globally optimal solutions to the optimization problems, see e.g. [14, 13]. We are aiming at solving the minimum compliance problems \((P^{c}_D)\) and \((P^{c}_N)\) and the minimum mass problem \((P^{m}_D)\) to proven global optimality. We present in this section two methods with this capability. The first method is based on the principle of outer approximation, see [21] and [23]. It consists of solving a sequence of nonlinear continuous problems and relaxed versions of a mixed 0–1 linear master program.

We also propose an alternative approach to perform the global optimization by a combination of the convergent outer approximation [21], [23] and the local branching [22] algorithms. Local branching has been very successful, both as a method and as a heuristic and is today implemented in commercial packages for mixed integer optimization, mainly due to its ability to achieve early updates of the incumbent. With local branching the solution neighborhoods are defined, by the introduction of linear inequalities (local branching cuts) in our mixed 0–1 program. We explore the solution neighborhoods using the outer approximation framework.

4.4.1 Outer approximation

We have developed a framework based on the principles of outer approximation that provides globally optimal solutions to the minimum compliance and minimum mass problems \((P^{c}_D)\) – \((P^{m}_D)\). For the minimum compliance problems \((P^{c}_D)\) and \((P^{c}_N)\) this framework consists of solving a finite sequence of relaxed versions of problem \((P^{c}_N)\). The idea behind outer approximation is to approximate the nonlinear objective and constraint functions with linear functions. Given a number of points \(x^p \in [0,1]^n\) the relaxation of the outer approximation master problem of \((P^{c}_N)\) is given by the linear mixed 0–1 problem

\[
\text{minimize}_{\eta \in \mathbb{R}, x \in \mathbb{R}^n} \eta \\
\text{subject to} \quad c(x^r) + \left(\nabla c(x^r)\right)^T(x - x^r) - \eta \leq 0, \quad r = 1, \ldots, p \\
\sum_{i=1}^{I} x_{ijk} = 1, \quad \forall (j,k) \\
m(x) \leq m_{\text{max}}, \\
x_{ijk} \in \{0,1\}, \quad \forall (i,j,k) \\
\eta \geq 0.
\]  

(4.9)

Outer approximation relies on the representation of convex sets by a collection of supporting planes and it requires convexity of the objective and constraint functions. Since the function \(c(x)\) is convex the linearization constraints represent supporting hyperplanes. The feasible set of problem \((4.9)\) is non-empty due to assumptions \([A1] [A3]\). Since the objective function in \((4.9)\) is bounded from below, there exists at least one optimal solution.

The outer approximation algorithm is stated for the minimum compliance problems \((P^{c}_D)\)
Algorithm 6: Outer approximation for the minimum compliance problems \((P_{cR})\) and \((P_{cD})\).

Solve the continuous relaxation of the relevant minimum compliance problem. Denote the optimal solution \(x^0\) with objective \(z_R\). Generate a compliance inequality on \(x^0\), i.e.

\[ c(x^0) + (\nabla c(x^0))^T(x - x^0) - \eta \leq 0. \]

and add to the outer approximation master problem (4.9).
Solve the rounding heuristic problem (4.2). Denote the optimal solution \(\hat{x}\).
Set the lower bound \(lb = z_R\).
Compute the displacement vectors \(u_l(\hat{x}) = K(\hat{x})^{-1}f_l \forall l\) and compute the upper bound

\[ ub = \sum_l w_l f_l^T u_l(\hat{x}). \]

Set \(p = 1\).
while \((ub - lb)/ub > \epsilon_0\) do

\begin{itemize}
  \item Solve the linear mixed 0–1 problem (4.9). Denote the optimal design \(\hat{x}^p\).
  \item Generate a compliance inequality on \(\hat{x}^p\) and add to (4.9).
  \item Update the lower bound \(lb \leftarrow \max\{lb, z_{OA}\}\).
  \item For the solution of the master problem (4.9), compute the corresponding displacement vector \(u_l(\hat{x}^p)\) and update the upper bound
  \[ ub \leftarrow \min\{ub, \sum_{l=1}^L w_l f_l^T u_l(\hat{x}^p)\}. \]
\end{itemize}

Set \(p \leftarrow p + 1\).
end

Due to assumptions (A1) – (A4), Algorithm 6 terminates with a feasible solution to the minimum compliance problem \((P_{cD})\) which has a relative optimality gap no larger than \(\epsilon_0\), see [23] for a proof.

Remark 1. In the implementation more than one compliance cut is generated at every iteration and added to the outer approximation master problem (4.9). The branch–and–cut method used to solve the relaxed master problem returns a pool of optimal or near optimal points from which cuts are also generated.

Remark 2. An outer approximation method for the worst case problem \((P_{wD})\) is obtained if the inequalities are modified to

\[ c_l(x^r) + (\nabla c_l(x^r))^T(x - x^r) - \eta \leq 0, \forall l, r = 0, \ldots, p. \]
The computation of the upper bound must be replaced by

\[ ub \leftarrow \min \{ ub, \max \{ f_l^T u_l(\hat{x}^p) \} \}. \]

### 4.4.2 Local branching

We have implemented a two–level local branching framework in order to achieve global optimality to the minimum compliance and minimum mass problems \((P_{cD}^c) – (P_{mD}^m)\). The procedure is in the spirit of well–known local search meta–heuristics. On the first level, the solution neighborhoods are defined by using the Hamming distance function

\[
\Delta(x, \hat{x}) = \sum_{q \in \mathcal{N}^1(\hat{x})} (1 - x_q) + \sum_{q \in \mathcal{N} \setminus \mathcal{N}^1(\hat{x})} x_q
\]

where \( \mathcal{N} = \{1, \ldots, n\} \) is the index set of the 0–1 variables and \( \mathcal{N}^1 = \{ q \in \mathcal{N} | \hat{x}_q = 1 \} \) is the binary support of a feasible reference point \( \hat{x} \). Given a feasible solution \( \hat{x} \) of either \((P_{cD}^c) – (P_{mD}^m)\) and a neighborhood size parameter \( \kappa \), we define the \( \kappa \)–opt neighborhood around \( \hat{x} \) as the set of feasible solutions of either \((P_{cD}^c) – (P_{mD}^m)\), satisfying the additional local branching constraint \( \Delta(x, \hat{x}) \leq \kappa \). In case of minimizing the compliance a subregion of problem \((P_{cN}^c)\) is defined by

\[
\begin{align*}
\text{minimize} & \quad c(x) \\
\text{subject to} & \quad m(x) \leq m_{\text{max}}, \\
& \quad \Delta(x, \hat{x}) \leq \kappa, \\
& \quad \sum_{i=1}^{f} x_{ijk} = 1, \quad \forall (j, k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k).
\end{align*}
\]

The set associated with the current branching is in this way partitioned by means of the disjunction

\[
\Delta(x, \hat{x}) \leq \kappa \; \text{(left branch)} \quad \text{or} \quad \Delta(x, \hat{x}) \geq \kappa + 1 \; \text{(right branch)}
\]

On the second level of local branching, we explore the set defined by the left branch. For this purpose we are making use of the outer approximation framework described in Section 4.4.1. The relaxation of the outer approximation master problem (4.10) is given by the linear 0–1 problem

\[
\begin{align*}
\text{minimize} & \quad \eta \\
\text{subject to} & \quad c(x^p) + (\nabla c(x^p))^T (x - x^p) - \eta \leq 0, \quad \forall p = 1, \ldots, P \\
& \quad m(x) \leq m_{\text{max}}, \\
& \quad \Delta(x, \hat{x}) \leq \kappa, \\
& \quad \sum_{i=1}^{f} x_{ijk} = 1, \quad \forall (j, k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k) \\
& \quad \eta \geq 0.
\end{align*}
\]

We reverse the last local branch constraint into \( \Delta(x, \hat{x}) \geq \kappa + 1 \) and given the solution of problem (4.12) we introduce a new local branching cut into problem (4.10). We repeat
the procedure until no further improvement of the upper bound can be achieved or problem \((P^c_D)\) becomes infeasible. We resort to outer approximation to explore the remaining of the feasible region of either \((P^c_D) - (P^m_D)\) and conclude the enumeration.

The local branching algorithm in case of the minimum compliance problem \((P^c_D)\) is described in Algorithm 7.

**Algorithm 7: Local branching for solving the minimum compliance problem \((P^c_D)\).**

- Solve the continuous relaxation \((P^c_R)\). Denote the optimal solution \(\overline{x}^0\) with objective \(z_R\).
- Solve the rounding heuristic problem \((4.2)\). Denote the optimal solution \(\hat{x}\).
- Generate a local branching inequality on \(\hat{x}\), i.e. \(\Delta(x, \hat{x}) \leq \kappa\).
- Set the lower bound \(lb = z_R\).
- Compute the displacement vectors \(u_l(\hat{x}) = K(\hat{x})^{-1}f_l\). Set the upper bound
  \[
  ub = \sum_{l=1}^{L} w_l f_l^T u_l(\hat{x}).
  \]

Set \(p = 1\).

**while** \((ub - lb)/ub > \epsilon_0\) **do**

- Solve \((4.10)\) by means of outer approximation.
  - **if** \((4.10)\) is infeasible **then**
    - Stop and exit.
  - **else**
    - Denote the optimal solution \(\hat{x}^p\).
  - **end**

  For the solution of \((4.10)\), compute the corresponding displacement vectors \(u_l(\hat{x}^p) \forall l\) and update the upper bound
  \[
  ub \leftarrow \min\{ub, \sum_{l=1}^{L} w_l f_l^T u_l(\hat{x}^p)\}.
  \]

- **if** the upper bound was not improved **then**
  - Stop and exit the while loop.
- **else**
  - Reverse the last local branching constraint.
  - Generate a local branching inequality on \(\hat{x}^p\), i.e. \(\Delta(x, \hat{x}^p) \leq \kappa\).
  - Set \(p \leftarrow p + 1\).
- **end**

**end**

Delete the last local branching constraint and solve the resulting mixed 0–1 program by outer approximation.

Algorithm 7 terminates with a feasible solution to the minimum compliance problem \((P^c_D)\) which has a relative optimality gap no larger than \(\epsilon_0\).
4.5 Implementation and parameters

All continuous relaxations \((P^c_R)\) and \((P^m_R)\) and all the nonlinear programs in the feasibility pump heuristics (4.6) and (4.8) are solved by a MATLAB implementation of a primal–dual interior point method for nonlinear programming that updates the barrier parameter adaptively following the ideas in [43]. The primal–dual saddle–point systems for computing the search directions are solved using the LDL–factorization provided with MATLAB. We developed and implemented a special purpose interior point method rather than using a general purpose software. This allowed us to use the exact Hessian of the compliance functions without explicitly computing it. The Hessian of compliance is both completely dense and very expensive to compute. The optimality tolerance in the interior point method is set to \(10^{-7}\) and the feasibility tolerance is set to \(10^{-9}\).

The heuristics, the outer approximation method, and the local branching algorithm presented in Algorithms 3–7 are also implemented in the numerical environment MATLAB. The local branching neighborhood search parameter \(\kappa\) is set to a value of 5.

The linear mixed 0–1 integer programs in the heuristics and the relaxation of the outer approximation master problem are solved by the commercial branch–and–cut software for mixed–integer programming IBM CPLEX version 12.5 [19]. The parameters in IBM CPLEX are set to default values.

The finite elements used in the numerical experiments are 9 node Mindlin type plate elements with 5 degrees of freedom per node and obtained by full Gaussian integration. The strain–displacement matrices \(B_j\) are obtained with numerical integration using Gauss quadrature, see e.g. [3], [8], [18] and [30]. The matrix \(B_j\) is written as

\[
B_j = \begin{bmatrix}
B_j^1 & B_j^2 & \ldots
\end{bmatrix}^T
\]

where \(B_j^\eta\) is the strain–displacement matrix in the \(\eta\)–th Gauss quadrature point, i.e.

\[
B_j^\eta = \begin{bmatrix}
B_m^{\eta} & B_b^{\eta} & B_s^{\eta}
\end{bmatrix}^T
\]

The membrane component \(B_m^{\eta}\), the bending component \(B_b^{\eta}\), and the shear component \(B_s^{\eta}\) are given by

\[
B_m^{\eta} = \begin{bmatrix}
\frac{\partial N}{\partial x} & 0 & 0 & 0 & 0 \\
0 & \frac{\partial N}{\partial y} & 0 & 0 & 0 \\
\frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} & 0 & 0 & 0
\end{bmatrix}, \quad B_b^{\eta} = \begin{bmatrix}
0 & 0 & 0 & \frac{\partial N}{\partial x} & 0 \\
0 & 0 & 0 & \frac{\partial N}{\partial y} & 0 \\
0 & 0 & 0 & \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x}
\end{bmatrix}, \quad B_s^{\eta} = \begin{bmatrix}
0 & 0 & \frac{\partial N}{\partial y} & N & 0 \\
0 & 0 & \frac{\partial N}{\partial x} & 0 & N
\end{bmatrix}
\]

with \(N\) being the shape functions describing the deformations within the finite elements. The constitutive matrix \(C_{ik}\) is block–diagonal and given by \(C_{ik} = \text{diag}(C_{ik}^{\eta})\), where \(C_{ik}^{\eta}\) is the constitutive matrix in the \(\eta\)–th Gauss quadrature point, i.e.

\[
C_{ik}^{\eta} = \begin{bmatrix}
A_{ik} & B_{ik} & 0 \\
B_{ik} & D_{ik} & 0 \\
0 & 0 & S_{ik}
\end{bmatrix}
\]

\([A_{ik}], [B_{ik}], [D_{ik}], [S_{ik}]\) are the extensional, coupling, bending and shear stiffness matrices respectively for the \(i\) – \(th\) given material for the \(k\) – \(th\) layer, see e.g. [23].
4.6 Numerical experiments

In this section we present the numerical experience with the outer approximation and local branching global optimization methods and the three heuristics applied to a set of discrete material optimization problems. The ability of our methods and heuristics to perform multi-material optimization is examined by solving minimum compliance and minimum mass problems of laminated composite plates.

All examples were run on an Intel® Xeon 5150 processor running at 2.66 GHz. The problems are considered solved (to global optimality) if the relative optimality gap ≤ 1%.

4.6.1 The benchmark problems

In Table 4.1 we present the set of problem instances intended for the heuristics. We consider the three different design domains and load conditions for multi-material optimization of laminated composite plates shown in Figures 4.1 – 4.3. The table reports the number of finite elements used for the mesh discretization, the total number of degrees of freedom, the number of design variables, the number of load cases considered and the maximum allowable mass $m_{\text{max}}$.

In Table 4.2 we present the set of reduced size problem instances intended for the global optimization methods. The same design domains and load conditions are considered as in Table 4.1 for smaller problem sizes (fewer design variables and degrees of freedom).

The compliance value provided by the continuous relaxation of the minimum compliance problem ($P_{\text{N}}$) is used as a bound on the maximum allowable compliance for the minimum mass problem ($P_{\text{D}}$).

For the problem instances reported in Table 4.3 the same design domains and load conditions as in Table 4.1 are considered with certain manufacturing limitations included in the problem formulations. We examine the case where the design variables are coupled with the material properties of an entire laminae, resulting in a layer wise constant choice of material. Finally, we extend our models to eliminate the placement of the soft isotropic material in the outer layers of the structure.

Table 4.1: Characteristics of the problems intended for the heuristics. The candidate materials are an orthotropic material oriented at the 4 distinct directions in the set $\{-45^0, 0^0, 45^0, 90^0\}$, and a soft isotropic material. The material properties are listed in Table 4.4.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
<th>Elements</th>
<th>DOF</th>
<th>Variables</th>
<th>Loads</th>
<th>Layers</th>
<th>$m_{\text{max}}[kg]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1H1</td>
<td>Clamped Uniform loading</td>
<td>16×16</td>
<td>5445</td>
<td>10240</td>
<td>1</td>
<td>8</td>
<td>49.4</td>
</tr>
<tr>
<td>P1H2</td>
<td>Clamped Uniform loading</td>
<td>32×32</td>
<td>21125</td>
<td>40960</td>
<td>1</td>
<td>8</td>
<td>49.4</td>
</tr>
<tr>
<td>P2H</td>
<td>Simply supported Point load</td>
<td>16×16</td>
<td>21780</td>
<td>10240</td>
<td>4</td>
<td>8</td>
<td>53.2</td>
</tr>
<tr>
<td>P3H</td>
<td>Clamped Bending – Torsion</td>
<td>16×48</td>
<td>32010</td>
<td>30720</td>
<td>2</td>
<td>8</td>
<td>80.2</td>
</tr>
</tbody>
</table>
4.6. NUMERICAL EXPERIMENTS

Table 4.2: Characteristics of the problems intended for the global optimization methods. The candidate materials are an orthotropic material oriented at the 4 distinct directions in the set \{-45^0, 0^0, 45^0, 90^0\}, and a soft isotropic material. The material properties are listed in Table 4.4.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
<th>Elements</th>
<th>DOF</th>
<th>Variables</th>
<th>Loads</th>
<th>Layers</th>
<th>(m_{\text{max}}[\text{kg}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1G1</td>
<td>Clamped</td>
<td>4x4</td>
<td>405</td>
<td>640</td>
<td>1</td>
<td>8</td>
<td>49.4</td>
</tr>
<tr>
<td>P1G2</td>
<td>Clamped</td>
<td>8x8</td>
<td>1445</td>
<td>2560</td>
<td>1</td>
<td>8</td>
<td>49.4</td>
</tr>
<tr>
<td>P2G1</td>
<td>Simply supported Point load</td>
<td>4x4</td>
<td>1620</td>
<td>640</td>
<td>4</td>
<td>8</td>
<td>53.2</td>
</tr>
<tr>
<td>P2G2</td>
<td>Simply supported Point load</td>
<td>8x8</td>
<td>5780</td>
<td>2560</td>
<td>4</td>
<td>8</td>
<td>53.2</td>
</tr>
<tr>
<td>P3G1</td>
<td>Clamped Bending – Torsion</td>
<td>4x12</td>
<td>2250</td>
<td>1920</td>
<td>2</td>
<td>8</td>
<td>80.2</td>
</tr>
<tr>
<td>P3G2</td>
<td>Clamped Bending – Torsion</td>
<td>8x24</td>
<td>8330</td>
<td>7680</td>
<td>2</td>
<td>8</td>
<td>80.2</td>
</tr>
</tbody>
</table>

Table 4.3: Characteristics for the problem instances with manufacturing constraints. The candidate materials are an orthotropic material oriented at the 4 distinct directions in the set \{-45^0, 0^0, 45^0, 90^0\}, and a soft isotropic material. The material properties are listed in Table 4.4. In the problem instances PxL the design variables are coupled with the material properties of an entire laminae. In the problem instances PxF our models include appropriate constraints that eliminate the placement of the soft isotropic material in the outer layers of the structure.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
<th>Elements</th>
<th>DOF</th>
<th>Variables</th>
<th>Loads</th>
<th>Layers</th>
<th>(m_{\text{max}}[\text{kg}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1L</td>
<td>Clamped</td>
<td>16x16</td>
<td>5445</td>
<td>40</td>
<td>1</td>
<td>8</td>
<td>49.4</td>
</tr>
<tr>
<td>P1F</td>
<td>Clamped</td>
<td>16x16</td>
<td>5445</td>
<td>9728</td>
<td>1</td>
<td>8</td>
<td>49.4</td>
</tr>
<tr>
<td>P2L</td>
<td>Simply supported Point load</td>
<td>16x16</td>
<td>21780</td>
<td>40</td>
<td>4</td>
<td>8</td>
<td>53.2</td>
</tr>
<tr>
<td>P2F</td>
<td>Simply supported Point load</td>
<td>16x16</td>
<td>21780</td>
<td>9728</td>
<td>4</td>
<td>8</td>
<td>53.2</td>
</tr>
<tr>
<td>P3L</td>
<td>Clamped Bending – Torsion</td>
<td>16x48</td>
<td>32010</td>
<td>40</td>
<td>2</td>
<td>8</td>
<td>80.2</td>
</tr>
<tr>
<td>P3F</td>
<td>Clamped Bending – Torsion</td>
<td>16x48</td>
<td>32010</td>
<td>29184</td>
<td>2</td>
<td>8</td>
<td>80.2</td>
</tr>
</tbody>
</table>

Clamped plate under uniform loading

The ability of our methods and heuristics to perform multi–material optimization is first examined on a clamped plate example. The dimensions of the plate are 1.0m x 1.0m x 0.04m, as depicted in Figure 4.1. The plate consists of 8 layers of equal thicknesses and the candidate materials are an orthotropic material oriented at 4 distinct directions \{-45^0, 0^0, 45^0, 90^0\} and
Table 4.4: Material properties in the principal material coordinate system for the candidate materials used in the numerical experiments.

<table>
<thead>
<tr>
<th></th>
<th>Isotropic Polymeric Foam</th>
<th>Orthotropic Material</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x [GPa]$</td>
<td>0.065</td>
<td>34.0</td>
</tr>
<tr>
<td>$E_y [GPa]$</td>
<td>8.2</td>
<td></td>
</tr>
<tr>
<td>$E_z [GPa]$</td>
<td>8.2</td>
<td></td>
</tr>
<tr>
<td>$G_{xy} [GPa]$</td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td>$G_{yz} [GPa]$</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>$G_{xz} [GPa]$</td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td>$\nu_{xy}$</td>
<td>0.47</td>
<td>0.29</td>
</tr>
<tr>
<td>$\rho [kg/m^3]$</td>
<td>200.0</td>
<td>1910.0</td>
</tr>
</tbody>
</table>

an isotropic polymeric foam. The material properties are listed in Table 4.4.

Figure 4.1: Design domain and boundary conditions for the eight layer clamped plate under uniform loading. The dimensions of the plate are $1.0 m \times 1.0 m \times 0.04 m$.

Simply supported plate subjected to a point load

In the second example we solve a multi layered plate subjected to a point load. The dimensions of the plate are $1.0 m \times 1.0 m \times 0.08 m$, as depicted in Figure 4.2. We are considering four independent load cases of equal magnitude and equal importance. In each load case the plate is simply supported along the edges. The plate consists of eight layers of equal thicknesses. The first candidate material is an isotropic polymeric foam and the remaining four candidate materials are an orthotropic material oriented at 4 distinct directions $\{-45^0, 0^0, 45^0, 90^0\}$. The material properties are listed in Table 4.4.

Clamped plate under flapwise bending and torsion

In the third and final example we solve a simplified plate model of a wind turbine blade. The dimensions of the plate are $3.0 m \times 1.0 m \times 0.04 m$ and the plate consists of 8 layers of
equal thicknesses as depicted in Figure 4.3. We consider two independent load cases of equal magnitude and equal importance. In each load case the plate is clamped at the root. In the first load case the plate is subjected to flapwise bending. This load case arises when the turbine is brought to a standstill condition due to extreme high wind (50 year gust scenario). Moreover, due to aerodynamic considerations, as the cross-section of the blade twists along the length, the structure will be subjected to a torsional load. The candidate materials are, as before, an orthotropic material oriented at 4 distinct directions \{−45^0, 0^0, 45^0, 90^0\} and an isotropic polymeric foam. The material properties are listed in Table 4.4.

Figure 4.3: Design domain and boundary conditions for the eight layer clamped plate under flapwise bending and torsion. The dimensions of the plate are $3.0m \times 1.0m \times 0.04m$. 

Figure 4.2: Design domain and boundary conditions for a simply supported plate with four point loads. The dimensions of the plate are $1.0m \times 1.0m \times 0.08m$. The plate consists of eight layers of equal thicknesses.
4.6.2 Numerical experiments with the heuristics

In Table 4.5 we are presenting the numerical results for the rounding heuristic as described in Algorithm 3 and the feasibility pump heuristic as described in Algorithm 4 for the different problem instances presented in Table 4.1. We report the number of the interior point iterations for solving the continuous relaxation, the number of the (rounding) heuristic iterations, the computation time, the objective of the continuous relaxation, the objective function value of the rounding heuristic, and finally the obtained relative optimality gap.

The obtained results showcase the excellent convergence properties and the ability of the primal–dual interior point method to react swiftly to changes of scale of our problems. The method managed to converge in 13 to 29 iterations in all the examined cases.

We obtain qualitatively different solutions for different mesh discretizations. This is expected since no regularization approach is included in the problem formulations. It is well-established that the infinite dimensional problem formulation for topology optimization may lack a solution in general, see e.g. [6]. Some sort of restriction method is needed in order to obtain well-posed design problems. For solid–void topology optimization alternatives include constraints on the perimeter, see e.g. [27] and [40]. Generalizations of perimeter constraints to discrete material optimization problems are currently being developed by the authors.

For many of the problem instances the obtained relative optimality gap is close to optimal. In two of the cases (problems P2H and P3H) the gap is below 1% and there is no need to continue with global optimization. Our heuristics are highly dependent on the considered design domain, loading and boundary conditions. The relative optimality gap varies by as low as 2% to as high as 22.8% depending on the considered case.

The material distributions for the examined problems are depicted in Figures 4.4 – 4.6. We get similar distributions of the minimum compliance problem and the minimum mass problem for the related problem instances. The obtained results are in good agreement with previous studies performed on the same benchmark examples, and presented in e.g. [45], [26] and [31]. The laminate configuration for all load cases is a symmetric lay–up configuration. The optimized designs use the soft material to form a sandwich structure, leading to a high bending stiffness to weight ratio for the composite. This corresponds to well known reinforcement techniques for sandwich panels, see e.g. [13]. In case of the simplified wind turbine blade model as shown in Figure 4.6 the dominant effect near the clamped root of the blade is bending. This is clearly interpreted by the placement of the $0^0$ plies near this region. The resulting shear due to the applied torsion becomes more significant near the free end of the blade where the placement of the $\pm 45^0$ plies takes place.

We attempt to find improved feasible designs after performing the heuristics for the minimum compliance problems ($P^{c}_D$) and ($P^{m}_D$) and the minimum mass problem ($P^{m}_D$) by applying the gap improvement method/heuristic described in Algorithm 5. In Table 4.6 we report the computation time, the number of mixed integer problems solved (problem (4.7)), the lower bound provided by the nonlinear relaxed master problems ($P^{c}_R$) and ($P^{m}_R$), the objective function value of the best found design and the obtained relative optimality gap.

In case of the layered clamped plate benchmark example (i.e. problems P1Hx), this approach was used as a global optimization method and was able to solve all the considered problem instances to global optimality. The remaining benchmark examples (i.e. problems P2H and P3H), were not solved to global optimality within the given time limit. Nevertheless, we managed to further improve the relative optimality gap significantly. In these cases, the algorithm has been used as a heuristic.
4.6. NUMERICAL EXPERIMENTS

Table 4.5: Numerical results on the rounding heuristic in Algorithm 3 for the minimum compliance problem ($P_{cN}$) and the feasibility pump heuristic Algorithm 4 for the minimum mass problem ($P_{mN}$) for the benchmark examples presented in Table 4.1. We report the number of interior point iterations for solving the continuous relaxation, the number of iterations performed in the heuristics (Algorithm 3 and Algorithm 4, respectively), the computation time, the objective function value of the continuous relaxation, the objective function value of the rounding heuristic, and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Objective</th>
<th>I.P.</th>
<th>Heuristic</th>
<th>Time</th>
<th>Bounds</th>
<th>Gap(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1H1C</td>
<td>C</td>
<td>20</td>
<td>1</td>
<td>00:00:44</td>
<td>1.797</td>
<td>1.933</td>
</tr>
<tr>
<td>P1H1M</td>
<td>M</td>
<td>21</td>
<td>17</td>
<td>00:00:47</td>
<td>42.560</td>
<td>43.428</td>
</tr>
<tr>
<td>P1H2C</td>
<td>C</td>
<td>22</td>
<td>1</td>
<td>00:03:57</td>
<td>1.787</td>
<td>1.923</td>
</tr>
<tr>
<td>P1H2M</td>
<td>M</td>
<td>20</td>
<td>10</td>
<td>00:03:21</td>
<td>37.803</td>
<td>38.920</td>
</tr>
<tr>
<td>P2HC</td>
<td>C</td>
<td>19</td>
<td>1</td>
<td>00:15:22</td>
<td>14.766</td>
<td>18.223</td>
</tr>
<tr>
<td>P2HM</td>
<td>M</td>
<td>13</td>
<td>1</td>
<td>00:02:31</td>
<td>51.337</td>
<td>51.402</td>
</tr>
<tr>
<td>P3HC</td>
<td>C</td>
<td>29</td>
<td>1</td>
<td>00:05:14</td>
<td>5.756</td>
<td>7.460</td>
</tr>
<tr>
<td>P3HM</td>
<td>M</td>
<td>18</td>
<td>2</td>
<td>00:04:11</td>
<td>92.402</td>
<td>92.733</td>
</tr>
</tbody>
</table>

Table 4.6: Numerical results on the gap improvement method (G.I.M.) in Algorithm 5 for the benchmark examples presented in Table 4.1. We report the computation time, the number of mixed integer problems solved (problem (4.7)), the lower bound provided by the nonlinear relaxed master problems ($P_{c,R}$) and ($P_{m,R}$), the objective function value of the best found design, and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Objective</th>
<th>Time</th>
<th>Itns</th>
<th>Bounds</th>
<th>Gap(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1H1C</td>
<td>C</td>
<td>00:09:22</td>
<td>4</td>
<td>1.797</td>
<td>7.0</td>
</tr>
<tr>
<td>P1H1M</td>
<td>M</td>
<td>00:19:34</td>
<td>30</td>
<td>42.560</td>
<td>2.0</td>
</tr>
<tr>
<td>P1H2C</td>
<td>C</td>
<td>15:32:12</td>
<td>102</td>
<td>1.787</td>
<td>7.1</td>
</tr>
<tr>
<td>P1H2M</td>
<td>M</td>
<td>12:25:45</td>
<td>97</td>
<td>37.803</td>
<td>2.9</td>
</tr>
<tr>
<td>P2HC</td>
<td>C</td>
<td>23:57:21</td>
<td>11</td>
<td>14.766</td>
<td>19.0</td>
</tr>
<tr>
<td>P3HC</td>
<td>C</td>
<td>23:58:02</td>
<td>33</td>
<td>5.756</td>
<td>22.8</td>
</tr>
</tbody>
</table>

4.6.3 Numerical experiments with the global optimization methods

In Table 4.7 we present the numerical results from the outer approximation framework for the different problem instances presented in Table 4.2. We report the computation time, the number of solved relaxations, the number of outer approximation cuts introduced, the lower bound provided by the nonlinear relaxed master problem, the objective function value of the best found design, and the obtained relative optimality gap.

The outer approximation method managed to solve only small scale problem instances. Nevertheless, outer approximation managed to improve the relative optimality gap significantly, compared to the value obtained from the heuristics, in all cases. The multiple load case problems (i.e. problems P2G1 and P3G1) were solved in only one outer approximation iteration.

In Table 4.8 we present the numerical results from the local branching framework for the
Figure 4.4: Design obtained by applying the rounding heuristic in Algorithm 3 on the minimum compliance problem $P_{cD}^*$ of the eight layer clamped plate, i.e. problem P1H1, under uniform loading. The finite element mesh has been discretized with 256 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 4.1. The proposed design has a relative optimality gap of 7.0% as presented in Table 4.5.

Figure 4.5: Design obtained by applying the rounding heuristic in Algorithm 3 on the minimum compliance problem $P_{cD}^*$ of the simply supported plate with four point loads, i.e. problem P2H. The finite element mesh has been discretized with 256 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 4.2. The proposed design has a relative optimality gap of 19.0% as presented in Table 4.5.

different problem instances presented in Table 4.2. We report the computation time, the number of left branches solved, the outer approximation iterations and cuts for the right branch, the lower bound provided by the nonlinear relaxed master problems $P_{cR}^*$ and $P_{mR}^*$, the objective function value of the best found design and the obtained relative optimality gap.

The results indicate that this approach is competitive to the outer approximation framework presented in Section 4.4.1 for this class of problems, due to the ability of local branching to provide us with early updates of the incumbent solution. The objective function value of the best found design is provided by solving the left branch. The feasible set defined by the right branch is explored to further improve the lower bound provided by the nonlinear relaxed master problems $P_{cR}^*$ and $P_{mR}^*$.

The gap improvement method/heuristic presented in Algorithm 5 clearly outperforms both outer approximation and local branching in handling large scale problem instances.
Figure 4.6: Design obtained by applying the rounding heuristic in Algorithm 3 on the minimum compliance problem \((P_{cD})\) of the eight layer clamped plate, i.e. problem P3H under flapwise bending and torsion. The finite element mesh has been discretized with 768 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 4.3. The proposed design has a relative optimality gap of 22.8% as presented in Table 4.5.
The material distributions for the examined problems closely resemble the designs depicted in Figures 4.4 – 4.6, slightly modified to achieve global optimality. The modifications result in designs lacking symmetry in–plane and through the thickness of the structure and their representation has therefore been omitted here.

Table 4.7: Numerical results from applying the outer approximation (O.A.) framework in Algorithm \[\text{6}\] for the problem instances presented in Table \[\text{4.1}\]. We report the computation time, the number of solved relaxations, the number of outer approximation cuts introduced, the lower bound provided by the nonlinear relaxed master problems \((P_{cR})\) and \((P_{mR})\), the objective function value of the best found design and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Obj.</th>
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<th>O.A.</th>
<th>Bounds</th>
<th>Gap(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td>Lower</td>
<td>Upper</td>
</tr>
<tr>
<td>P1G1</td>
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<td>00:00:02</td>
<td>2</td>
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<td>1.654</td>
</tr>
<tr>
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<td>00:02:57</td>
<td>76</td>
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<td>1.832</td>
</tr>
<tr>
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<td>00:16:15</td>
<td>119</td>
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<td>37.484</td>
</tr>
<tr>
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<td>16.524</td>
</tr>
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<td>57.681</td>
</tr>
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<td>602</td>
<td>15.436</td>
<td>16.711</td>
</tr>
<tr>
<td>P3G1</td>
<td>C</td>
<td>23:59:14</td>
<td>1608</td>
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<td>6.068</td>
</tr>
<tr>
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<td>M</td>
<td>00:00:03</td>
<td>1</td>
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<td>92.934</td>
</tr>
<tr>
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<td>23:59:01</td>
<td>579</td>
<td>5.764</td>
<td>6.329</td>
</tr>
</tbody>
</table>

Table 4.8: Numerical results from applying the local branching (L.B.) framework in Algorithm \[\text{7}\] for the problem instances presented in Table \[\text{4.1}\]. We report the computation time, the number of left branches solved, the outer approximation iterations and cuts for the right branch, the lower bound provided by the nonlinear relaxed master problems \((P_{cR})\) and \((P_{mR})\), the objective function value of the best found design and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Prob.</th>
<th>Obj.</th>
<th>Time [h:m:s]</th>
<th>Left Branch</th>
<th>Right Branch</th>
<th>Bounds</th>
<th>Gap(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Time [h:m:s]</td>
<td>Num</td>
<td>Itns</td>
<td>Cuts</td>
</tr>
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<td>00:00:21</td>
<td>00:00:21</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>C</td>
<td>00:03:24</td>
<td>00:03:11</td>
<td>7</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>P1G2</td>
<td>M</td>
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<td>00:00:30</td>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
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<td>00:00:35</td>
<td>00:00:35</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>P2G1</td>
<td>M</td>
<td>00:00:31</td>
<td>00:00:24</td>
<td>1</td>
<td>1</td>
<td>24</td>
</tr>
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<td>23:58:25</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>03:15:21</td>
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<td>1853</td>
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</tr>
<tr>
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<td>09:01:12</td>
<td>00:00:04</td>
<td>2</td>
<td>872</td>
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<tr>
<td>P3G2</td>
<td>C</td>
<td>23:56:47</td>
<td>23:56:47</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

4.6.4 Numerical experiments with the manufacturing constraints

In terms of manufacturability, the designs shown in Figures 4.4 – 4.6 do not comply with basic design rules that prevent failure such as delamination and matrix cracking problems.
The changes in the fibre orientation throughout the plane of a layer are too abrupt. The formulations presented in Section 4.2 are general enough to cover also the situation that the design variables are coupled to patches (larger areas). Alternatively, the design variables can be coupled with the material properties of an entire laminae. This possibility is showcased in Figure 4.7 for the eight layer clamped problem, i.e. problem P1L.

We include additional appropriate (manufacturing) constraints in our problem formulations to eliminate the placement of the soft material in the outer layers. This possibility is illustrated for the eight layer clamped plate problem, i.e. problem P1F, in Figure 4.8. The two skin layers are coupled with the design variables associated only with the orthotropic material.

In Table 4.9 we present the numerical results from the heuristics for the different problem instances presented in Table 4.3. We report the number of interior point iterations, the number of the heuristic iterations, the computation time, the objective of the continuous relaxation, the objective function value of the heuristic, and the obtained relative optimality gap.

In three of the examined cases the relative optimality gap is below 1% and there is no need to continue further on with the application of the gap improvement method/heuristic or the global optimization methods. The relative optimality gap varies by as low as 1.3% to as high as 15.5% depending on the considered case.

In Table 4.10 we present the numerical results by applying the gap improvement method/heuristic in Algorithm 5 for the problem instances of Table 4.3. We report the computation time, the number of mixed integer problems solved (problem (4.7)), the lower bound provided by the nonlinear relaxed master problems \( P_\text{cR} \) and \( P_\text{mR} \), the objective function value of the best found design, and the obtained relative optimality gap. Seven out of nine problem instances were actually solved to global optimality, while for the rest of the cases the heuristic managed to further improve the relative optimality gap significantly.

In Table 4.11 we present the numerical results from the outer approximation framework for the different problem instances presented in Table 4.3. We report the computation time, the number of solved relaxations, the number of outer approximation cuts introduced, the lower bound provided by the nonlinear relaxed master problems \( P_\text{cR} \) and \( P_\text{mR} \), the objective function value of the best found design, and the obtained relative optimality gap. The method managed to solve to global optimality seven out of nine problem instances and in most of the cases in just one iteration.

In Table 4.12 the numerical results from the local branching framework are presented for the different problem instances presented in Table 4.3. The table contains the computation time, the number of left branches solved, the outer approximation iterations and cuts for the right branch, the lower bound provided by the nonlinear relaxed master problems \( P_\text{cR} \) and \( P_\text{mR} \), the objective function value of the best found design, and the obtained relative optimality gap. Like outer approximation, local branching managed to solve seven out of nine of the problem instances and further improve the relative optimality gap for the remaining problems.

### 4.7 Concluding remarks

We have developed several special purpose global optimization methods and heuristics to solve discrete material optimization problems for optimal lay-up design of laminated composite
structures. Our design parameterization follows the discrete material optimization scheme proposed in [26] and [22]. That is, the design variables are binary and indicate locally presence or absence of a material in the structure from a list of pre–defined materials. By applying a reformulation technique we re–state the considered optimization problems as mixed 0–1 problems that exhibit convexity properties.

Several efficient heuristic techniques are designed to obtain discrete feasible solutions,
Table 4.9: Numerical results on the rounding heuristic in Algorithm 3 for the minimum compliance problem \((P_{CN}^c)\) and the feasibility pump heuristic in Algorithm 4 for the minimum mass problem \((P_{RN}^m)\) for the benchmark examples presented in Table 4.3. We report the number of the interior point iterations for solving the continuous relaxation, the number of the (rounding) heuristic iterations, the computation time, the objective of the continuous relaxation, the objective function value of the rounding heuristic and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Objective</th>
<th>Itns</th>
<th>Time</th>
<th>Bounds</th>
<th>Gap(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I.P.</td>
<td>Heuristic</td>
<td>I.P.</td>
<td>Heuristic</td>
<td>Lower</td>
</tr>
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<td>00:00:32</td>
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<td>2.926</td>
</tr>
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<td>M</td>
<td>10</td>
<td>00:00:47</td>
<td>37.349</td>
<td>42.000</td>
</tr>
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<td>P1F</td>
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<td>22</td>
<td>00:00:47</td>
<td>1.798</td>
<td>1.822</td>
</tr>
<tr>
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<td>19</td>
<td>00:00:40</td>
<td>42.962</td>
<td>43.395</td>
</tr>
<tr>
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</tr>
<tr>
<td>P2L</td>
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<td>11</td>
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</tr>
<tr>
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</tr>
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<tr>
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<td>M</td>
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<td>00:06:25</td>
<td>48.999</td>
<td>49.650</td>
</tr>
<tr>
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<td>C</td>
<td>26</td>
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<td>5.836</td>
<td>6.057</td>
</tr>
<tr>
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<td>M</td>
<td>16</td>
<td>00:03:18</td>
<td>92.174</td>
<td>92.266</td>
</tr>
</tbody>
</table>

Table 4.10: Numerical results on the gap improvement method (G.I.M.) in Algorithm 5 for the problem instances presented in Table 4.3. We report the computation time, the number of mixed integer problems solved, the lower bound provided by the nonlinear relaxed master problems \((P_{RN}^c)\) and \((P_{RN}^m)\), the objective function value of the best found design, and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Objective</th>
<th>Time</th>
<th>Itns</th>
<th>Bounds</th>
<th>Gap(%)</th>
</tr>
</thead>
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<td>Upper</td>
<td>Heuristic</td>
<td>G.I.M.</td>
<td></td>
</tr>
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<td>41.855</td>
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</tr>
<tr>
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<td>00:03:45</td>
<td>9</td>
<td>1.798</td>
<td>1.809</td>
</tr>
<tr>
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<td>23.754</td>
<td>23.992</td>
</tr>
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</tr>
<tr>
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<td>15.726</td>
</tr>
<tr>
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<td>6.686</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>C</td>
<td>23:56:23</td>
<td>1</td>
<td>5.836</td>
<td>6.057</td>
</tr>
</tbody>
</table>

based on the optimal solution to the continuous relaxations of the considered problem. These heuristics have advantageous theoretical properties and can be successfully applied to large-scale problems. The heuristics are based on solving sequences of well-posed optimization problems for which efficient methods and robust implementations exist. Our methods and heuristics provide feasible 0–1 designs to the considered problem, without the use of any
Table 4.11: Numerical results from applying the outer approximation (O.A.) framework in Algorithm 6 for the problem instances presented in Table 4.3. We report the computation time, the number of solved relaxations, the number of outer approximation cuts introduced, the lower bound provided by the nonlinear relaxed master problems (PcR) and (PmR), the objective function value of the best found design, and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Problem</th>
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<th>Time [h:m:s]</th>
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<th>Bounds</th>
<th>Gap(%)</th>
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<tr>
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<td>Itns</td>
<td>Cuts</td>
<td>Lower</td>
<td>Upper</td>
</tr>
<tr>
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<td>3</td>
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</tr>
<tr>
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<td>00:01:02</td>
<td>8</td>
<td>21</td>
<td>1.798</td>
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<td>4</td>
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</tr>
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<td>5</td>
<td>33.100</td>
</tr>
<tr>
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</tr>
<tr>
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<td>1</td>
<td>4</td>
<td>49.650</td>
</tr>
<tr>
<td>P3F</td>
<td>C</td>
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<td>216</td>
<td>355</td>
<td>5.836</td>
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</table>

Table 4.12: Numerical results from applying the local branching (L.B.) framework in Algorithm 7 for the problem instances presented in Table 4.3. We report the computation time, the number of left branches solved, the outer approximation iterations and cuts for the right branch, the lower bound provided by the nonlinear relaxed master problems (PcR) and (PmR), the objective function value of the best found design, and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
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<tbody>
<tr>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td>Heur.</td>
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<tr>
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<td></td>
<td></td>
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<td>00:01:04</td>
<td>3</td>
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<td>4</td>
<td>2.695</td>
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</tr>
<tr>
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<td>00:00:10</td>
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<td>1</td>
<td>3</td>
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</tr>
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<td>C</td>
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<td>00:04:52</td>
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<td>0</td>
<td>0</td>
<td>1.798</td>
<td>1.3</td>
</tr>
<tr>
<td>P2L</td>
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<td>00:01:54</td>
<td>1</td>
<td>1</td>
<td>3</td>
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<td>15.5</td>
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<td>00:01:37</td>
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<td>5</td>
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</tr>
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<td>08:07:26</td>
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<td>4</td>
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<td>15.216</td>
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</tr>
<tr>
<td>P3L</td>
<td>C</td>
<td>00:01:21</td>
<td>00:01:04</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>P3L</td>
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<td>00:00:45</td>
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<td>1</td>
<td>4</td>
<td>49.650</td>
<td>1.3</td>
</tr>
<tr>
<td>P3F</td>
<td>C</td>
<td>23:57:19</td>
<td>23:57:19</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>5.836</td>
<td>3.6</td>
</tr>
</tbody>
</table>

intricate material interpolation schemes or continuation approaches. The numerical results indicate that for many of the problem instances, our heuristics provide us with solutions that are provably close to optimal.

Our aim is to solve the considered problems to global optimality. We have therefore proposed a combination of the convergent outer approximation and local branching algorithms that provides globally optimal solutions. The numerical experience of the methods and heuristics are reported on a set of discrete material optimization problems. The design examples have not before been solved in the literature.
One important future aspect is to extend the developed models, methods, and heuristics to perform global optimization for lay-up design of laminated composite structures with manufacturing and/or local stress and/or displacement constraints. The inclusion of such constraints poses several theoretical and computational challenges that have to be overcome before they can be included in practical problems with many design variables.

Acknowledgements

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Bibliography


Chapter 5

Article 2

Optimal Design of Laminated Composite Structures Under Manufacturing Constraints by Mixed 0–1 Nonlinear Optimization Techniques

Konstantinos Marmaras∗ Mathias Stolpe†

Abstract

Special purpose global optimization methods and heuristic techniques for discrete multi–material optimization of laminated composites have been previously developed and implemented by the authors in Chapter 4. In this manuscript we extend the methodology proposed in Chapter 4 in order to perform simultaneous multi–material and thickness optimization of laminated composites. The objective is either a weighted sum of the individual static compliances subject to a mass constraint or the total mass of the structure which is minimized with constraints on the individual static compliances. We extend the problem formulations with additional constraints to model manufacturing limitations for laminated composite structures as linear or mildly nonlinear constraints. The theoretical implications introduced by the addition of the manufacturing constraints are examined on a set of well–defined benchmark examples originating from the literature. The obtained results showcase the excellent convergence properties of the global optimization methods when manufacturing considerations are included in the design problem. Our methods and heuristics prove to be competitive with existing methods for discrete multi–material and thickness optimization of laminated composite structures.

Mathematical Subject Classification (2000): 90C59, 90C90, 74P15, 90C26, 74E30

Keywords: Structural Optimization, Laminated Composites, Discrete Material and Thickness Optimization, Heuristic Techniques, Global Optimization

5.1 Introduction

The use of composite materials in engineering applications has shown a rapid increase due to their excellent mechanical properties such as very high strength to weight ratio. Furthermore, composite structures can be stiffened tailored to meet local design requirements, offering more design freedom than metallic structures. This tailoring can be achieved by varying the number of plies in the laminate, the ply material, and the thickness of each individual lamina.

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5.1. INTRODUCTION

However, this highly attractive feature is accompanied by the complexity of the resulting design problem, especially when manufacturing aspects are being considered, making design optimization an appropriate tool for the design of laminated composite structures.

In this manuscript we are considering simultaneous material and ply thickness optimization of laminated composite structures subject to manufacturing constraints. The modelling of the considered problems is based on the parameterization proposed in [30]. The methodology is an immediate extension of the original Discrete Material Optimization (or DMO) methodology introduced in [26] and [22], which constitutes a generalization of the modelling techniques used in structural topology optimization, see e.g. [6], [55], [13] and [54]. The mechanical properties of each layer in a composite laminate are computed as a weighted sum of the properties of a finite number of candidate materials. The objective is to drive the influence of all but one of these materials to zero at each point of the structure by penalizing mixtures of materials. This can be achieved either through a constitutive interpolation scheme, see e.g. [11], [13], [6], [52] and [59], by employing an explicit quadratic concave constraint function, see e.g. [31], or through an exact penalty function in the objective, see e.g. [33] and [16].

Evolutionary techniques have been the most popular method for optimizing laminated composite structures. They are global optimization methods that have the ability to work directly with integer variables. In addition, evolutionary techniques can be applied in cases where sensitivities of structural responses are difficult to calculate. This feature is a theoretical and computational significant advantage in composite laminate design where derivative calculations are often costly or impossible to obtain. The use of evolutionary methods was first adopted in [18], [41], [51] and [4] for stacking sequence design of laminated composite structures. They have been successfully applied to design problems with structural criteria such as strength, stiffness, buckling loads and fundamental frequencies, see e.g. [51] and [49]. An exhaustive review regarding the application of evolutionary methods on optimization of composite structures can be found in [14], [13]. However, the use of evolutionary techniques has been limited to small scale problem instances because of the exhaustive computing cost, especially when the individual layer thickness and material are optimized simultaneously.

Literature studies covering optimal design of composite structures under manufacturing constraints are scarce and involve mainly the use of evolutionary methods and heuristic techniques. The issue of achieving a blended design was first addressed in [34]. A genetic algorithm was applied for a minimum weight design problem. The composite structure was divided into a number of non–overlapping patches and the fiber orientation of all patches was adjusted according to the orientations of the patch with the maximum number of layers. A similar heuristic approach was proposed in [3] to ensure fiber continuity through the structure. A number of contiguous inner or outer plies were removed from a guide laminate and the laminate sequence of all adjacent panels was determined accordingly. A more sophisticated method to control the variations of the laminate sequence in adjacent panels was employed in [64] and [35] in the form of continuity constraints known as blending rules. A genetic algorithm with a recessive repair strategy was implemented in [62], in order to deal with contiguity constraints. A similar repair approach was used to enforce the required number of plies of given orientations. Ply drop–off constraints were considered in [63] and [20] to optimize a wing structure using the method of feasible directions.

In this present work we extend the methods and heuristics proposed in Chapter 4 to perform simultaneous material and ply thickness optimization of laminated composite structures subject to manufacturing constraints. We follow a relaxation methodology for solving the considered optimal design problems. As opposed to the original Discrete Material Optimization
methodology we obtain discrete feasible solutions to the stated mixed 0–1 convex problems by the application of heuristic techniques without the use of material interpolation schemes. Our heuristics are theoretically well-posed and either provide a discrete feasible solution or correctly determine that the original problem is infeasible. We are aiming at solving the optimization problems to global optimality by extending the outer approximation and the local branching frameworks implemented in Chapter 4 that provide with guarantee globally optimal solutions (if given enough time).

This manuscript is organized as follows. Section 5.2 describes the statement and motivation of the chosen design parameterization. In Section 5.3 we list the manufacturing constraints which can be modelled with the chosen parameterization. The statement and motivation of the relevant optimal design problems is described in Section 5.4. Our solution approach for solving this class of problems is presented in Section 5.5. The numerical experiments with our methods and heuristics are demonstrated in Section 5.7.

5.2 Design Parameterization

We consider the simultaneous topology and material selection problem in a fixed reference design domain \( \Omega \in \mathbb{R}^3 \). This particular problem has been previously modeled in [55] and [26] using a three-phase topology optimization method. This approach was later extended in [26] and [22] to handle any number of phases. In this present work our models follow the parameterization scheme proposed in [30]. This method constitutes a direct extension of the original Discrete Material Optimization (or DMO) parameterization, see e.g. [26] and [22] and allows the simultaneous determination of the appropriate laminate thickness and the material choice in the structure. The layered design domain is partitioned into finite elements, see e.g. [5], with each element representing an identical number of layers, with uniform thicknesses. The structural behavior of the laminate is described using the first-order shear deformation theory (or FSDT), see e.g. [23]. Equilibrium is formulated using the principle of stationary total potential energy (or TPE), see e.g. [5].

The individual finite elements can be further arranged into a number of design subdomains with a constant number of layers, in order to define regions (also called patches) within the structure where the lamination sequence is uniform, such that \( \Omega = \bigcup_{j,k} \Omega_{jk} \), where the index \( j \) denotes the design domain number and the index \( k \) denotes the layer number. The idea of collecting design variables in patches is inspired from the manufacturing process of laminated composites where fiber mats covering larger areas are often used. The material selection variables \( x_{ijk} \in \{0,1\} \) are binary and represent locally presence or absence of a material \( i \) in layer \( k \) of design domain \( j \) from a list of pre-defined candidate materials that are represented by their constitutive matrices. The total number of binary material selection variables is \( n = J \cdot K \cdot I \), where \( I \) is the number of candidate materials, \( J \) is the number of design subdomains associated with the material selection variables and \( K \) is the total number of layers. In our numerical experiments the set of candidate materials consists of orthotropic materials with different fiber orientations, but other materials such as isotropic, quasi-isotropic and anisotropic materials, can also be included. The choice among the candidate materials supposes the selection of a single material in each design domain \( \Omega_{jk} \). This condition is enforced by employing the following linear inequality constraints, also called generalized upper bound constraints

\[
\sum_{i=1}^{I} x_{ijk} \leq 1, \quad \forall (j,k).
\]
Together with the integrality conditions \( x_{ijk} \in \{0,1\} \) these constraints ensure that at most one material can be chosen in each design domain \( j \).

5.3 Manufacturing constraints

The use of optimal design of composites is primarily focused on the structural design of parts or sub–structures excluding manufacturing constraints. We can explicitly model manufacturing limitations as linear or mildly nonlinear inequality constraints by following the design parameterization described in Section 5.2. The inclusion of manufacturing constraints along with structural considerations in the early design phase will result in structures with better structural performance, limiting the need of manually post–processing the found designs.

5.3.1 Fiber angles variations control

Manufacturing requirements for laminated composite structures may place additional restrictions on the complexity of the lamination sequence that must be included in the optimization problem. The implementation will follow well established design rules of laminated composite structures that prevent laminate failure initiation such as matrix cracking and delamination.

Adaptive patch formation

The manufacturing technology of laminated composite structures dictates the partitioning of the design domain into regions (or patches) with a uniform lamination sequence. To model these restrictions on the design space, we impose limitations on the in–plane fiber angles variations between adjacent design domains. The implementation of these constraints is based on the perimeter method for variable–topology shape optimization of elastic structures, see e.g. [27].

It is well–established that the infinite dimensional 0–1 minimum compliance problem which is approximated with the discretized problem may lack a solution in general, see e.g. [37] and [19]. Several methods have been proposed to achieve a well–posed design problem, see e.g. [12], [53] and [47]. The perimeter method [27] has been successfully used as a regularization technique for compliance optimization problems by imposing an upper–bound perimeter constraint. In the particular case of controlling the fiber angles variation in the plane of each individual layer of the laminate the perimeter constraints are formulated as

\[
\sum_{j=1}^{J} \sum_{j' \in \mathcal{L}_j} \sum_{i=1}^{I} |x_{ijk} - x_{ij'k}| l_{jj'} \leq P_k, \quad \forall k, \tag{5.2}
\]

where \( P_k > 0 \) is the maximum "perimeter" value and \( \mathcal{L}_j \) is the unique index set with the design subdomains surrounding design domain \( \Omega_{jk} \). The interface length \( l_{jj'} \) between the design domains \( \Omega_{jk} \) and \( \Omega_{j'k} \) is introduced in order to ensure the independence of the obtained designs from the finite element mesh discretization. By varying the maximum perimeter value \( P_k \), we can adjust the boundaries of the different regions (or patches) within the structure with a uniform fiber angle distribution.

It is obvious from (5.2) that the perimeter function is non–monotonous. We can circumvent the inherent problem of non–differentiability by introducing additional continuous variables \( s_{ijjk} \) with the interpretation
\[ |x_{ijk} - x_{ij'k}| = s_{ijj'k}, \forall (i, j, j' \in \mathcal{L}_j, k). \] (5.3)

The resulting (linear) constraints become

\[
\sum_{i,j,j' \in \mathcal{L}_j} s_{ijj'k} l_{jj'} \leq P_k, \quad \forall k
\]

\[-s_{ijj'k} \leq x_{ijk} - x_{ij'k} \leq s_{ijj'k}, \quad \forall (i, j, k) \text{ and } j' \in \mathcal{L}_j, \]

\[s_{ijj'k} \geq 0, \quad \forall (i, j, k) \text{ and } j' \in \mathcal{L}_j
\]

\[x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k). \]

Even though the size of the problem increases both in the total number of design variables and in the number of constraints, the additional constraints are sparse and there exist specialized algorithms and data structures that can efficiently handle the sparse structure of these equations. The total number of the additional continuous variables is denoted \( n_s \).

**Remark 3.** A variant perimeter constraint of quadratic form was proposed in [65] to restrict variations of element densities over the whole design region. This formulation is characterized by its smoothness and positive semi–definite Hessian matrix. We can follow this alternative approach to overcome any issues with non–differentiability of (5.2). In this case the perimeter constraints become

\[
\sum_{j=1}^{J} \sum_{j' \in \mathcal{L}_j} \sum_{i=1}^{I} (x_{ijk} - x_{ij'k})^2 l_{jj'} \leq P_k, \quad \forall k. \quad (5.5)
\]

**Ply blocking**

The lay–up design of laminated composite structures has a strong influence on the formation of interlaminar normal and shear stresses near the laminate edge zones. Depending on the stacking sequence utilized, laminates with the same number of plies at each orientation can have significant differences in the resulting interlaminar stress distributions, see e.g. [42]. It has been demonstrated in [48, 44] that the existence of interlaminar stresses initiates two different failure modes at a free edge: (1) crack initiation through the thickness of the interior plies and (2) free–edge delamination along the midsurface of the laminate. We can reduce the risk of such a failure by employing limitations on the allowable number of identical contiguous uni–directional (UD) fiber material candidates through the thickness of the laminate

\[
\sum_{k=k_1}^{k_1+CL} x_{ijk} \leq CL, \quad \forall (i, j), \quad k_1 = 1, 2, \ldots, K - CL, \quad (5.6)
\]

where \( CL \) is the contiguity limit \( 0 < CL < K \). Note that non–UD candidate materials are not subject to this constraint.

**Balanced condition**

In many cases the engineer should also seek a symmetric and balanced laminate in order to eliminate the occurrence of any couplings between the structural responses of the laminate. The presence of such couplings in general leads to an increase of the structural deflections which results in a reduction of the effective stiffness and the buckling strength of the laminate. Moreover, a symmetric and balanced lay–up limits the occurrence of manufacturing induced
shape distortions such as warping of the composite structure during the cool–down from the
cure temperature, see e.g. [60].

The balanced condition requires that for every angle ply (those at any angle θ other than
0° and 90°) with a positive fiber orientation angle θ, there is a corresponding ply with the
negative fiber orientation angle −θ

\[
\sum_{k=1}^{K} x_{\theta jk} = \sum_{k=1}^{K} x_{(-\theta)j k}, \quad \forall j.
\]  

(5.7)

Adjacency constraints

Abrupt fiber angle variations between adjacent design subdomains, result in excessive trans-
itions in the in–plane stiffness and strength of the laminate structure. The so–called adjacency
constraints prevent 90° fiber angle alternations between adjacent design subdomains by lim-
iting the allowable choices of material candidates to a reduced set of options.

\[
x_{\theta jk} + x_{(\theta+90^\circ)j'k} \leq 1, \quad \forall (j, k) \text{ and } j' \in L_j.
\]  

(5.8)

5.3.2 Thickness variation control

We will impose additional restrictions in the problem formulation that prevent the formation
of interior holes in the laminate structure. Moreover, we will employ limitations on the
thickness variation rate in order to make sure that the ply–drops will take place in a smooth
and consistent manner throughout the entire laminate structure.

Material Deposition

Laminated composite structures are manufactured by laying plies over one another up to
the required thickness, with no interior holes in the ply. The material deposition constraint
used here to support the manufacturing process of composite structures is inspired from the
corresponding constraint used in topology optimization of molded parts, see e.g. [67] and [66].
This constraint does not allow the creation of interior holes in the structure by enforcing the
placement of material to decrease monotonously in each design subdomain Ω_{jk}, see e.g. [56] and [57]

\[
\sum_{i=1}^{I} x_{ij(k+1)} \leq \sum_{i=1}^{I} x_{ijk}, \quad \forall (j, k).
\]  

(5.9)

Furthermore, the bottom layer of the laminate is solid, i.e. does not contain any holes

\[
\sum_{i=1}^{I} x_{ij1} = 1, \quad \forall j.
\]  

(5.10)

Ply–drop constraint

Laminate thickness variation throughout the entire structure results in the termination of
plies at different locations, known as ply–drops. We impose the following limitations on the
thickness variation rate in order to reduce the resulting stress concentrations appearing at
the drop locations, see e.g. [56] and [57].
\[ -S \leq \sum_{i,k} (x_{ijk} - x_{ij'k}) \leq S, \quad \forall j, j' \in L_j, \]  

(5.11)

where \( S \) is the limit, 0 < \( S \) < \( K \) controlling the thickness variation across adjacent plies. A common design guideline for choosing the slope limit value \( S \) dictates any changes in the laminate thickness normal to the primary loading directions to occur at a taper ratio of at least 20:1, while thickness changes normal to secondary loading directions to occur at a taper ratio of at least 10:1.

5.4 Problem statements and assumptions

In the first problem formulation a weighted sum of \( L \) individual static compliances \( f_l^T u_l \) is minimized, where \( f_l \) are the static design independent loads and \( u_l \) is the corresponding displacement vector satisfying the linear elasticity equilibrium equations \( K(x)u_l = f_l \). The problem is modelled as a nonconvex mixed 0–1 nonlinear program by following the approach of simultaneous analysis and design, see e.g. \[16\]

\[
\begin{align*}
\text{minimize} & \quad \sum_{l=1}^{L} w_l f_l^T u_l \\
\text{subject to} & \quad K(x)u_l - f_l = 0, \quad \forall l,
\end{align*}
\]

(5.12)

where \( Q \) denotes the set

\[ Q = \{ \sum_{i=1}^{I} x_{ijk} \leq 1, \quad \forall (j, k), \}
\]

\[ A x \leq b, \]

\[ \sum_{i,j,j' \in L_j} s_{ijj'} k_{jj'} \leq P_k, \quad \forall k, \]

\[ -s_{ijj'} k \leq x_{ijk} - x_{ij'k} \leq s_{ijj'} k, \quad \forall (i, j, k) \text{ and } j' \in L_j, \]

\[ s_{ijj'} k \geq 0, \quad \forall (i, j, k) \text{ and } j' \in L_j, \]

\[ x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k) \}

where \( K(x) \in \mathbb{R}^{n_d \times n_d} \) is the stiffness matrix, \( n_d \) denotes the number of free finite element degrees of freedom, \( m^{\max} > 0 \) is the maximum allowable mass and \( A \in \mathbb{R}^{n_c \times n} \) is the Jacobian of the (linear) manufacturing constraints with \( n_c \) denoting the total number of constraints. The mass of the structure \( m(x) \) is computed by

\[ m(x) = \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{i=1}^{I} x_{ijk} t_k a_j \rho_i, \]

where \( t_k \) is the thickness of layer \( k \), \( a_j \) is the area of element \( j \), \( \rho_i \) is the density of the given material \( i \). The related minimum mass problem formulation is

\[
\begin{align*}
\text{minimize} & \quad m(x) \\
\text{subject to} & \quad K(x)u_l - f_l = 0, \quad \forall l,
\end{align*}
\]

(5.13)

\[ f_l^T u_l \leq \epsilon_l^{\max}, \quad \forall l, \]

\[ (x, s) \in Q, \]
where \( c_{l}^{\text{max}} > 0 \) is the maximum allowable compliance for the \( l \) – \( th \) load condition.

### 5.4.1 Assumptions

In order to establish the advantageous theoretical properties of the chosen parameterization and overcome the theoretical implications introduced by the addition of the manufacturing constraints described in Section 5.3, we extend the assumptions on our models and problem data already stated in Chapter 4.

(A1) The structure is sufficiently constrained, i.e. constrained against rigid body displacements. In case of simultaneous material and thickness optimization, a material decomposition constraint as described in Section 5.3 is always included in the problem formulations (5.12) and (5.13), such that the final design does not contain any interior holes. The stiffness matrix \( K(x) \) is symmetric and positive definite for all \( x \in [0, 1]^n \). Moreover, the stiffness matrix is linear (or possibly affine) in the design variables.

\[
K(x) = \sum_{i,j,k} x_{ijk} K_{ijk} = \sum_{i,j,k} x_{ijk} B^T_j C_{ik} B_j = \sum_j B^T_j (\sum_{i,k} x_{ijk} C_{ik}) B_j, \tag{5.14}
\]

where \( K_{ijk} \) is the stiffness matrix for the \( k \) – \( th \) layer of the \( j \) – \( th \) domain and the \( i \) – \( th \) material, \( C_{ik} = C^T_{ik} \succeq 0 \) is the constitutive matrix for the \( i \) – \( th \) given material for the \( k \) – \( th \) layer in the global coordinate system and \( B_j \) is the strain–displacement matrix for the \( j \)–th domain, see e.g. [5] and [23].

(A2) The external loads \( f_l \in \mathbb{R}^{n_d} \setminus \{0\} \) for \( l = 1, \ldots, L \). Furthermore, we assume that the load vectors are independent of the design variables.

(A3) The mass limit, \( m_{\text{max}} \) satisfies

\[
\sum_{j=1}^J \sum_{k=1}^K t_k a_{j \min} \{ q_i \} < m_{\text{max}} < \sum_{j=1}^J \sum_{k=1}^K t_k a_{j \max} \{ q_i \}.
\]

(A4) The compliance limits \( c_{l}^{\text{max}} > 0 \) are chosen such that there exists a design vector \( x \in Q' \) that satisfies the compliance inequalities with strict inequality, i.e.

\[
f_l^T K^{-1}(x) f_l < c_{l}^{\text{max}} \quad \forall \ l.
\]

where \( Q' \) is the set resulting from relaxing the integer constraints on the design variables in the set \( Q \).

(A5) The weighting factors \( w_l \geq 0 \) for all load cases \( l \).

Assumption (A1) guarantees that there is a unique displacement solution of the equilibrium equations \( u_l = K(x)^{-1} f_l \forall l \). This assumption will later be used to perform the reformulations of the optimal design problems (5.12) and (5.13) and to state them into their nested form, see e.g. [6]. Moreover, assumption (A1) together with assumption (A5) will be utilized to ensure convexity of the objective in case of the minimum compliance problem.

Assumption (A2) is stated to avoid the trivial situation in which the design domain is not subjected to any load. Assumptions (A3) – (A5) will be used to assure that the feasible set of the continuous relaxations of (5.12) and (5.13) is non–empty.
5.5 Solution Approach

Problems (5.12) and (5.13) are classified as nonconvex mixed 0–1 optimization problems. We resort to reformulation techniques to state them as mixed 0–1 convex problems. This way we can make use of the optimization methods and heuristic techniques already developed and implemented in Chapter 4 for solving this class problems.

5.5.1 Nested problem formulations

The minimum compliance problem (5.12) can be reformulated as a linear semi–definite program, see e.g. [8, 9, 10], or as a second order cone program, see e.g. [7], or alternatively as a non–smooth problem that exhibits special convexity properties, see e.g. [2] and [6]. Assumption (A1) is sufficient to guarantee that there is a unique displacement solution of the equilibrium equations $u_l = K(x)^{-1} f_l \forall l$ and so it is possible to eliminate the nodal displacement variables from the problem formulations (5.12) and (5.13) and state them into their nested form, see e.g. [6]. The minimum compliance problem is now stated as a 0–1 program with a nonlinear and convex objective function, see e.g. [61]

$$\begin{align*}
\text{minimize} & \quad c(x) \\
\text{subject to} & \quad m(x) \leq m^{\text{max}}, \\
& \quad (x, s) \in Q,
\end{align*}$$

(5.15)

where $c(x)$ is the weighted sum of the individual static compliances and is computed by

$$c(x) = \sum_{l=1}^{L} w_l f_l^T K(x)^{-1} f_l.$$

The corresponding nested formulation of the minimum mass problem (5.13) is

$$\begin{align*}
\text{minimize} & \quad m(x) \\
\text{subject to} & \quad c_l(x) \leq c_l^{\text{max}}, \quad \forall l \\
& \quad (x, s) \in Q,
\end{align*}$$

(5.16)

where $c_l(x) = f_l^T K(x)^{-1} f_l$ is the individual static compliance.

5.5.2 Convex continuous relaxations

A common approach of attacking problems (5.15) and (5.16) is by relaxing the integer constraints on the design variables. This results in a continuous relaxation of the minimum compliance problem (5.15)

$$\begin{align*}
\text{minimize} & \quad c(x) \\
\text{subject to} & \quad m(x) \leq m^{\text{max}}, \\
& \quad (x, s) \in Q',
\end{align*}$$

(5.17)

The main motivation in following this approach relies on the fact that the feasible set of the relaxed problem (5.17) is larger than the feasible set of the 0–1 program (5.15) and therefore a lower bound of problem (5.15) and subsequently also of (5.12) can be obtained.
5.5. SOLUTION APPROACH

in a relative fast way and with reasonable resources. Due to assumptions \((A1) - (A3)\) it is guaranteed that the feasible set of problem \((5.17)\) is non-empty and also compact (closed and bounded), see e.g. \([1]\). Since the objective function is convex and twice continuous differentiable, there exists at least one optimal solution of \((5.17)\), see e.g. \([1]\). Finding a KKT-point to \((5.17)\) assures local and global optimality, see e.g. \([29]\). If the optimal solution to the relaxed problem \((5.17)\) is binary, then it is also a global optimal solution to the original discrete problems \((5.12)\) and \((5.15)\).

In the case of the minimum mass problem \((5.16)\), the continuous relaxation is

\[
\begin{align*}
\text{minimize} & \quad m(x) \\
\text{subject to} & \quad c_l(x) \leq c_{l_{\text{max}}}, \quad \forall l \\
& \quad (x, s) \in Q'.
\end{align*}
\]

\((5.18)\)

Assumptions \([A1], [A2], \text{and} [A4]\) guarantee that the convex feasible set of problem \((5.18)\) is non-empty. Slater’s (refined) constraint qualification is also satisfied since the manufacturing constraints and the generalized upper bound constraints are linear and the constraints on the individual static compliances hold with strict inequality due to assumption \([A4]\) see e.g. \([17]\). Thus every KKT-point of \((5.18)\) is both a local and a global minimizer.

5.5.3 Obtaining a discrete feasible solution

The optimal solution of the continuous problems \((5.17)\) and \((5.18)\) will provide the basis for obtaining integer feasible designs to the original mixed 0–1 problems \((5.12)\) and \((5.13)\). We will for this reason extend the implementation of the heuristic techniques developed in Chapter 4.

Rounding heuristic for minimum compliance problems

The first heuristic rounds the optimal solution of the continuous relaxation of the relevant minimum compliance problem \((5.17)\) to an integer feasible solution. Given a non-discrete solution \(\bar{x}\) of the continuous relaxation problem \((5.17)\) a feasible design is obtained by solving the 0–1 program

\[
\begin{align*}
\text{minimize} & \quad \|\bar{x} - x\|_1 \\
\text{subject to} & \quad m(x) \leq m_{\text{max}}, \\
& \quad (x, s) \in Q.
\end{align*}
\]

\((5.19)\)

If the feasible set of the original minimum compliance problem \((5.12)\) is non-empty, then the feasible set of \((5.19)\) is also non-empty. If \((5.12)\) is infeasible it is concluded that \((5.19)\) is also infeasible.

A feasibility pump for minimum mass problems

The rounding heuristic as described in Section 5.5.3 cannot handle nonlinear constraints in the implementation and therefore cannot be applied in the case of the minimum mass problem \((5.13)\). We attempt to find feasible solutions to problem \((5.13)\) by applying a special purpose version of the feasibility pump heuristic proposed in \([15]\). At iteration \(p\) of the feasibility pump we attempt to find an integer feasible point \(\hat{x}_p\) by performing an outer approximation of the convex region defined by the nonlinear constraints of the problem.
\[ \begin{align*}
\text{minimize} \quad & \|x - \mathbf{x}^{p-1}\|_1 \\
\text{subject to} \quad & c_l(x^r) + (\nabla c_l(x^r))^T(x - x^r) \leq c^\text{max}_l, \quad \forall r = 0, \ldots, p - 1, \quad \forall l \\
& (x, s) \in Q. 
\end{align*} \] (5.20)

The gradient of the individual static compliance \(c_l(x)\) with respect to the design variables \(x_{ijk}\) can be derived to be, see e.g. [39] and [45]:

\[ \frac{\partial c_l(x)}{\partial x_{ijk}} = -u_l^T \frac{\partial K(x)}{\partial x_{ijk}} u_l. \]

Since the function \(c_l(x)\) is convex the linearization constraints represent supporting hyper-planes. Thus, if the feasible set of the original minimum mass problem (5.13) is non–empty, then the feasible set of (5.20) is also non–empty. If (5.13) is infeasible, it is concluded that (5.20) is also infeasible.

A linear mixed 0–1 problem formulation of (5.20) can be obtained by introducing additional continuous variables \(z_{ijk}\) with the interpretation

\[ |x_{p-1} - x_{ijk}| = z_{ijk} \forall (i, j, k). \] (5.21)

The resulting 0–1 problem is

\[ \begin{align*}
\text{minimize} \quad & \sum_{i,j,k} z_{ijk} \\
\text{subject to} \quad & c_l(x^r) + (\nabla c_l(x^r))^T(x - x^r) \leq c^\text{max}_l, \quad \forall r = 0, \ldots, p - 1, \quad \forall l \\
& (x^r - \mathbf{x}^p)^T(x - x^r) \geq 0, \quad \forall r = 1, \ldots, p - 1 \\
& -z_{ijk} \leq x_{ijk} - x_{p-1}^{ijk} \leq z_{ijk}, \quad \forall (i, j, k) \\
& z_{ijk} \geq 0, \quad \forall (i, j, k), \\
& (x, s) \in Q. 
\end{align*} \] (5.22)

Since the objective function is bounded from below and problem (5.22) is linear, if the feasible set of (5.13) is non–empty, it follows that there exists at least one optimal solution of problem (5.22).

Then we compute the point \(\mathbf{x}^p\), which satisfies the constraints in the continuous relaxation (5.18) of the minimum mass problem (5.16)

\[ \begin{align*}
\text{minimize} \quad & \|x - \mathbf{x}^p\|_2^2 \\
\text{subject to} \quad & c_l(x) \leq c^\text{max}_l, \quad \forall l \\
& (x, s) \in Q'. 
\end{align*} \] (5.23)

Assumptions (A1), (A2) and (A4) guarantee that problem (5.23) has non–empty feasible set. Assumption (A4) states that the nonlinear inequalities constraints hold with strict inequalities. Since problem (5.23) is also convex, Slater’s (refined) constraint qualification is satisfied, see e.g. [17]. Thus, finding a KKT–point to (5.23) assures local and global optimality.
Remark 4. The enhanced version of the feasibility pump [15] used here includes the inequalities
\[(\hat{x}^r - \tilde{x}^r)^T(x - \hat{x}^r) \geq 0, \forall r = 1, \ldots, p - 1.\]

Each of these inequalities represents a supporting hyperplane that separates \(\hat{x}^r\) from the convex region defined by the nonlinear constraints. For a proof on the validity of these inequalities see e.g. Chapter 7.

5.5.4 Global optimization

Our primary aim is to develop modern optimization methods which are capable of solving the considered mixed 0–1 problems to proven global optimality. We will extend the outer approximation [21], [23] and local branching [22] algorithms developed in Chapter 4 that will provide us with globally optimal solutions to the minimum compliance problem (5.12) and the minimum mass problem (5.13).

Outer approximation

The main idea behind outer approximation relies on the representation of convex sets by a collection of supporting planes. It consists of solving a finite sequence of relaxed versions of problem (5.15). Given a number of points \(x^p \in [0, 1]^n\) the relaxation of the outer approximation master problem of (5.15) is given by the linear mixed 0–1 problem

\[
\begin{align*}
\min_{\eta \in \mathbb{R}, x \in \mathbb{R}^n, s \in \mathbb{R}^{ns}} & \quad \eta \\
\text{subject to} & \quad c(x^r) + (\nabla c(x^r))^T(x - x^r) - \eta \leq 0, \quad r = 1, \ldots, P \\
& \quad m(x) \leq m_{\text{max}}, \\
& \quad (x, s) \in Q, \\
& \quad 0 \leq \eta \leq \min_r \left\{ \sum_{l=1}^L w_l f_l^T u_l(x^r) \right\}.
\end{align*}
\]

(5.24)

Since the function \(c(x)\) is convex these constraints represent supporting hyperplanes. If problem (5.24) is infeasible, then the original minimum compliance problem (5.15) is also infeasible. Since the objective function in (5.24) is bounded from below and the problem is linear, if problem (5.15) is feasible, then there exists at least one optimal solution of problem (5.24). Note that an upper bound on variable \(\eta\) has been set in order to ensure that a previously found feasible design \(x^r\) with \(r = 1, \ldots, P\) is not replicated by the algorithm, see e.g. [23].

The projection of (5.12) onto the space of the binary design variables gives rise to the following linear continuous problem

\[
\begin{align*}
\min_{x^p \in \mathbb{R}^n} & \quad c(x^p) \\
\text{subject to} & \quad (x^p, s) \in Q.
\end{align*}
\]

(5.25)

Since the objective function is bounded from below and problem (5.25) is linear, it follows that there exists at least one optimal solution of problem (5.25). In case of the minimum mass problem (5.16) the relaxation of the outer approximation master problem is given by the linear mixed 0–1 program
CHAPTER 5. ARTICLE 2

As opposed to the minimum compliance problem formulation (5.24), problem (5.26) can generate a design that might be infeasible with respect to the original minimum mass problem (5.13). We check the feasibility of the obtained designs with respect to problem (5.13) by solving the (projected) linear continuous problem

\[
\begin{align*}
\text{minimize} \quad & m(x) \\
\text{subject to} \quad & c_l(x^r) + (\nabla c_l(x^r))^T (x - x^r) \leq c_l^{\text{max}}, \quad \forall l, \ r = 1, \ldots, P \\
& (x, s) \in Q, \\
& 0 \leq \eta \leq \min_r \{\sum_{l=1}^L w_l f_l^T u_l(x^r)\}.
\end{align*}
\]

(5.26)

If problem (5.27) is infeasible the generated compliance cuts will be introduced as additional feasibility cuts in problem (5.26). With each new solution obtained a new linearization constraint is added to the following master problems (5.24) and (5.26), each time cutting away regions of the feasible set within which the optimal solution is to be located.

Local branching

An alternative approach is to perform the global optimization by a combination of the convergent outer approximation [21], [23] and local branching [22] algorithms. This procedure initially defines the solution neighborhoods by the introduction of linear inequalities (local branching cuts) in our mixed 0–1 program. Given a feasible solution \( \hat{x} \) of either (5.12) or (5.13) and a neighborhood size parameter \( \kappa \), we define the \( \kappa \)-opt neighborhood around \( \hat{x} \) as the set of feasible solutions of either (5.12) or (5.13), satisfying the additional local branching constraint \( \Delta(x, \hat{x}) \leq \kappa \), where \( \Delta(x, \hat{x}) \) is the Hamming distance function defined as

\[
\Delta(x, \hat{x}) = \sum_{s \in \mathcal{N}^1(\hat{x})} (1 - x_s) + \sum_{s \in \mathcal{N} \setminus \mathcal{N}^1(\hat{x})} x_s
\]

where \( \mathcal{N} = \{1, \ldots, n\} \) is the index set of the 0–1 variables and \( \mathcal{N}^1 = \{s \in \mathcal{N} | \hat{x}_s = 1\} \) is the binary support of a feasible reference point \( \hat{x} \). The set associated with the current branching is in this way partitioned by means of the disjunction

\[
\Delta(x, \hat{x}) \leq \kappa \quad \text{left branch} \quad \lor \quad \Delta(x, \hat{x}) \geq \kappa + 1 \quad \text{right branch}
\]

(5.28)

A subregion of problem (5.15) is defined by the addition of the local branching cuts

\[
\begin{align*}
\text{minimize} \quad & c(x) \\
\text{subject to} \quad & m(x) \leq m^{\text{max}}, \\
& \Delta(x, \hat{x}) \leq \kappa, \\
& (x, s) \in Q.
\end{align*}
\]

(5.29)

We resort to outer approximation to explore the solution neighborhood defined by the left branch. With each new solution of problem (5.29) we introduce a new local branching cut.
5.6. IMPLEMENTATION AND PARAMETERS

and reverse the last local branch constraint into $\Delta(x, \hat{x}) \geq \kappa + 1$. Local branching solves a finite sequence of linear 0–1 problems until no further improvement of the upper bound can be achieved or problem (5.29) becomes infeasible. We explore the remaining of the feasible region of either (5.12) or (5.13) and conclude the enumeration.

5.6 Implementation and parameters

The continuous relaxations of the mixed 0–1 problems (5.17) and (5.18) and the nonlinear program in the feasibility pump heuristic (5.23) are solved by a special purpose primal–dual interior point method [43]. The primal–dual saddle–point systems are solved using an LDL–factorization with partial pivoting. The linear mixed 0–1 integer programs in the heuristics and the relaxation of the outer approximation master problem are solved by the commercial branch–and–cut software for mixed–integer programming CPLEX version 12.5 [19]. The continuous problem (5.27) is solved using the barrier method which is available within CPLEX version 12.5 [19].

The primal–dual interior point method, the heuristic techniques and the global optimization methods presented in Sections 5.5.3 – 5.5.4 are implemented within the numerical environment MATLAB. The optimality tolerance in the interior point method is set to $10^{-7}$ and the feasibility tolerance is set to $10^{-9}$. The local branching neighborhood search parameter is set to $\kappa = 5$. The parameters in CPLEX are set to default values.

5.7 Numerical experiments

The ability of our global optimization methods and heuristics to perform simultaneous multi–material and thickness optimization is examined on a set of discrete material optimization problems of laminated composite plates as presented in Table 5.1. The problems are considered solved (to global optimality) if the relative optimality gap $\leq 1\%$. All examples were run on an Intel® Xeon™ 5150 processor running at 2.66 GHz.

5.7.1 The benchmark problems

The performance of our methods and heuristics will be examined on a set of benchmark examples originating from the literature, see e.g. [31]. The first example is a layered clamped plate as depicted in Figure 5.1. The dimensions of the plate are $1.0m \times 1.0m \times 0.02m$ and the candidate materials are an orthotropic material (glass epoxy) oriented at 4 distinct directions $\{-45^0, 0^0, 45^0, 90^0\}$ and void. The material properties are listed in Table 5.2. We then solve the hinged plate example shown in Figure 5.2. The plate is subject to a center point load and the candidate materials are again glass fiber reinforced epoxy oriented at 4 distinct directions $\{-45^0, 0^0, 45^0, 90^0\}$ and void. In our third and final example we apply our methods and heuristics on the simply supported plate example shown in Figure 5.3, where we consider four independent load cases of equal magnitude and equal importance. We use the same candidate materials with the previous examples.

In the problems presented in Table 5.1 we examine the activation of the perimeter constraints (5.4) with a perimeter value $P_k = 15$ along with the application of the material deposition constraints (5.9) and (5.10) and the ply–drop constraints (5.11) with a slope limit $S = 1$. Moreover, we seek to obtain a symmetric and balanced laminate by activating the
balanced condition \((5.7)\) and enforcing symmetry conditions throughout the plane of each individual layer of the structure. Finally, we apply the ply blocking constraints \((5.6)\) with a contiguity limit \(CL = 2\) and the so–called adjacency constraints \((5.8)\) in the plane of each individual lamina. We report the number of finite elements used for the mesh discretization, the total number of degrees of freedom, the number of design variables, the total number of constraints, the number of load cases considered and the number of laminate layers. The compliance value provided by the continuous relaxation of the minimum compliance problem \((5.15)\) is used as a bound on the maximum allowable compliance for the minimum mass problem \((5.13)\). The maximum allowable mass is \(m_{\text{max}} = 28.65\text{[kg]}\) for all the examined cases.

The exact value of \(P_k\) was chosen after a trial and error process. The effect of varying the perimeter value \(P_k\) is illustrated in Figure 5.10 for a single layer clamped plate example. The activation of the perimeter constraints clearly restricts the fiber angle variations through the plane of each individual layer in a very elegant manner. By tightening the perimeter constraints we adaptively partition the design domain into patches with a uniform fiber angle distribution which clearly resembles the manufacturing technology of laminated composite structures where fiber mats covering larger areas are often used.

Table 5.1: Characteristics of the problems when the perimeter constraints \((5.4)\), the material deposition constraints \((5.9)\) and \((5.10)\), the ply–drop constraints \((5.11)\), the balanced condition \((5.7)\), the ply blocking constraints \((5.6)\) and the adjacency constraints \((5.8)\) are active.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
<th>Elements</th>
<th>DOF</th>
<th>Variables</th>
<th>Constraints</th>
<th>Loads</th>
<th>Layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1G1</td>
<td>Clamped plate Uniform loading</td>
<td>8×8</td>
<td>1445</td>
<td>4608</td>
<td>16013</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>P1G2</td>
<td>Clamped plate Uniform loading</td>
<td>16×16</td>
<td>5445</td>
<td>19456</td>
<td>68269</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>P2G1</td>
<td>Hinged plate Point load</td>
<td>8×8</td>
<td>1445</td>
<td>4608</td>
<td>16013</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>P2G2</td>
<td>Hinged plate Point load</td>
<td>16×16</td>
<td>5445</td>
<td>19456</td>
<td>68269</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>P3G1</td>
<td>Simply supported plate Point load</td>
<td>8×8</td>
<td>5780</td>
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<td>21780</td>
<td>19456</td>
<td>68269</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

5.7.2 Numerical results with the heuristics

In Table 5.3 we present the numerical results for the rounding heuristics as described in Sections 5.5.3 and 5.5.3 for the problem instances presented in Table 5.1. We report the number of the interior point iterations for solving the continuous relaxation, the number of the (rounding) heuristic iterations, the computation time, the objective of the continuous
Table 5.2: Material properties in the principal material coordinate system for the candidate materials used in the numerical experiments.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value [GPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x$</td>
<td>38.0</td>
</tr>
<tr>
<td>$E_y$</td>
<td>9.0</td>
</tr>
<tr>
<td>$E_z$</td>
<td>9.0</td>
</tr>
<tr>
<td>$G_{xy}$</td>
<td>3.6</td>
</tr>
<tr>
<td>$G_{yz}$</td>
<td>3.5</td>
</tr>
<tr>
<td>$G_{xz}$</td>
<td>3.6</td>
</tr>
<tr>
<td>$\nu_{xy}$</td>
<td>0.3</td>
</tr>
<tr>
<td>$\rho$ [kg/m$^3$]</td>
<td>1870.0</td>
</tr>
</tbody>
</table>

Figure 5.1: Design domain and boundary conditions for the layered clamped plate under uniform loading. The dimensions of the plate are $1.0 \text{m} \times 1.0 \text{m} \times 0.02 \text{m}$.

Figure 5.2: Design domain and boundary conditions for the layered hinged plate subjected to a center point load. The dimensions of the plate are $1.0 \text{m} \times 1.0 \text{m} \times 0.02 \text{m}$.

relaxation, the objective function value of the rounding heuristic, and finally the obtained relative optimality gap. The obtained results in Table 5.3 showcase the excellent convergence properties and the ability of the interior point method to react swiftly to changes of scale in the problem. Our heuristics did not manage to solve any of the problem instances to global
optimal. Moreover, the relative optimality gap obtained is quite high for all the problem instances and dictates for the application of a global optimization method.

Table 5.3: Numerical results on the rounding heuristic for the minimum compliance problem \((\text{5.15})\) (Section 5.5.3) and the feasibility pump heuristic for the minimum mass problem \((\text{5.16})\) (Section 5.5.3) for the benchmark examples presented in Table 5.1. We report the number of the interior point iterations for solving the continuous relaxation, the number of iterations performed in the heuristics, the computation time, the objective function value of the continuous relaxation, the objective function value of the rounding heuristic and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Objective</th>
<th>Itns</th>
<th>I.P.</th>
<th>Heuristic</th>
<th>Time</th>
<th>Bounds</th>
<th>Gap(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>I.P.</td>
<td>Heuristic</td>
<td></td>
<td>Lower</td>
<td>Upper</td>
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<td>51.458</td>
</tr>
<tr>
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<td>M</td>
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<td>3</td>
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<td>37.400</td>
</tr>
<tr>
<td>P1G2</td>
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<tr>
<td>P2G1</td>
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<td>25.242</td>
<td>57.394</td>
</tr>
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<td>M</td>
<td>29</td>
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<tr>
<td>P2G2</td>
<td>C</td>
<td>52</td>
<td>1</td>
<td></td>
<td>00:03:15</td>
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<tr>
<td>P2G2</td>
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<td>9</td>
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<tr>
<td>P3G1</td>
<td>C</td>
<td>55</td>
<td>1</td>
<td></td>
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<td>4.018</td>
<td>7.810</td>
</tr>
<tr>
<td>P3G1</td>
<td>M</td>
<td>27</td>
<td>3</td>
<td></td>
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<td>15.491</td>
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<td>P3G2</td>
<td>C</td>
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<td>P3G2</td>
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<td>5</td>
<td></td>
<td>00:02:42</td>
<td>15.491</td>
<td>37.400</td>
</tr>
</tbody>
</table>

5.7.3 Numerical results with the global optimization methods

In Table 5.4 we present the numerical results from the outer approximation framework for the problem instances presented in Table 5.1. We report the computation time, the number of solved relaxations, the number of outer approximation cuts introduced, the lower bound provided by the nonlinear relaxed master problem, the objective function value of the best found design, and the obtained relative optimality gap.
The outer approximation method has showcased excellent convergence properties for this particular class of problems. Several of the problem instances were solved to global optimality with a small number of outer approximation iterations performed. The method managed to solve problem instances of up to 19456 design variables, 68269 constraints and 21780 degrees of freedom to global optimality. It is important to note the significance of including the compliance cuts at the values obtained from solving the continuous relaxation. The numerical influence on the convergence speed of the global optimization methods when high Pareto value cuts are included, i.e. compliance cuts obtained from designs with a low objective value such as the value obtained from the continuous relaxation was elaborated in [40].

In Table 5.5 we present the numerical results from the local branching framework for the different problem instances presented in Table 5.1. We report the computation time, the number of left branches solved, the outer approximation iterations and cuts for the right branch, the lower bound provided by the nonlinear relaxed master problems (5.17) and (5.18), the objective function value of the best found design and the obtained relative optimality gap. Outer approximation clearly outperforms local branching for solving this class of problems. In fact, with outer approximation we managed to solve more problem instances to global optimality.

The material distributions and the thickness variation throughout the entire laminate structure for the examined problems are depicted in Figures 5.4 – 5.9. We get similar distributions of the minimum compliance problem and the minimum mass problem for the related problem instances. The activation of the manufacturing constraints results in designs with a uniform lamination sequence where the placement of the material takes place in an additive manner. The activation of the ply drop constraints makes sure that the ply drops will take place in a smooth and consistent manner throughout the entire laminate structure.

Note, that we obtain similar qualitatively solutions for the different mesh discretizations. This is to be expected since the application of the perimeter constraints constitutes a regularization technique for the particular problem cases, see e.g. [27] and [46]. In Figure 5.11 we demonstrate this important property of the perimeter method on a single layer clamped plate example.

5.8 Concluding remarks

In this work we have extended the global optimization methods and heuristic techniques proposed in Chapter 4 to perform simultaneous material and thickness optimization of laminated composite structures including manufacturing considerations. Our models follow an extension of the original Discrete Material Optimization parameterization, see e.g. [26], [22], proposed in [30]. In the optimal design problems we want to minimize either a weighted sum of the individual static compliances or the total mass of the structure.

The problem formulations are appropriately extended to handle the addition of manufacturing considerations as explicit linear constraints. We impose limitations on the fiber angles variations through an implementation that is based on the perimeter method for variable topology shape optimization of elastic structures. A material deposition constraint has been implemented together with restrictions on the laminate thickness variation rate resembling the manufacturing technology of laminated composites. Our models include additional structural considerations according to well established design rules for composite structures.

By applying a reformulation technique the considered optimization problems are stated
Table 5.4: Numerical results from applying the outer approximation (O.A.) framework (Section 5.5.4) for the problem instances presented in Table 5.1. We report the computation time, the number of solved relaxations, the number of outer approximation cuts introduced, the lower bound provided by the nonlinear relaxed master problems (5.17) and (5.18), the objective function value of the best found design and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Obj.</th>
<th>Time [h:m:s]</th>
<th>O.A.</th>
<th>Bounds</th>
<th>Gap(%)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>104</td>
<td>44.832</td>
</tr>
<tr>
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<td>2</td>
<td>32</td>
<td>37.400</td>
</tr>
<tr>
<td>P1G2</td>
<td>C</td>
<td>23:57:42</td>
<td>14</td>
<td>374</td>
<td>38.781</td>
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<tr>
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<td>2</td>
<td>23</td>
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<td>11</td>
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<td>3</td>
<td>34</td>
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</tbody>
</table>

Table 5.5: Numerical results from applying the local branching (L.B.) framework (Section 5.5.4) for the problem instances presented in Table 5.1. We report the computation time, the number of left branches solved, the outer approximation iterations and cuts for the right branch, the lower bound provided by the nonlinear relaxed master problems (5.17) and (5.18), the objective function value of the best found design and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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<td>00:00:03</td>
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<td>2</td>
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<td>37.400</td>
<td>&lt;1</td>
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<td>2</td>
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<td>30.971</td>
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<tr>
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<td>C</td>
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<td>6</td>
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<td>31.476</td>
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<tr>
<td>P2G2</td>
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<td>00:00:02</td>
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<td>2</td>
<td>11</td>
<td>30.971</td>
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<tr>
<td>P3G1</td>
<td>C</td>
<td>02:24:41</td>
<td>00:00:02</td>
<td>1</td>
<td>38</td>
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<td>6.267</td>
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</tr>
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<td>M</td>
<td>00:05:52</td>
<td>00:00:02</td>
<td>1</td>
<td>3</td>
<td>32</td>
<td>20.453</td>
<td>58.6</td>
</tr>
<tr>
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<td>00:00:04</td>
<td>1</td>
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<td>00:00:02</td>
<td>1</td>
<td>3</td>
<td>32</td>
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<td>58.6</td>
</tr>
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</table>

in their nested form as mixed 0–1 convex problems. A primal–dual interior point method for nonlinear programming has been developed and implemented to solve the continuous relaxation of the mixed integer problems. The numerical experiments exhibit the robust
5.8. CONCLUDING REMARKS

Figure 5.4: Fiber angle variations of the best found designs by applying the outer approximation method (Section 5.5.4) and the local branching method (Section 5.5.4) on the minimum compliance problem (5.12) of the layered clamped plate by application of the material deposition constraints (5.9) and (5.10), the ply drop constraints (5.11), the balanced condition (5.7), the ply blocking constraints (5.6), the adjacency constraints (5.8) and the perimeter constraints (5.4). The finite element mesh has been discretized with 256 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 5.1.

Figure 5.5: Thickness variation of the best found design by applying the outer approximation method (Section 5.5.4) and the local branching method (Section 5.5.4) on the minimum compliance problem (5.12) of the layered clamped plate (P1G2) by application of the material deposition constraints (5.9) and (5.10), the ply drop constraints (5.11), the balanced condition (5.7), the ply blocking constraints (5.6), the adjacency constraints (5.8) and the perimeter constraints (5.4). The finite element mesh has been discretized with 256 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 5.1.

characteristics of the chosen method. Based on the solution from the continuous relaxation a discrete feasible solution is obtained by the application of heuristic techniques. The heuristics
Figure 5.6: Fiber angle variations of the best found designs by applying the outer approximation method (Section 5.5.4) and the local branching method (Section 5.5.4) on the minimum compliance problem (5.12) of the layered hinged plate by application of the material deposition constraints (5.9) and (5.10), the ply drop constraints (5.11), the balanced condition (5.7), the ply blocking constraints (5.6), the adjacency constraints (5.8) and the perimeter constraints (5.4). The finite element mesh has been discretized with 256 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 5.2.

Figure 5.7: Thickness variation of the best found design by applying the outer approximation method (Section 5.5.4) and the local branching method (Section 5.5.4) on the minimum compliance problem (5.12) of the layered hinged plate (P2G2) by application of the material deposition constraints (5.9) and (5.10), the ply drop constraints (5.11), the balanced condition (5.7), the ply blocking constraints (5.6), the adjacency constraints (5.8) and the perimeter constraints (5.4). The finite element mesh has been discretized with 256 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 5.2.

either provide us with a discrete feasible solution or correctly determine that the original mixed 0–1 problem is infeasible. Our primary aim is to solve the considered problems to
5.8. CONCLUDING REMARKS

Figure 5.8: Fiber angle variations of the best found designs by applying the outer approximation method (Section 5.5.4) and the local branching method (Section 5.5.4) on the minimum compliance problem (5.12) of the layered simply supported plate by application of the material deposition constraints (5.9) and (5.10), the ply drop constraints (5.11), the balanced condition (5.7), the ply blocking constraints (5.6), the adjacency constraints (5.8) and the perimeter constraints (5.4). The finite element mesh has been discretized with 256 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 5.3.

Figure 5.9: Thickness variation of the best found design by applying the outer approximation method (Section 5.5.4) and the local branching method (Section 5.5.4) on the minimum compliance problem (5.12) of the layered simply supported plate (P3G2) by application of the material deposition constraints (5.9) and (5.10), the ply drop constraints (5.11), the balanced condition (5.7), the ply blocking constraints (5.6), the adjacency constraints (5.8) and the perimeter constraints (5.4). The finite element mesh has been discretized with 256 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 5.3.
Figure 5.10: Globally optimal designs obtained by applying the outer approximation method (Section 5.5.4) on the minimum compliance problem \((5.12)\) of a single layer clamped plate by application of the perimeter constraints \((5.4)\) with a varying perimeter value. The finite element mesh has been discretized with 256 Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 5.1. The candidate materials are an orthotropic material (glass epoxy) oriented at the 4 distinct directions in the set \(\{-45^0, 0^0, 45^0, 90^0\}\). The material properties are listed in Table 5.2.

![Figure 5.10](image)

(a) \(P_k = 10\)  
(b) \(P_k = 20\)  
(c) \(P_k = 30\)

Figure 5.11: Globally optimal designs obtained by applying the outer approximation method (Section 5.5.4) on the minimum compliance problem \((5.12)\) of a single layer clamped plate by application of the perimeter constraints \((5.4)\) with a perimeter value \(P_k = 20\) and a varying mesh density. The finite element mesh has been discretized with Q9 plate elements. The domain geometry, loading and boundary conditions are depicted in Figure 5.1. The candidate materials are an orthotropic material (glass epoxy) oriented at the 4 distinct directions in the set \(\{-45^0, 0^0, 45^0, 90^0\}\). The material properties are listed in Table 5.2.

![Figure 5.11](image)

(a) 64 Elements  
(b) 256 Elements  
(c) 1024 Elements

global optimality and apply two different methods with this capability.

The numerical experience of the methods and heuristics are reported on a set of discrete material and thickness optimization problems originating from the literature. The obtained results showcase the excellent convergence properties of the global optimization methods when manufacturing constraints are included in the design problem. Problem instances of up to 19456 design variables, 68269 constraints and 21780 degrees of freedom were solved to global optimality. In the numerical experiments it is proven that the application of the perimeter constraints constitutes a regularization technique for this class of problems. The effect of varying the perimeter value is reflected in the obtained designs.

Future research on this topic will concentrate on extending the current parameterization...
to examine the failure mechanisms of laminated composite structures. Although many failure theories exist in the literature, the establishment of failure analysis models which can accurately predict the strength of laminated composites under complex loading conditions constitutes still one of the most critical problems in the analysis of composite structures.

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Bibliography


Chapter 6

Article 3

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Optimal Design of Laminated Composite Structures Including Local Failure Criteria by Mixed 0–1 Nonlinear Optimization Techniques

Konstantinos Marmaras

Abstract

We consider discrete multi–material optimization of laminated composite structures including local failure criteria. The objective is a weighted sum of the individual static compliances subject to a mass constraint. The optimal design problems that arise are stated as nonconvex mixed integer problems. We resort to different reformulation techniques and state the original nonconvex mixed–integer problems as either linear or nonlinear convex mixed 0–1 programs. The chosen parameterization offers significant advantages in modeling local failure criteria. These additional constraints are introduced as a set of linear inequalities, which guarantees the favorable mathematical properties that have been achieved by performing the reformulations. The ability of our models to perform multi–material optimization including local failure criteria is examined on a set of well–defined benchmark examples originating from the literature.

Mathematical Subject Classification (2000): 90C59, 90C90, 74P15, 90C26, 74E30

Keywords: Structural Optimization, Laminated Composites, Discrete Material and Thickness Optimization, Global Optimization

6.1 Introduction

Failure in a composite structure can be caused by the evolution of different types of damage mechanisms, such as matrix transverse cracking, fiber fracture and delamination. The particular damage modes depend on the applied loading, the composite lay–up and the stacking sequence. Although many failure theories exist in the literature, see e.g. [29], [25], [9] and [10], the establishment of accurate failure analysis models that can be used for the strength assessment of laminated composites under complex loading conditions constitutes still one of the most critical problems in the analysis of composite structures. The failure criteria developed for predicting the first ply failure in the laminate can be divided into two main categories: stress–based criteria and fracture mechanics based criteria. The stress–based criteria may be further divided into three categories: non–interactive such as the maximum stress
and strain criteria, partially interactive such as the Tsai–Hill and Hashin–Rotem criteria, and fully interactive such as the Tsai–Wu criterion. An exhaustive literature review regarding the application of failure criteria on optimization of composite structures can be found in [14, 13].

In structural optimization, two different approaches of formulating the relevant optimal design problems have been used extensively. The first approach is called the nested analysis and design (or NAND), see e.g. [6], where only the structural design variables are treated as the optimization variables. The second set of formulations is known as simultaneous analysis and design (or SAND), see e.g. [16], where the optimization problem is formulated and solved simultaneously in design and state variables (the nodal displacement vector), while the equilibrium equations are stated as a constraint to the optimization problem. A comparative evaluation of the two formulations for structural and mechanical system optimization problems was presented in [3]. Several alternative but equivalent formulations for modelling structural optimization problems were presented in [21].

In this manuscript we consider discrete multi–material optimization of laminated composite structures including local failure criteria by following the approach of simultaneous analysis and design [16]. One of the earliest attempts to use explicit methods in structural optimization was presented in [11] [12] for optimum design of linear elastic trusses. In [16] and [17] it was concluded that the SAND formulations are competitive with the conventional (or NAND) formulations. In recent years, the simultaneous approach has been successfully applied in modelling topology optimization problems, see e.g. [7], [28] and [6].

This manuscript is organized as follows. Section 6.2 describes the statement and motivation of the chosen design parameterization. Section 6.3 presents the considered failure criteria, which can be included in the problem formulations as a linear set of constraints. Section 6.4 describes the statement and motivation of the relevant optimal design problems. The numerical experiments with our models are demonstrated in Section 6.6.

**6.2 Design parameterization**

The design parameterization of the optimal design problems closely follows the Discrete Material Optimization (or DMO) scheme, see e.g. [26] and [22]. The reference design domain $\Omega \in \mathbb{R}^3$ is partitioned into a number of design subdomains with each domain representing a constant number of layers, such that $\Omega = \bigcup_{jk} \Omega_{jk}$, where the index $j$ denotes the design domain number and the index $k$ denotes the layer number. Each design subdomain is further partitioned into finite elements for performing the analysis, see e.g. [5], [8], [18] and [30], with each element representing an identical number of layers with uniform thicknesses. The finite elements implementation follows the first order shear deformation theory (or FSDT), see e.g [23].

The design variables $x_{ijk} \in \{0, 1\}$ are in our case binary and represent locally presence or absence of a material $i$ in layer $k$ of design domain $j$, from a given set of candidate materials. The total number of binary material selection variables is $n = J \cdot K \cdot I$, where $I$ is the number of candidate materials, $J$ is the number of design subdomains associated with the material selection variables and $K$ is the total number of layers. We enforce the choice of at most one material in each design subdomain by employing additional linear equality constraints, also called generalized upper bound constraints

$$\sum_{i=1}^{I} x_{ijk} = 1, \quad \forall (j,k). \quad (6.1)$$
6.2.1 Assumptions

The optimal design problems that arise are stated as nonconvex mixed integer programs. We will resort to different reformulation techniques to state them as mixed integer 0–1 linear problems or as nonlinear convex mixed 0–1 programs. In order to perform these reformulations we state a number of assumptions on our analysis models and problem data.

(A1) The topology of the structure does not change and the stiffness matrix \( K(x) \) is symmetric and positive definite for all \( x \in [0,1]^n \). The stiffness \( K(x) \) matrix is linear (or possibly affine) in the design variables and

\[
K(x) = K_0 + \sum_{i,j,k} x_{ijk} K_{ijk} = K_0 + \sum_{i,j,k} x_{ijk} B_j^T C_{ik} B_j
\]

where \( C_{ik} = C_{ik}^T \succeq 0 \) is the constitutive matrix for the \( i-th \) given material for the \( k-th \) layer and \( B_j \) is the strain–displacement matrix for the \( j-th \) domain. \( K_{ijk} \) is the stiffness matrix for the \( k-th \) layer of the \( j-th \) domain and the \( i-th \) material and \( K_0 \) is a given symmetric and positive semidefinite matrix. The matrix \( K_0 \) can be used to model the situation that the structure also contains domains which are not part of the design domain.

(A2) The external loads \( f_l \in \mathbb{R}^{n_d} \backslash \{0\} \) for \( l = 1, \ldots, L \). This assumption is stated to avoid the trivial situation in which the design domain is not subjected to any load. Furthermore, we assume that the load vectors are independent of the design variables.

(A3) The relative importance of each load case is given by a non–negative weighting factor \( w_l \geq 0 \) for all load cases \( l \).

(A4) The state variables are explicitly bounded, i.e. there exists bounds \( u_{l,\min} \) and \( u_{l,\max} \), such that \( u_{l,\min} \leq u_l \leq u_{l,\max} \forall l \).

6.2.2 Reformulation of the equilibrium equations

In this manuscript we will follow the approach of simultaneous analysis and design [16] to state our optimal design problems. By following this approach the optimal design problems are formulated and solved simultaneously in design and state variables, while the equilibrium equations are stated as a constraint to the optimization problem

\[
K(x)u_l = f_l, \quad \forall l
\]

where \( K(x) \in \mathbb{R}^{n_d \times n_d} \) is the stiffness matrix, \( n_d \) denotes the number of free finite element degrees of freedom, \( f_l \) are the static design independent loads and \( u_l \) is the corresponding displacement vector satisfying the linear elasticity equilibrium equations for the applied load case \( l \). The equilibrium equations (6.2) are bilinear in the design and state variables. Based on the reformulation–linearization technique, see e.g. [24], we can reduce the nonconvex formulation (6.2) to a linear formulation. The main idea of this technique is to first disaggregate the nonlinear equilibrium equations using additional continuous state variables \( q_{ijkl} \in \mathbb{R}^{r_j} \) representing the strain state in layer \( k \) of element \( j \), where \( r_j \) depends on the spatial direction of the structure under consideration, the finite element used, and the number of Gauss quadrature points used in the numerical integration, see e.g. [5], [8], [18], [30]. In case of the
first order shear deformation theory of laminated composite structures $r_j = 8 \times H$, where $H$ is the total number of Gauss quadrature points. For ease of notation we throughout assume that the weighting factors in the Gauss quadrature rule are all equal to one.

$$q_{ijkl} = x_{ijk} B_j u_l, \forall l.$$ \hfill (6.3)

The equilibrium equations then become linear in the state variables $q_{ijkl} \in \mathbb{R}^{r_j}$

$$K(x) u_l = \sum_{ijk} B_j^T C_{ijk} q_{ijkl} = f_l, \forall l. \hfill (6.4)$$

The last part of the reformulation follows the ‘Big–M’ approach, see e.g. [15], where the bilinear equality constraints (6.3) can equivalently be written as a number of linear inequality constraints. The reformulation relies on the fact that the design variables are binary and the state variables are explicitly bounded.

$$x_{ijk} c_{ijkl}^{\min} \leq q_{ijkl} \leq x_{ijk} c_{ijkl}^{\max}, \quad \forall (i,j,k), \forall l,$$ \hfill (6.5)

where the vectors $c_{ijkl}^{\min} \in \mathbb{R}^{r_j}$ and $c_{ijkl}^{\max} \in \mathbb{R}^{r_j}$ denote the lower and upper bounds on the components in the vector $B_j u_l$ and they can be computed as, see e.g. [27]

$$c_{ijkl}^{\min} = \min_u \{ B_j u \mid u_{l,\min} \leq u_l \leq u_{l,\max} \} = B_j^+ u_{l,\min} + B_j^- u_{l,\max},$$

and

$$c_{ijkl}^{\max} = \max_u \{ B_j u \mid u_{l,\min} \leq u_l \leq u_{l,\max} \} = B_j^+ u_{l,\max} + B_j^- u_{l,\min}. \hfill (6.6)$$

The notations $(\cdot)^+$ and $(\cdot)^-$ are defined as follows: If $\alpha \in \mathbb{R}$ then $\alpha^+ = \max\{\alpha, 0\}$. Similarly, $\alpha^-$ is defined as $\alpha^- = \min\{\alpha, 0\}$. By performing this reformulation, the optimal design problems are made accessible to general robust and efficient branch–and–cut methods developed for solving this class of problems, which can provide us with guarantee global or nearly global optimal solutions (if given enough time). However, with the suggested reformulation the size of the problems is increased substantially, both in terms of variables and constraints, which constitutes these problems difficult to solve. Moreover, the ‘Big–M’ approach [15] we are following during the reformulations introduces large constants, i.e. the vectors $c_{ijkl}^{\min} \in \mathbb{R}^{r_j}$ and $c_{ijkl}^{\max} \in \mathbb{R}^{r_j}$, which give rise to weak continuous relaxations and result in slow convergence of the method.

### 6.3 Failure criteria

The scope of an extensive macromechanical laminate failure analysis consists of the selection of the laminate failure theory for an accurate prediction of failure initiation, i.e. first ply failure (or FPF) in the laminate, the selection of a failure progression scheme in the laminate following the failure initiation and the selection of a criterion for predicting the ultimate laminate failure (or ULF). Based on available experimental data it was concluded in [29] that all failure criteria provide reasonable and nearly identical prediction of failure initiation in the laminate structure. This is particularly true for the first ply failure envelope which is approximately ellipsoidal for all criteria. After occurrence of the first–ply failure, the stiffness of the ply is substantially reduced by either matrix or fiber failures. The strength of the
laminate is evaluated again to check if the laminate is able to carry any additional load. This ply-to-ply analysis is repeated until the ultimate strength of the laminate is reached. Strictly speaking, failure criteria for multidirectional laminants are valid up to the first-ply failure envelope. After occurrence of the first-ply failure, the plies within the laminate suffer internal damage such as transverse cracks and delaminations. The classical laminated plate theory is no longer valid for broken, discontinuous materials.

In this manuscript the maximum strain and maximum stress failure criteria will be utilized for predicting the first ply failure. Using the current design parameterization these criteria are formulated as a set of linear inequality constraints, which guarantees the favorable mathematical properties that have been achieved by performing the reformulations, as described in Section 6.2.

6.3.1 Maximum strain failure criterion

In this criterion, it is assumed that the structure will fail if one of the strain components in the principal material direction reaches a critical value. It is formulated mathematically as follows

\[ \epsilon_{jkl} \leq \sum \limits_{i} x_{ijk} \epsilon_{\text{max},i}, \quad \forall (j,k), \forall l, \]

\[ -\epsilon_{jkl} \leq \sum \limits_{i} x_{ijk} \epsilon_{\text{max},i}, \quad \forall (j,k), \forall l, \]

where \( \epsilon_{jkl} \in \mathbb{R}^5 = (\epsilon_{11,jkl}, \epsilon_{22,jkl}, \gamma_{12,jkl}, \gamma_{13,jkl}, \gamma_{23,jkl})^T \) represents the strain tensor related to the load condition \( l \) and \( \epsilon_{\text{max},i} \in \mathbb{R}^5 \) is the upper bound vector on the strain tensor related with the candidate material \( i \). The strain limits can be different for each component of the strain tensor and they can be different according to the state of stress (i.e. tension or compression).

According to the first order shear deformation theory, see e.g. [23], the displacements in the global coordinate system are given as

\[
\begin{align*}
    u(x, y, z) &= u_0(x, y) + z\theta_z(x, y), \\
    v(x, y, z) &= v_0(x, y) + z\theta_y(x, y), \\
    w(x, y, z) &= w_0(x, y).
\end{align*}
\]

Differentiating the displacements (6.8) yields the strains in the global Cartesian coordinate system

\[
\epsilon_{xyz} = \begin{bmatrix}
    \epsilon_{xx} \\
    \epsilon_{yy} \\
    \gamma_{xy} \\
    \gamma_{xz} \\
    \gamma_{yz}
\end{bmatrix} = \begin{bmatrix}
    \frac{\partial u}{\partial x} \\
    \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
    \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\
    \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \\
    \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y}
\end{bmatrix}
\]

Introducing the middle-surface strains \( \epsilon_{xx}^0, \epsilon_{yy}^0, \gamma_{xy}^0 \) and the middle-surface curvatures \( \kappa_{xx}, \kappa_{yy}, \kappa_{xy} \) we obtain
6.3. FAILURE CRITERIA

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{xy}^0
\end{bmatrix} + z
\begin{bmatrix}
\kappa_{xx} \\
\kappa_{yy} \\
\kappa_{xy}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u_0}{\partial x} \\
\frac{\partial v_0}{\partial y} + \frac{\partial v_0}{\partial x} \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}
\end{bmatrix}
+ z
\begin{bmatrix}
\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \\
\frac{\partial \theta_x}{\partial y} \\
\frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_v}{\partial x}
\end{bmatrix}
\] (6.10)

\[
\begin{bmatrix}
\gamma_{xx} \\
\gamma_{yz}
\end{bmatrix} =
\begin{bmatrix}
\gamma_{xx}^0 \\
\gamma_{yz}^0
\end{bmatrix} + z
\begin{bmatrix}
\partial w_0 + \theta_x \\
\partial w_0 + \theta_y
\end{bmatrix}
\] (6.11)

Based on the fact that the continuous state variables \(q_{ijkl} \in \mathbb{R}^{7}\) correspond to the middle-surface strains and curvatures, in the \(\eta\)-th Gauss quadrature point it holds

\[
\begin{bmatrix}
\varepsilon_{xx,ijkl}^\eta \\
\varepsilon_{yy,ijkl}^\eta \\
\gamma_{xy,ijkl}^\eta
\end{bmatrix} =
\begin{bmatrix}
\sum_i q_{ijkl,1}^\eta \\
\sum_i q_{ijkl,2}^\eta \\
\sum_i q_{ijkl,3}^\eta
\end{bmatrix} + z
\begin{bmatrix}
\sum_i q_{ijkl,4}^\eta \\
\sum_i q_{ijkl,5}^\eta \\
\sum_i q_{ijkl,6}^\eta
\end{bmatrix}, \forall (j, k), \forall l
\] (6.11)

The strain tensor related to the load condition \(l\) is calculated in the principal material coordinate system as

\[
\varepsilon_{ijkl} = T_{\theta(i)}^{-T} \begin{bmatrix}
\varepsilon_{xyz,ijkl}^1 \\
\varepsilon_{xyz,ijkl}^2 \\
\ldots
\end{bmatrix}^T
\] (6.12)

where \(T_{\theta(i)}\) represents the matrix used to perform the transformation of the strain components from the global Cartesian coordinate system to the principal material coordinate system, see e.g. [20]. By denoting \(c = \cos(\theta(i))\) and \(s = \sin(\theta(i))\) the transformation matrix \(T_{\theta(i)}\) is obtained as follows

\[
T_{\theta(i)} = \begin{bmatrix}
c^2 & s^2 & cs & 0 & 0 \\
s^2 & c^2 & -cs & 0 & 0 \\
-2cs & 2cs & c^2 - s^2 & 0 & 0 \\
0 & 0 & 0 & c & -s \\
0 & 0 & 0 & s & c
\end{bmatrix}
\] (6.13)

The constraints (6.7) on the strain components \(\gamma_{13,ijkl}, \gamma_{23,ijkl}\)^T further tighten the 'Big-M' formulations (6.5) which results in several of these constraints becoming redundant. In the implementation, only the more stringent of the constraints (6.5) and (6.7) on the strain components \(\gamma_{13,ijkl}, \gamma_{23,ijkl}\)^T will be included in the problem formulations.

6.3.2 Maximum stress failure criterion

The maximum stress criterion has a similar form with the maximum strain criterion (6.7). It assumes that the structure will fail as soon as one of the strain components in the principal material direction reaches a critical value. It is stated mathematically as follows
\[
\sigma_{jkl} \leq \sum_i x_{ijk} \sigma_{\text{max},i}, \quad \forall (j, k), \forall l, \quad \text{(6.14)}
\]

\[
-\sigma_{jkl} \leq \sum_i x_{ijk} \sigma_{\text{max},i}, \quad \forall (j, k), \forall l,
\]

where \( \sigma_{jkl} \in \mathbb{R}^5 = (\sigma_{11,jkl}, \sigma_{22,jkl}, \tau_{12,jkl}, \tau_{13,jkl}, \tau_{23,jkl})^T \) represents the stress tensor related to the load condition \( l \) and \( \sigma_{\text{max},i} \in \mathbb{R}^5 \) is the upper bound vector on the stress tensor related with the candidate material \( i \).

The constitutive relations are described in the principal material coordinate system using the first order shear deformation theory, see e.g. \cite{23}

\[
\begin{bmatrix}
\sigma_{11,jkl} \\
\sigma_{22,jkl} \\
\tau_{12,jkl} \\
\tau_{13,jkl} \\
\tau_{23,jkl}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 & 0 & 0 \\
Q_{21} & Q_{22} & 0 & 0 & 0 \\
0 & 0 & Q_{44} & 0 & 0 \\
0 & 0 & 0 & Q_{55} & 0 \\
0 & 0 & 0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11,jkl} \\
\epsilon_{22,jkl} \\
\gamma_{12,jkl} \\
\gamma_{13,jkl} \\
\gamma_{23,jkl}
\end{bmatrix}, \quad \forall (j, k), \forall l \quad \text{(6.15)}
\]

where the material constants may be described in terms of the engineering constants for a given material \( i \)

\[
\frac{E_{1,i}}{1 - \nu_{21,i} \nu_{12,i}}, \quad Q_{12} = \frac{\nu_{12,i} E_{2,i}}{1 - \nu_{21,i} \nu_{12,i}}, \quad Q_{21} = Q_{12}, \quad Q_{22} = \frac{E_{2,i}}{1 - \nu_{21,i} \nu_{12,i}}, \quad \text{(6.16)}
\]

and the strain tensor \( \epsilon_{jkl} \in \mathbb{R}^5 = (\epsilon_{11,jkl}, \epsilon_{22,jkl}, \gamma_{12,jkl}, \gamma_{13,jkl}, \gamma_{23,jkl})^T \) is computed by using equations (6.11) and (6.12).

### 6.4 Problem statements

In this manuscript we will follow the approach of simultaneous analysis and design \cite{16} to formulate the optimal design problems. We will exploit the structure of the considered problems to state them as either linear or nonlinear convex mixed 0–1 programming problems.

#### 6.4.1 Linear formulations

The first multiple load minimum compliance problem is stated as a linear mixed 0–1 program (or MILP), where a a weighted sum of \( L \) individual static compliances is minimized subject to local failure criteria and manufacturing constraints

\[
\begin{align*}
\text{minimize} & \quad \sum_{l=1}^L w_l f_l^T u_l \\
\text{subject to} & \quad m(x) \leq m_c, \\
& \quad (x, q, u) \in Q,
\end{align*}
\]

\text{(6.17)}
where \( Q \) denotes the set
\[
Q = \left\{ \sum_{i=1}^{I} x_{ijk} = 1, \forall (j, k), \sum_{i,j,k} B^T_j C_{ik} q_{ijkl} = f_I, \forall l, \right. \\
x_{ijk}^{\text{min}} \leq q_{ijkl} \leq x_{ijk}^{\text{max}}, \forall (i, j, k), \forall l, \\
(1 - x_{ijk}) q_{ijkl}^{\text{min}} \leq B_j u_l - q_{ijkl} \leq (1 - x_{ijk}) q_{ijkl}^{\text{max}}, \forall (i, j, k, l), \\
x_{ijk} \in \{0, 1\}, \forall (i, j, k), q_{ijkl} \in \mathbb{R}^r, \forall (i, j, k, l). \right\}
\]
(6.18)

The constraints \( F(x, q) \leq 0 \) indicate the non failure condition for the external load \( f_I \). The mass of the structure \( m(x) \) is computed by
\[
m(x) = \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{i=1}^{I} x_{ijk} t_k a_j \rho_i,
\]
where \( t_k \) is the thickness of layer \( k \), \( a_j \) is the area of element \( j \), \( \rho_i \) is the density of the given material \( i \). We assume that the mass limit \( m^{\text{max}} \) satisfies
\[
\sum_{j=1}^{J} \sum_{k=1}^{K} t_k a_j \min_i \{ \rho_i \} < m^{\text{max}} < \sum_{j=1}^{J} \sum_{k=1}^{K} t_k a_j \max_i \{ \rho_i \}.
\]

### 6.4.2 Quadratic formulations

We can further exploit the structure of problem (6.17) by reformulating it as a nonlinear convex mixed 0–1 program. With this reformulation we anticipate an improvement in the convergence properties of the branch–and–cut method by obtaining tighter bounds from the continuous relaxations. The quadratic reformulation of compliance has previously been used for truss topology optimization problems in [1] and [2]. Using the fact that \( q_{ijkl} = 0 \) if \( x_{ijk} = 0 \) and that \( x_{ijk} = x_{ijk}^2 \), the compliance can be written as a convex quadratic function in the state variables \( q_{ijkl} \in \mathbb{R}^r 
\]
\[
f_I^T u_l = u_I^T K(x) u_l = u_I^T \left( \sum_{i,j,k} x_{ijk} B_j^T C_{ik} B_j \right) u_l = u_I^T \left( \sum_{i,j,k} x_{ijk}^2 B_j^T C_{ik} B_j \right) u_l = \sum_{i,j,k} q_{ijkl}^T C_{ik} q_{ijkl}
\]

The minimum compliance problem (6.17) can now equivalently be written as the quadratic mixed 0–1 program (or MIQP)
\[
\begin{align}
\text{minimize} & \quad \sum_{l=1}^{L} w_l \left( \sum_{i,j,k} q_{ijkl}^T C_{ik} q_{ijkl} \right) \\
\text{subject to} & \quad m(x) \leq m^c, \\
& \quad (x, q, u) \in Q.
\end{align}
\]
(6.19)

### 6.5 Implementation and parameters

The linear and nonlinear convex mixed 0–1 integer programs (6.17) and (6.19) are solved by the commercial branch–and–cut software for mixed–integer programming CPLEX version 12.5 [19]. The parameters in CPLEX are set to default values.
The finite elements used in the numerical experiments are 9 node Mindlin type plate elements with 5 degrees of freedom per node and obtained by full Gaussian integration. The strain and stresses are calculated at the reduced integration points. The reduced integration points are superconvergent points, i.e. points where the stresses are an order of magnitude more accurate than in any other point within the element, see e.g. [4]. Moreover, the strains and stresses are calculated at the top and bottom location of each individual ply of the laminate. These are the locations within each individual lamina, where the maximum and minimum strain and stress values occur.

The strain–displacement matrices $B_j$ are obtained with numerical integration using Gauss quadrature, see e.g. [5], [8], [18] and [30]. The matrix $B_j$ is written as

$$B_j = [B^1_j \ B^2_j \ \ldots]^T$$

where $B^\eta_j$ is the strain–displacement matrix in the $\eta$–th Gauss quadrature point, i.e.

$$B^\eta_j = [B^\eta_{m} \ B^\eta_{b} \ B^\eta_{s}]^T$$

The membrane component $B^\eta_{m}$, the bending component $B^\eta_{b}$, and the shear component $B^\eta_{s}$ are given by

$$B^\eta_{m} = \begin{bmatrix} \frac{\partial N}{\partial x} & 0 & 0 & 0 \\ 0 & \frac{\partial N}{\partial y} & 0 & 0 \\ \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} & 0 & 0 \end{bmatrix}, \quad B^\eta_{b} = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial N}{\partial x} \\ 0 & 0 & 0 & \frac{\partial N}{\partial y} \\ 0 & 0 & 0 & \frac{\partial N}{\partial y} \end{bmatrix}, \quad B^\eta_{s} = \begin{bmatrix} 0 & 0 & \frac{\partial N}{\partial x} & N \\ 0 & \frac{\partial N}{\partial y} & 0 & N \end{bmatrix}$$

with $N$ being the shape functions describing the deformations within the finite elements. The constitutive matrix $C_{ik}$ is block–diagonal and given by $C_{ik} = \text{diag}(C^\eta_{ik})$, where $C^\eta_{ik}$ is the constitutive matrix in the $\eta$–th Gauss quadrature point, i.e.

$$C^\eta_{ik} = \begin{bmatrix} A_{ik} & B_{ik} & 0 \\ B_{ik} & D_{ik} & 0 \\ 0 & 0 & S_{ik} \end{bmatrix}$$

$[A_{ik}], [B_{ik}], [D_{ik}]$ and $[S_{ik}]$ are the extensional, coupling, bending and shear stiffness matrices respectively for the $i$–th given material for the $k$–th layer, see e.g. [23].

### 6.6 Numerical experiments

The ability of our models to perform multi–material optimization when manufacturing limitations and failure criteria are considered is examined on a set of discrete material optimization problems of laminated composite plates as presented in Table 6.1. The problems are considered solved (to global optimality) if the relative optimality gap $\leq 1\%$. All examples were run on an Intel® Xeon™ 5150 processor running at 2.66 GHz.

#### 6.6.1 The benchmark problems

In the first example we solve a layered clamped plate under uniform loading. The dimensions of the plate are $1.0m \times 1.0m \times 0.02m$, as depicted in Figure 6.1. We then consider a layered
hinged plate subject to a center point load as shown in Figure 6.2. Finally, we solve a layered simply supported plate subject to a center point load as depicted in Figure 6.3. A preliminary static analysis of the given loading condition is performed, where a complete mixture of the candidate materials prevails in each design subdomain. Based on the obtained results from the analysis we set the bounds on the displacement variables to the values $u_{\text{max}} = u_{\text{min}} = 1.5 \max_d |u_{l,d}|, \forall l$.

Table 6.1: Characteristics of the problems for multi–material optimization. The candidate materials are glass epoxy oriented at the 4 distinct directions in the set $\{-45^0, 0^0, 45^0, 90^0\}$ and a Divinycell foam H130. The material properties are listed in Table 6.2.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
<th>Elems</th>
<th>DOF</th>
<th>Design Variables</th>
<th>State Variables</th>
<th>Layers</th>
<th>$m_{\text{max}}$ $\text{[kg]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>Clamped plate</td>
<td>4×4</td>
<td>405</td>
<td>320</td>
<td>23285</td>
<td>4</td>
<td>28.65</td>
</tr>
<tr>
<td></td>
<td>Uniform loading</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P2</td>
<td>Hinged plate</td>
<td>4×4</td>
<td>405</td>
<td>320</td>
<td>23433</td>
<td>4</td>
<td>28.65</td>
</tr>
<tr>
<td></td>
<td>Center point load</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P3</td>
<td>Simply supported plate</td>
<td>4×4</td>
<td>405</td>
<td>320</td>
<td>23341</td>
<td>4</td>
<td>28.65</td>
</tr>
<tr>
<td></td>
<td>Center point load</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 6.1: Design domain and boundary conditions for the layered clamped plate under uniform loading. The dimensions of the plate are $1.0m \times 1.0m \times 0.02m$.

6.6.2 Numerical results

In Table 6.3 we present the numerical results with the linear (6.17) and quadratic (6.19) formulations respectively, for the problem instances presented in Table 6.1. We report the computation time, the number of branch and cut nodes visited, the lower bound provided by the relaxed master problems, the objective function value of the best found design and the obtained relative optimality gap. As expected the quadratic formulations clearly outperform the linear formulations, due to their ability to provide tighter lower bounds during the mixed integer optimization. With the MIQP framework, several of the problem instances were solved to global optimality. The activation of the failure conditions results in designs with a larger compliance value. According to the failure condition utilized there is a numerical influence on the convergence speed of the branch–and–cut method.
Table 6.2: Material properties in the principal material coordinate system for the candidate materials used in the numerical experiments.

<table>
<thead>
<tr>
<th></th>
<th>Divinycell Foam H130</th>
<th>Glass Epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1[\text{GPa}]$</td>
<td>0.148</td>
<td>38.0</td>
</tr>
<tr>
<td>$E_2[\text{GPa}]$</td>
<td>9.0</td>
<td>9.0</td>
</tr>
<tr>
<td>$E_3[\text{GPa}]$</td>
<td>9.0</td>
<td>9.0</td>
</tr>
<tr>
<td>$G_{12}[\text{GPa}]$</td>
<td>3.6</td>
<td></td>
</tr>
<tr>
<td>$G_{23}[\text{GPa}]$</td>
<td>3.5</td>
<td></td>
</tr>
<tr>
<td>$G_{13}[\text{GPa}]$</td>
<td>3.6</td>
<td></td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.48</td>
<td>0.3</td>
</tr>
<tr>
<td>$\rho[\text{kg/m}^3]$</td>
<td>130.0</td>
<td>1870.0</td>
</tr>
<tr>
<td>$\sigma_{\text{max},11}[\text{GPa}]$</td>
<td>0.003</td>
<td>0.93</td>
</tr>
<tr>
<td>$\sigma_{\text{max},22}[\text{GPa}]$</td>
<td>0.003</td>
<td>0.033</td>
</tr>
<tr>
<td>$\sigma_{\text{max},12}[\text{GPa}]$</td>
<td>0.003</td>
<td>0.07</td>
</tr>
<tr>
<td>$\sigma_{\text{max},13}[\text{GPa}]$</td>
<td>0.002</td>
<td>0.07</td>
</tr>
<tr>
<td>$\sigma_{\text{max},23}[\text{GPa}]$</td>
<td>0.002</td>
<td>0.04</td>
</tr>
<tr>
<td>$\epsilon_{\text{max},11}$</td>
<td>2.027E-2</td>
<td>2.447E-2</td>
</tr>
<tr>
<td>$\epsilon_{\text{max},22}$</td>
<td>2.027E-2</td>
<td>0.367E-2</td>
</tr>
<tr>
<td>$\epsilon_{\text{max},12}$</td>
<td>4.400E-2</td>
<td>1.944E-2</td>
</tr>
<tr>
<td>$\epsilon_{\text{max},13}$</td>
<td>4.400E-2</td>
<td>1.944E-2</td>
</tr>
<tr>
<td>$\epsilon_{\text{max},23}$</td>
<td>4.400E-2</td>
<td>1.200E-2</td>
</tr>
</tbody>
</table>

Figure 6.2: Design domain and boundary conditions for the layered hinged plate subjected to a center point load. The dimensions of the plate are $1.0m \times 1.0m \times 0.02m$.

6.7 Concluding remarks

In this manuscript we consider multi–material optimization of laminated composite structures subject to constraints on local failure criteria. The optimal design problems that arise are stated as nonconvex mixed integer problems. We resort to different reformulation techniques
6.7. CONCLUDING REMARKS

Figure 6.3: Design domain and boundary conditions for the layered simply supported plate subjected to a center point load. The dimensions of the plate are \(1.0m \times 1.0m \times 0.02m\).

Table 6.3: Numerical results from applying the MILP and MIQP frameworks for the problem instances presented in Table 6.1. We report the computation time, the number of branch and cut nodes visited, the lower bound provided by the relaxed master problems, the objective function value of the best found design and the obtained relative optimality gap.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Formulation</th>
<th>Failure Criterion</th>
<th>Time [h:m:s]</th>
<th>Nodes</th>
<th>Bounds</th>
<th>Gap(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Lower</td>
<td>Upper</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P1</td>
<td>MILP</td>
<td>–</td>
<td>47:59:11</td>
<td>2599373</td>
<td>-378.1250</td>
<td>0.3735</td>
</tr>
<tr>
<td>P1</td>
<td>MILP</td>
<td>Max. Strain</td>
<td>47:59:21</td>
<td>1924750</td>
<td>-6.8742</td>
<td>0.5546</td>
</tr>
<tr>
<td>P1</td>
<td>MILP</td>
<td>Max. Stress</td>
<td>47:58:52</td>
<td>1870856</td>
<td>-267.3915</td>
<td>–</td>
</tr>
<tr>
<td>P1</td>
<td>MIQP</td>
<td>–</td>
<td>47:59:21</td>
<td>3017929</td>
<td>0.2865</td>
<td>0.3191</td>
</tr>
<tr>
<td>P1</td>
<td>MIQP</td>
<td>Max. Strain</td>
<td>47:58:54</td>
<td>2041090</td>
<td>0.2769</td>
<td>0.3209</td>
</tr>
<tr>
<td>P1</td>
<td>MIQP</td>
<td>Max. Stress</td>
<td>47:59:25</td>
<td>2093814</td>
<td>0.2672</td>
<td>0.3319</td>
</tr>
<tr>
<td>P2</td>
<td>MILP</td>
<td>–</td>
<td>47:58:47</td>
<td>2319409</td>
<td>-40.0000</td>
<td>0.4826</td>
</tr>
<tr>
<td>P2</td>
<td>MILP</td>
<td>Max. Strain</td>
<td>47:59:11</td>
<td>1619229</td>
<td>-40.0000</td>
<td>–</td>
</tr>
<tr>
<td>P2</td>
<td>MILP</td>
<td>Max. Stress</td>
<td>47:59:14</td>
<td>1344883</td>
<td>-40.0000</td>
<td>–</td>
</tr>
<tr>
<td>P2</td>
<td>MIQP</td>
<td>–</td>
<td>47:58:44</td>
<td>1504213</td>
<td>0.2768</td>
<td>0.3421</td>
</tr>
<tr>
<td>P2</td>
<td>MIQP</td>
<td>Max. Strain</td>
<td>47:59:04</td>
<td>1360434</td>
<td>0.2768</td>
<td>0.3504</td>
</tr>
<tr>
<td>P3</td>
<td>MILP</td>
<td>–</td>
<td>47:58:56</td>
<td>2410509</td>
<td>-60.0000</td>
<td>0.2766</td>
</tr>
<tr>
<td>P3</td>
<td>MILP</td>
<td>Max. Strain</td>
<td>47:59:14</td>
<td>1927263</td>
<td>-60.0000</td>
<td>0.6431</td>
</tr>
<tr>
<td>P3</td>
<td>MILP</td>
<td>Max. Stress</td>
<td>47:59:32</td>
<td>1821679</td>
<td>-60.0000</td>
<td>0.4986</td>
</tr>
<tr>
<td>P3</td>
<td>MIQP</td>
<td>–</td>
<td>21:12:34</td>
<td>2358564</td>
<td>0.2218</td>
<td>0.2218</td>
</tr>
<tr>
<td>P3</td>
<td>MIQP</td>
<td>Max. Strain</td>
<td>27:21:54</td>
<td>2934027</td>
<td>0.2421</td>
<td>0.2421</td>
</tr>
<tr>
<td>P3</td>
<td>MIQP</td>
<td>Max. Stress</td>
<td>32:54:31</td>
<td>2501611</td>
<td>0.2421</td>
<td>0.2421</td>
</tr>
</tbody>
</table>

and state the original nonconvex mixed–integer problems as either linear or nonlinear convex mixed 0–1 programs. By performing the reformulations, the optimal design problems are made accessible to general robust and efficient global optimization methods developed for solving this class of problems. The chosen parameterization offers significant advantages in modeling failure criteria. In this manuscript we consider the maximum strain and maximum stress failure criteria which are introduced into the problem formulation as a set of linear
inequalities. The addition of the local failure criteria in our models preserves the favorable mathematical properties that have been achieved by performing the reformulations. The numerical experience of our models are reported in a set of discrete material optimization problems originating from the literature. We examine the performance of the current parameterization in the case of multi–material optimization under different loading and boundary conditions. In the numerical experiments it is shown that the quadratic formulations clearly outperform the linear formulations.

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Bibliography


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