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Minimum Q Electrically Small Antennas

Oleksiy S. Kim

Abstract—Theoretically, the minimum radiation quality factor $Q$ of an isolated resonance can be achieved in a spherical electrically small antenna by combining $TM_{1m}$ and $TE_{1m}$ spherical modes, provided that the stored energy in the antenna spherical volume is totally suppressed. Using closed-form expressions for the stored energies obtained through the vector spherical wave theory, it is shown that a magnetic-coated metal core reduces the internal stored energy of both $TM_{1m}$ and $TE_{1m}$ modes simultaneously, so that a self-resonant antenna with the $Q$ approaching the fundamental minimum is created. Numerical results for a multiarm spherical helix antenna confirm the theoretical predictions. For example, a 4-arm spherical helix antenna with a magnetic-coated perfectly electrically conducting core ($ka = 0.284$) exhibits the $Q$ of 0.66 times the Chu lower bound, or 1.25 times the minimum $Q$.

Index Terms—Chu lower bound, circular polarization, electric dipole, electrically small antennas, fundamental limitations, magnetic dipole, quality factor.

I. INTRODUCTION

The radiation quality factor $Q$ is one of the key characteristics that measures the performance of an electrically small antenna. It relates the total stored energy $W$—the sum of electric $W_E$ and magnetic $W_M$ stored energies—to the radiated power $P$ and is defined at a resonance as

$$Q = \frac{W}{P} = 2\omega \frac{\max(W_E, W_M)}{P}. \quad (1)$$

The second expression applies to the case when an antenna is not self-resonant, but tuned to the resonance by an ideal lumped component (capacitor or inductor).

The fractional bandwidth of an isolated resonance is inverse proportional to the $Q$, and since the latter is bounded from below, the former appears to be limited from above. This, however, does not preclude an antenna to have a wider fractional bandwidth than dictated by the limit, if several resonances occur close enough to each other [1]. On the other hand, the $Q$ of a passive linear time-invariant antenna can never be less than a fundamental lower bound [2], and therefore there have always been intriguing questions about how low the $Q$ can theoretically be for a given antenna size and how close this $Q$ can be approached in practice.

A. Single-Mode $Q$

Wheeler and Chu were the first to systematically address the problem. In his paper from 1947 [3], Wheeler considered electrically small cylindrical antennas as circuit elements—an electric dipole antenna as a capacitor and a magnetic dipole antenna as an inductor. For these antennas, he derived simple expressions for the “radiation power ratio,” which in fact is the inverse of $Q$. Chu attacked the problem differently. He introduced an imaginary minimum sphere of radius $a$ enclosing an antenna and disregarded the fields inside (and, thus, the internal stored energy) by assuming “the most favorable conditions to exist inside the sphere” [4]. The external fields were represented by the vector spherical wave functions—$TM_{mn}$ and $TE_{mn}$. Then, an equivalent circuit was defined for each spherical wave, and the corresponding quality factors $Q_n$ were obtained, though with some approximations. Since, the dipole modes ($n = 1$) have the lowest $Q$. $Q_1$ represents the lower bound on $Q$ for a single-mode antenna. Later, in 1996, McLean showed how $Q_1$ can be exactly obtained from Chu’s equivalent circuit [5].

As a matter of fact, Collin and Rothschild presented the exact expression for the lower bound already in 1964 [6]. Contrary to Chu, they did not consider the equivalent circuits, but operated directly with fields and energy densities. Thus, the Chu lower bound, as we know it today, was obtained as

$$Q_{Chu} = \frac{1}{(ka)^3} + \frac{1}{ka}. \quad (2)$$

where $k$ is the free-space wave number. Collin and Rothschild [6] also gave exact expressions for the $Q$ of higher order modes ($n > 1$).

A realistic $Q$ accounting for the stored energy inside the minimum sphere is inevitably larger than the $Q_{Chu}$. In 1958, Wheeler derived the $Q$ for vanishingly small spherical antennas filled with a magnetodielectric material with relative permittivity $\varepsilon_r$ and permeability $\mu_r$ as [7]

$$Q = \left(1 + \frac{\varepsilon_r}{2}\right) \frac{1}{(ka)^3} \quad (3a)$$

$$Q = \left(1 + \frac{2}{\mu_r}\right) \frac{1}{(ka)^3} \quad (3b)$$

for electric and magnetic dipole antennas, respectively. Expressions (3) have several important implications:

1) for electric dipole antennas with air core, $Q \rightarrow 1.5Q_{Chu}$, as $ka \rightarrow 0$;
2) for magnetic dipole antennas with air core, $Q \rightarrow 3.0Q_{Chu}$, as $ka \rightarrow 0$;
3) dielectric material with $\varepsilon_r > 1$ increases the $Q$ of electric dipole antennas;
4) magnetic material of infinitely high permeability allows a vanishingly small magnetic dipole antenna to reach the Chu lower bound.

In 2006, using the Chu equivalent circuits extended for the antenna internal region, Thal numerically computed the exact $Q$ for air-core spherical antennas of finite size excited by a surface electric current [8]. However, exact explicit expressions for the $Q$ of such antennas were provided later by Hansen and Collin in 2009 [9]. Numerical and experimental results reported for

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various air-core electric and magnetic spherical dipole antennas are
in full agreement with the theoretical bounds [10]–[15].

The resonances in a finite-size antenna core caused by finite
internal fields may be of little importance in the air-core case
(the first cavity resonance is at \( k_a = k \alpha = 2.744 \) for
TM_{1m} modes), but they start appearing in the electrically small
antenna regime (usually, \( k \alpha < 0.5 \)), as the permittivity and/or
permeability of the core increases. The cavity resonances are
characterized by zero radiated power, and thus infinite \( Q \).
Therefore, Wheeler’s formulas (3) are not valid for finite-size
antennas. Instead, exact expressions, as derived in [16] for
magnetic dipole (TE_{1m} mode) antennas and in [17] for general
TM_{nm} - and TE_{nm}-mode antennas, must be applied. Moreover,
as shown in [16] and [18], the infinite permeability is no longer
optimal for a solid homogeneous core of a finite-size magnetic
dipole antenna. In this case, the optimal value of permeability
is limited by the first cavity resonance to
\[
\mu_r = \frac{1}{\varepsilon_r} \left( \frac{2.816}{(ka)^2} \right)
\]
which corresponds to the lower bound [18]
\[
Q = \left( 1 + 0.366(ka)^2 \right) Q_{Chu}.
\]

In general, a solid homogeneous core cannot be regarded as
an optimal distribution of material inside an antenna. For ex-
ample, magnetic coating on the surface of a metal core can
have much higher permeability than a solid core, before the
first cavity resonance occurs. Consequently, the \( Q \) of a magnetic
dipole antenna with such a core can drop below the bound (5)
and even reach the Chu lower bound in the limit of infinitely thin
coating and infinitely high permeability [19], [20], note, for ar-
bitrary \( k \alpha \). Remarkably, the same limit applies to electric dipole
antennas as well [21].

In fact, the metal core is not essential for an antenna to ap-
proach the Chu lower bound. A thin shell of high-permeable
magnetic material performs equally well [22]–[24].

### B. Dual-Mode \( Q \)

Although the Chu lower bound (2) has proven reachable by
both electric and magnetic dipole antennas, it is not the lowest
\( Q \) an antenna of a given size \( k \alpha \) can achieve. Already Chu noted
that a combination of TM_{1m} and TE_{1m} modes potentially yields
the \( Q \) that is approximately one-half of the \( Q_{Chu} \) (again, ne-
glecting the internal stored energy) [4]. In 1996, McLean pre-
sented the exact lower bound for a dual-mode TM-TE antenna as
\[
Q_{\text{min}} = \frac{1}{2} \left( \frac{1}{(ka)^2} + \frac{2}{ka} \right).
\]

These results can also be obtained from the general expres-
sions by Fante [25], who also noted that this is the lowest \( Q \) that can be
achieved by a passive linear time-invariant antenna.\(^1\)

When electric and magnetic dipole modes radiate equal
power, electric and magnetic stored energy balance each other,
and the antenna becomes self-resonant [28]. However, the bal-
cance does not hold if the internal energy is taken into account,

\(^1\)This is also shown by Collin [26] and Geyi [27].

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\( Q \) for the same radiated power, the TE_{1m} mode stores four
times more energy inside the minimum sphere than the TM_{1m}
mode does (for an air-core antenna and \( k \alpha \to 0 \)). According
to (3), the balance can be restored by a solid magnetodielectric
core with \( \varepsilon_r \mu_r = 4 \) [7]. Of all possible combinations, \( \varepsilon_r = 1 \)
\( 4 \) is the most advantageous, since it provides the lowest
\( Q = 0.75Q_{Chu} \), as \( k \alpha \to 0 \).

Obviously, an even more advantageous core for a dual-mode
antenna is ensuring no stored energy inside the minimum sphere
for both TM_{1m} and TE_{1m} modes simultaneously. A magnetic-
coated metal sphere is the core that can do the job [29].

This paper shows how the \( Q \) of a dual-mode TM-TE spher-
ical antenna can be minimized using a material-coated perfectly
electrically conducting (PEC) core. In Section II, the theory
based on the vector spherical wave functions is presented.
Optimal geometrical and material parameters of the antenna
leading to the lowest possible \( Q \) are discussed in Section III. It
is shown that the minimum \( Q_{\text{min}} \) can be reached in the limit of
infinitely thin coating with infinitely high permeability. An ex-
pression for the excitation current for a dual-mode TM_{10}-TE_{10}
antenna is derived in Section IV, and a numerical example for
a multi-arm spherical helix antenna with an optimally designed
magnetic-coated PEC core is given in Section V. Finally, the
summary and conclusions are provided in Section VI.

In this work, the vector spherical wave functions in notation
by Hansen [30] are used. The time factor \( \exp(-i\omega t) \), where
\( \omega \) denotes the angular frequency and \( t \) the time variable, is
assumed and suppressed throughout the paper.

### II. Theory

The geometry of the problem is depicted in Fig. 1. It consists
of a PEC spherical core of radius \( b \) covered by a uniform layer
of a homogeneous isotropic and lossless material. The outer
radius of the material shell is \( \alpha \), and the relative permittivity and
permeability of the shell material are \( \varepsilon_r \) and \( \mu_r \), respect-
ively. The Cartesian coordinate system \( \{x, y, z\} \) and the corre-
sponding spherical coordinate system \( \{r, \theta, \phi\} \) are defined in the usual
manner so that the origin coincides with the core center. The
antenna is excited by a superposition of two electric current den-
sities \( J_{TM_{nm}} \) and \( J_{TE_{nm}} \) radiating the TM_{nm} and TE_{nm} modes,
respectively. Since we are aimed at the minimum \( Q \), our consid-
eration is restricted to \( n = 1 \) modes, and thus \( m \) can take values
0 and \( \pm 1 \). The energy and power relations are independent on
\( m \), and therefore it sufficient to consider \( m = 0 \) modes. Thus,
the excitation currents are (see the Appendix for details)

\[
\begin{align*}
J_{TM_{10}} &= J_{TM_{10}} \hat{a}_\theta \sin \theta \\
J_{TE_{10}} &= J_{TE_{10}} \hat{a}_\phi \sin \theta
\end{align*}
\]

where \( \hat{a}_\theta \) and \( \hat{a}_\phi \) are the unit vectors, \( J_{TM_{10}} \) and \( J_{TE_{10}} \) are the amplitudes of the current density for TM_{10} and TE_{10} modes,
respectively.

The radiated fields and the fields in the shell are superposition
of the corresponding TM_{10} and TE_{10} fields with modal expansion
coefficients [20, eq. (4)], [21, eq. (4)]

\[
\begin{align*}
C_{TM_{10}} &= J_{TM_{10}} \hat{a} \frac{4\sqrt{\pi}}{\eta_s} \frac{1}{\sqrt{6}} \left\{ \frac{(k \alpha)^2}{h_1^{(1)}(k \alpha)} \right\}' I_{TM_{10}}^{-1} \\
C_{TM_{10}}^+ &= J_{TM_{10}} \hat{a} \frac{4\sqrt{\pi}}{\eta_s} \frac{1}{\sqrt{6}} \left\{ \frac{(k \alpha)^2}{h_1^{(1)}(k \alpha)} \right\}' I_{TM_{10}}^{-1}
\end{align*}
\]
Fig. 1. A spherical antenna with a material-coated PEC core. The excitation is a time-harmonic electric surface current density $J$ distributed over the surface of the antenna.

$$C_{TE} = J_{TE} e^{i a} \frac{\sqrt{\eta a}}{\eta} \frac{4}{\sqrt{\varepsilon_0}} k a h_{1}^{(1)}(ka) I_{1}^{1}$$  \hspace{1cm} (8c)

$$C_{TE}^{+} = J_{TE} e^{i a} \frac{1}{\sqrt{\varepsilon_0}} \frac{4}{\sqrt{\varepsilon_0}} k a z_{1}^{(6)}(ka) I_{1}^{1}$$  \hspace{1cm} (8d)

where

$$I_{TM} = \frac{\eta}{\eta} k a h_{1}^{(1)}(ka) \left\{ (ka) z_{1}^{(6)}(ka) \right\}^{2}$$  \hspace{1cm} (9a)

$$I_{TE} = \frac{\eta}{\eta} k a h_{1}^{(1)}(ka) \left\{ (ka) z_{1}^{(6)}(ka) \right\}^{2}$$  \hspace{1cm} (9b)

In (8) and elsewhere, the superscripts “+” and “−” denote quantities corresponding to radiated fields in free space and fields in the shell, respectively; $\eta$ and $\eta_s$ are the intrinsic admittances of free space and the shell, respectively; and $ka = \sqrt{\varepsilon_0 \mu_0 k}$. The radial functions $z_{1}^{(6)}(ka)$ and $z_{1}^{(6)}(ka)$, being a linear combination of the spherical Bessel $j_{1}$ and Neumann $n_{1}$ functions, ensure that the boundary condition on the surface of the PEC sphere is satisfied for TM$_{10}$ and TE$_{10}$ modes, respectively:

$$z_{1}^{(6)}(ka) = y_{1}(ka)j_{1}(ka)$$  \hspace{1cm} (10a)

$$z_{1}^{(6)}(ka) = y_{1}(ka)j_{1}(ka)$$  \hspace{1cm} (10b)

The first derivative of the function $f(x)$ with respect to its argument $x$ is denoted by the prime symbol, as $f'(x)$.

The radiated power for each mode is expressed in terms of modal expansion coefficients of the radiated field $C^{\pm}$ as [30]

$$P_{TM} = \frac{1}{2} |C_{TM}^{\pm}|^2, \hspace{1cm} P_{TE} = \frac{1}{2} |C_{TE}^{\pm}|^2$$  \hspace{1cm} (11)

By imposing the condition

$$P_{TM} = P_{TE}$$  \hspace{1cm} (12)

which is necessary to achieve the minimum $Q$ (6), and using (8b), (8d), and (11), we can derive the relation between the amplitudes of the TM$_{10}$ and TE$_{10}$ excitation currents as

$$J_{TM} = \pm J_{TE} \frac{\eta}{\eta} \frac{k a z_{1}^{(6)}(ka)}{(ka) z_{1}^{(6)}(ka)} I_{1}^{1}$$  \hspace{1cm} (13)

Formally, the condition (12) allows the right part of (13) to be multiplied by a factor $e^{i \xi}$, where $\xi$ is an arbitrary phase. In other words, the phasing between the TM$_{10}$ and TE$_{10}$ excitation currents is not essential to achieve the balance between the respective radiated powers. However, as far as the minimum $Q$ is the goal, $J_{TM}$ and $J_{TE}$ must be in phase or opposite in phase ($\xi = 0$ or $\pi$), because any other phase shift would require additional reactive energy, and thus, would increase the $Q$ [26], [31].

The balance of the radiated power implies also the balance for the external electric and magnetic stored energies, $W_{E} = W_{H}$. Indeed, denoting $C^{+} = C_{TM}^{+} = C_{TE}^{+}$, we obtain [20], [21]

$$W_{E} - W_{E,TM} + W_{E,TE} = \frac{1}{4\omega} (C^{+})^2 \left( \frac{1}{(ka)^2} + \frac{2}{ka} \right)$$

$$W_{H} - W_{H,TM} + W_{H,TE} = \frac{1}{4\omega} (C^{+})^2 \left( \frac{1}{(ka)^2} + \frac{2}{ka} \right)$$  \hspace{1cm} (14)

where $W_{E,H,TM}$ and $W_{E,H,TE}$ are external electric/magnetic stored energy of the TM$_{10}$ and TE$_{10}$ modes, respectively.

The internal stored energies do not possess the same symmetry [20], [21]:

$$W_{E/H,TM} - W_{E/H,TE} = \frac{1}{4\omega} \left( C_{TM}^{+}, C_{TM}^{+} \right)^2 \left( C_{TE}^{+}, C_{TE}^{+} \right)^2$$  \hspace{1cm} (15)

with

$$W_{E/H,TM} = \frac{1}{2} \left\{ L(k a) - L(k a) R_{1}(k a) \right\}^2$$

$$W_{E/H,TE} = \frac{1}{2} \left\{ L(k a) - L(k a) R_{2}(k a) \right\}^2$$

where

$$L(k a) = R_{1}(k a) - R_{2}(k a)$$

$$R_{1}(k a) = \frac{(ka)j_{1}(ka)}{(ka)j_{1}(ka)}$$

$$R_{2}(k a) = \frac{(ka)j_{1}(ka)}{(ka)j_{1}(ka)}$$

$$U_{1}(k a) = y_{1}(ka)j_{1}(ka)$$

Consequently, to make the antenna self-resonant, another condition,

$$W_{E} = W_{H}$$  \hspace{1cm} (16)
in addition to (12), must be satisfied. Upon substituting (15) and (8), (16) reduces to

\[
\frac{(ka)^2}{(ka)^2} \left( \frac{W_{E_\infty} - W_{H_\infty}}{z_1^{(5)}(ka)} \right)^2 \left( W_{E_\infty}^2 - W_{H_\infty}^2 \right) = \left( \frac{(ka)z_1^{(6)}(ka)}{(ka)z_1^{(6)}(ka)} \right)^2 \left( W_{H_\infty}^2 - W_{E_\infty}^2 \right). \tag{17}
\]

Finally, the Q is found by using (14), (15), and (11) together with (8) in (1), so that

\[
Q = 2 \omega \frac{W_{E_\infty}^2 + W_{H_\infty}^2}{P_{TM_{\infty}} + P_{E_\infty}} = \max(Q_E, Q_H).
\tag{18}
\]

III. MINIMUM Q PARAMETERS

A standard way to minimize the ratio \(Q/Q_{\min}\) would be to take the derivative of it with respect to an antenna parameter, such as \(\mu_r\) or \(b/a\), set the result equal to zero, and solve the equation. This approach has been successfully applied to electric [29] and magnetic [21] dipole antennas, that is, to antennas with clearly dominating electric or magnetic energy. In the present case, however, it is expected that the minimum of Q is located in the vicinity of the energy balance, where the (18) becomes nondifferentiable due to the \(\max\) function. The problem is illustrated by an example in Fig. 2(a), where the \(Q/Q_{\min}\), \(Q_E/Q_{\min}\), and \(Q_H/Q_{\min}\) are plotted as functions of \(\mu_r\) for \(ka = 0.5\), \(b/a = 0.8\), and \(\varepsilon_r = 1\). It is seen that the minimum of \(Q/Q_{\min}\) is found at a salient point, that is, at the resonance [Fig. 2(b)]. In Fig. 2(b), it is also observed that these are the odd (radiating) resonances that yield low Q, whereas the even resonances, corresponding to the cavity resonances in the core, make the Q go to infinity.

Obviously, we cannot be certain that \(Q/Q_{\min}\) is always minimized at the radiating resonance. Therefore, we compare parameters found by a direct minimum search in (18) and by solving (17) in Fig. 3(a), where \(k_a\) found by both methods in the vicinity of the first four odd (radiating) resonances are plotted versus \(b/a\) for \(ka = 0.25\). We can see that the optimal and resonant parameters coincide for the first resonance \(p = 1\); in the entire range of \(b/a\), whereas for higher odd resonances \(p = 2, 3, 4\) a slight deviation occurs for small radii of the PEC sphere \(b/a < 0.2\). Nevertheless, the corresponding ratios \(Q/Q_{\min}\) shown in Fig. 3(b) clearly illustrate that the lowest \(Q/Q_{\min}\) can always be achieved by the resonant parameters, and thus with a self-resonant antenna, for any \(b/a\); for \(b/a \geq 0.13\), with \(p = 1\) parameters, and then, for \(b/a > 0.13\), with parameters for higher radiating resonances \(p > 1\). As the thickness of the material shell decreases and \(b/a\) goes to 1, the Q approaches \(Q_{\min}\) for all radiating resonances with \(p > 1\); among them, the lowest one \((p = 2)\) is preferable, since it is based on the lowest \(k_a\), and thus, the lowest \(\mu_r\).

Note that the optimal parameters do not depend separately on \(\varepsilon_r\) and \(\mu_r\), but only on the product \(\varepsilon_r\mu_r\), and thus sufficiently defined by a value of \(k_a\) for given \(ka\) and \(b/a\). Consequently, if the permittivity of the shell \(\varepsilon_r\) is higher than 1, the optimal \(\mu_r\) should be decreased to keep \(k_a\) unchanged, and the corresponding Q increases.

As shown in Fig. 4(a), the variation of resonant \(k_a\) with the antenna electrical size \(ka\) is rather marginal for \(p > 1\), whereas for \(p = 1\), the \(k_a\) scales almost linearly with \(ka\). The latter is due to the fact that the first resonance is located in the vicinity of \(\varepsilon_r\mu_r \approx 4\) for \(ka < 0.5\), which is also in agreement with (3) when \(b/a \to 0\). The corresponding variation of the ratio \(Q/Q_{\min}\) is shown in Fig. 4(b), where we can see that for \(p = 1\) and 2 (and for other \(p\), as well), the lowest achievable \(Q/Q_{\min}\) depends weakly on the antenna electrical size \(ka\).

Thus, from Fig. 4(a), given the antenna electrical size \(ka\) and radius of the PEC core \(b\), we can always find optimal resonant material parameters of the shell \(\varepsilon_r\mu_r = (k_a\mu_r)^2\), ensuring the lowest \(Q/Q_{\min}\). If the material parameters \(\varepsilon_r\) and \(\mu_r\) are fixed, which is a more realistic situation in practice, the optimal radius of the PEC core \(b\) can also be determined from Fig. 4(a) for a given \(k_a\mu_r = k_a\sqrt{\varepsilon_r\mu_r}\), but only if \(k_a\mu_r > 3.65\) so that the radiating resonance \(p = 2\) can be used. This resonance is always
preferable to higher resonances \((p > 2)\), because it corresponds to the largest \(h/\alpha\), and thus to the lowest \(Q/Q_{\text{min}}\), as illustrated in Fig. 5 for \(ka = 0.25\).

If the parameters are such that \(k_\alpha < 3.65\), then either a solid material core or no core at all can be more advantageous. However, in this case, either the condition of equal radiated power for the TE\(_{10}\) and TM\(_{10}\) modes (12) or the self-resonant condition (16) have to be discarded in order to achieve a low \(Q/Q_{\text{min}}\) (Fig. 5).

**IV. Dual-Mode TM\(_{10}\)–TE\(_{10}\) Spherical Antenna**

The total current density for a combination of TM\(_{10}\) and TE\(_{10}\) modes on the surface of a spherical antenna can be written using (7) and (13) as

\[
\mathbf{J} = J_\ell \sin \theta (\gamma \hat{a}_\theta + \hat{a}_\varphi)
\]

where \(J_\ell\) is an amplitude and \(\gamma\) is as defined in (13). Such an antenna yields perfect circular polarization—left-hand or right-hand, depending on the sign of \(\gamma\)—in all directions, except the poles, where the radiation is zero.\(^2\)

In (19), the terms in the parentheses in fact describe a set of spherical helices (Fig. 6) as

\[
\theta = 21\tan^{-1} \left( e^{\gamma(\varphi + c)} \right)
\]

where \(c\) is an arbitrary real constant. The argument \(\psi\) runs from \(-\infty\) to \(\infty\), where \(\theta\) asymptotically approaches \(0\) and \(\pi\), respectively. This implies that each helix contains an infinite number of turns. In practice, the helices can be truncated by setting the range of \(\theta\) from \(\theta_0\) to \(\pi - \theta_0\), which corresponds to \(\psi \in [\ln(\tan(\beta_0/2))/\gamma + c : -\ln(\tan(\beta_0/2))/\gamma + c]\). Then, the number of turns \(N_1\) in each helix is

\[
N_1 = -\frac{\ln(\tan(\beta_0/2))}{\pi \gamma}.
\]

\(^2\)The maximum directivity \(D = 1.5\) occurs at \(\theta = 90^\circ\). Consequently, for the antenna parameters ensuring \(Q \to Q_{\text{min}}\), the ratio \(D/Q\) approaches the limit \(1.5/Q_{\text{min}}\) found in [4], [27] for omnidirectional antennas radiating dipole modes.
If we put conducting strips or wires along these helices, we obtain a multi-arm spherical helix antenna. This antenna resembles similar folded spherical helix antennas reported in [10], [11], [14], [29], [32], but with a different winding governed by (20), which has neither \( z \)-uniform pitch, as in [10] and [11], nor \( \beta \)-uniform pitch, as in [14], [29], and [32].

V. SPHERICAL HELIX ANTENNA WITH A MATERIAL-COATED PEC CORE

Here, we consider a numerical example, for which we choose the outer radius of the core to be \( r_0 = 39 \text{ mm} \) and the material of the coating to be perfectly magnetic with relative permeability \( \mu_r = 900 \) and permittivity \( \varepsilon_r = 1 \). At 300 MHz, the electrical radius of the core becomes \( \rho = 7.351 \). Given this, from the data in Fig. 4(a), we can determine the resonant \( b/a = 0.782 \) for \( p = 2 \), which corresponds to the radius of the PEC core \( b = 30.5 \text{ mm} \). Finally, from (13), we find \( \gamma = 0.26 \).

To excite the antenna, we wind wire helices around the core according to (20) at the radius \( a_0 = 40 \text{ mm} \), as sketched in Fig. 7; the diameter of the wires is 1 mm. An offset of 1 mm between the wire center and the outer surface of the coating was introduced to ensure numerical stability of the surface integral equation technique [33] employed to solve the problem. Thus, an overall electrical radius of the antenna becomes \( k_{ru} \equiv k(a + 1.5 \text{ mm}) = 0.254 \).

By terminating the helices at \( \theta_0 = 7^\circ \), we limit the number of turns to \( N_t = 3.43 \), as determined from (21). However, the final \( N_t \) has to be fine-tuned to correct for the finite diameter of the wires and the air-gap between the wires and the core.

Table I summarizes the results for two, three and four helices, or helical arms \( (N_a = 2, 3, 4) \). We can see that the number of turns \( N_t \) obtained in each case is very close to the theoretically predicted value. The input resistance \( R_0 \) varies between 41 and 196 \( \Omega \), depending on the number of arms. As the number of arms increases, the wire current approximation of the ideal excitation current distribution (19) improves and the \( Q/Q_{\text{min}} \) approaches the limit. It should be noted that the bounds \( Q_{\text{chu}} \) and \( Q_{\text{min}} \) in Table I are calculated for the overall antenna electrical size \( k_{ru} = 0.254 \).

The frequency variation of the input impedance and the corresponding reflection coefficient (when matched at the resonance) for the four-arm configuration are shown in Fig. 8(a) and (b), respectively. A clear resonance is observed at 300 MHz with the fractional bandwidth of 1.5% at the -10-dB level and 2.7% at the -6-dB level of the reflection coefficient.

The radiation pattern is omnidirectional [Fig. 9(a)]—typical for an electrically small dipole antenna. The maximum directivity of 1.76 dBi occurs in the \( \theta = 90^\circ \) plane with the azimuthal variation within \( \pm 0.06 \) dBi due to the coarse approximation of the ideal excitation (19) by the wire currents. The polarization is circular over the entire far-field sphere except the polar regions, where the radiation is low anyway [Fig. 9(b)]. The axial ratio in the \( \theta = 90^\circ \) plane averaged over \( \phi \) is \(-0.33 \) dB with the variation \( \pm 0.02 \) dB.

VI. DISCUSSION

As the Chu lower bound (2) defines the lowest \( Q \) for a single-mode antenna, the minimum \( Q \) (6) sets the lower bound on ra-
radiation $Q$ for any linear passive time-invariant lossless antenna. To achieve the minimum $Q$, an antenna must satisfy certain conditions, as follows.

1) It must excite at least one electric dipole mode (TM$_{10}$ or TM$_{11}$) and one magnetic dipole mode (TE$_{10}$ or TE$_{11}$).

2) The amplitudes of the excitation currents must be related by the factor $\gamma$, as defined in (13). The latter must be real, because any phase shift other than $0^\circ$ or $180^\circ$ introduces extra stored energy into the system, and thus increases the $Q$ [26], [31]. If an antenna, besides being excited as prescribed above, also has the moments of the electric and magnetic dipoles aligned, its radiation is perfectly circular over the entire far-field sphere with the sense of polarization (left-hand or right-hand) depending on the sign of $\gamma$.

3) There must be no energy stored inside the antenna minimum sphere, which implies that 1) electric and magnetic dipole modes, being considered individually, both reach the Chu lower bound and 2) the antenna becomes self-resonant.

The exact theory presented in this paper shows that condition 3) can be satisfied in a spherical antenna having a PEC spherical core coated with infinitely thin layer of magnetic material of infinitely high permeability and excited by an electric current density impressed on the outer surface of the core. With finite values of the coating thickness and permeability, the $Q$ does not reach the $Q_{\text{min}}$ but still can closely approach it. Using closed-form expressions for the stored energies, radiated power, and the $Q$, the optimal parameters of the coating minimizing the $Q$ and ensuring a self-resonant configuration are found and presented in Fig. 4(a). Given an antenna electrical size $k_a$ and the relative radius of the PEC core $\ell/a$, the optimal material parameters of the coating can be found from the plot, as $\varepsilon \mu = (k_a \ell)^2/(k_a)^2$. Out of a variety of materials satisfying the obtained value $\varepsilon \mu$, the one with the lowest permittivity $\varepsilon_r$ is preferable, since it ensures the lowest stored energy for both electric and magnetic dipole modes [20], [21], and thus the lowest $Q$. Alternatively, an optimum radius of the PEC core $h$ can be determined from the same Fig. 4(a) for given $k_a$ and $k_a h = k n \sqrt{\varepsilon / \mu}$.

The relative importance of the PEC core in reducing the internal stored energy for electric and magnetic dipole modes was discussed in [21] and [24], respectively. In both cases, an infinitely thin shell of infinitely permeable magnetic material reduces the internal stored energy to zero with and without PEC core. However, a central metal core has a number of other advantages. First, as noted in [24], it ensures mechanical robustness of the antenna. Second, if made hollow, it provides a screened volume inside the antenna that can room various devices, such as batteries, transmitters–receivers, integrated circuits, and sensors.

The multi-arm spherical helix antenna with a magnetic-coated PEC core, presented as a numerical example, illustrates how a minimum Q antenna can be implemented. The antenna is of the overall electrical size $k_{\text{na}} = 0.254$, and its radiation quality factor is $Q = 0.66Q_{\text{Chu}} = 1.25Q_{\text{min}}$.
practice, the $Q$ can be further improved by placing thin conducting strips right on the surface of the coating, for example, by direct writing with conductive ink [34] or by direct transfer patterning [35].

**APPENDIX**

**EXCITATION CURRENTS FOR TM$_{10}$ AND TE$_{10}$ SPHERICAL MODES**

An excitation current for the TM$_{10}$ spherical mode must satisfy the following boundary condition on the surface of the spherical antenna (Fig. 1):

$$J_{TM_{10}}(\theta, \phi) = \vec{a}_r \times (H_{TM_{10}} - H_{TM_{10}})$$

(22)

where $\vec{a}_r$ is the radial unit vector and $H_{TM_{10}}$ is the magnetic field of the TM$_{10}$ spherical mode. Using [21, eq. (4)] for $H_{TM_{10}}$, expression (22) can be reduced to

$$J_{TM_{10}}(\theta, \phi) = \frac{i}{4\pi} \sqrt{k_0 h_{TM_{10}}} \left( \alpha_0 - k_0 h_{TM_{10}} \right) \sin \theta$$

(23a)

$$J_{TM_{10}}(\theta, \phi) = \sqrt{k_0 h_{TM_{10}}} \sin \theta$$

(23b)

Due to the spherical geometry of the problem (Fig. 1) and the orthogonality property of the spherical wave functions on any spherical surface concentric to the origin, no other modes except TM$_{10}$ are excited by the current (23b).

Similarly, using [20, eq. (4)], it can be shown that the current (7b) excites the TE$_{10}$ spherical mode.

**REFERENCES**


