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Extending the random-phase approximation for electronic correlation energies: The renormalized adiabatic local density approximation

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The adiabatic connection fluctuation-dissipation theorem with the random phase approximation (RPA) has recently been applied with success to obtain correlation energies of a variety of chemical and solid state systems. The main merit of this approach is the improved description of dispersive forces while chemical bond strengths and absolute correlation energies are systematically underestimated. In this work we extend the RPA by including a parameter-free renormalized version of the adiabatic local-density (ALDA) exchange-correlation kernel. The renormalization consists of a (local) truncation of the ALDA kernel for wave vectors \( q > 2k_F \), which is found to yield excellent results for the homogeneous electron gas. In addition, the kernel significantly improves both the absolute correlation energies and atomization energies of small molecules over RPA and ALDA. The renormalization can be straightforwardly applied to other adiabatic local kernels.

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Increasing computational resources has recently boosted a major interest in calculating electronic correlation energies from first principles using the adiabatic connection fluctuation-dissipation theorem (ACDF).1–4 The computational cost for such methods is much higher than traditional correlation functionals in density functional theory, but has the great advantage that it includes nonlocal effects and does not rely on error cancellation between exchange and correlation. The random phase approximation (RPA) represents the simplest approach to ACDF calculations and has already been applied to a broad range of electronic structure problems.5–11 While the approach to ACDF calculations and has already been applied in this Rapid Communication we derive a nonlocal exchange-correlation kernel which does not contain any fitted parameters. The construction is based on a renormalization of the HEG correlation hole, which removes the divergence of the pair-distribution function and brings total correlation energies much closer to experimental values than both RPA and local approximations for the kernel.

From the adiabatic connection and fluctuation-dissipation theorem, it follows that the correlation energy of an electronic system can be written

\[
E_c[n] = -\int_0^1 d\lambda \int_0^\infty \frac{d\omega}{2\pi} \text{Tr}[\pi v \chi^{\lambda}(i\omega) - v \chi^{KS}(i\omega)].
\] (1)

Here \( \chi^{KS} \) is the exact Kohn-Sham response function and \( \chi^{\lambda} \) is the interacting response function of a system where the electron-electron interaction \( v \) has been replaced by \( \lambda v \). Using TDDFT, one may express the interacting response function in terms of the Kohn-Sham response function as

\[
\chi^{\lambda} = \chi^{KS} + \lambda v f^{KS}_{\text{Hxc}} \chi^{\lambda},
\] (2)

where \( f^{\text{Hxc}}_{\lambda} = \lambda v + f^{\text{Hxc}} \) is the Hartree-exchange-correlation kernel at coupling strength \( \lambda \). The simplest approximation for \( f^{\text{Hxc}}_{\lambda} \) is the random phase approximation where the exchange-correlation part is neglected. A natural next step is to include an approximation for the exchange-correlation kernel. In particular, one could try the adiabatic local density approximation (ALDA) kernel

\[
f^{\text{ALDA}}_{\text{Hxc}}[n](r, r') = \delta(r - r') f^{\text{ALDA}}_{\text{Hxc}}[n],
\] (3)
where $f_{xc}^{ALDA}[n] = \frac{d}{dr} \left( n \epsilon_{xc}^{ALDA} \right)_{\text{local}}(r)$. In the following we will only consider the exchange part of the adiabatic kernel, since it has the simplifying property that $f_{xc}^{\lambda} = \lambda f_{xc}$. Additionally, we expect the effect of including a kernel in Eq. (2) will be dominated by the exchange contributions. As it turns out, the kernel Eq. (3) does not improve on total correlation energies or molecular atomization energies and is plagued by convergence problems related to the divergence of the pair-distribution function.

For the homogeneous electron gas the correlation energy is essentially given by the integral of the correlation hole. It can be regarded as an effective local-field potential and gives a reasonable description at small $q$ but overestimates the value for $q > 2k_F$. The ALDA correlation hole becomes zero when $f_{xc}^{\lambda} = 0$, which happens exactly at $q = 2k_F$. The divergence of the pair-distribution function originates from the slow decay tail at large $q$ where $f_{xc}^{\lambda}$ is complete dominated by the $q$-independent $f_{xc}^{ALDA}$. The full ALDA correlation hole is very similar to the ALDA correlation hole displayed here.

The correlation energy is essentially given by the integral of the coupling constant averaged correlation hole. Despite the divergent pair-distribution function, the ALDA correlation energy is well defined but converges very slowly due to the slow decay of $\tilde{g}(q)$ at large $q$. From the shape of the correlation hole it is expected that RPA underestimates the correlation energy, while ALDA overestimates it. Since the bad behavior of ALDA primarily comes from large values of $q$, it is now tempting to introduce a renormalized ALDA correlation energy obtained by cutting the correlation hole displayed here.

For the homogeneous electron gas, the cutoff is equivalent to using the Hartree-exchange-correlation kernel

$$f_{Hxc}^{\text{ALDA}}[n](q) = \theta(2k_F - q) f_{Hxc}^{\text{ALDA}}[n].$$

Since $k_F$ is related to the density, it is now straightforward to generalize this to inhomogeneous systems. We simply take $r \rightarrow |r - \mathbf{r}'|$ and $k_F \rightarrow [3\pi^2 n(\mathbf{r}, \mathbf{r}')]^{1/3}$ with $n(\mathbf{r}, \mathbf{r}') = |n(\mathbf{r}) + n(\mathbf{r}')|/2$. Thus, we obtain a nonlocal functional with no free parameters by performing a simple local renormalization of the correlation hole. It can be regarded as an ALDA functional where the delta function in Eq. (3) has acquired a density-dependent broadening. At large separation it reduces to the pure Coulomb interaction and it is expected to retain the accurate description of van der Waals interactions characteristic of RPA. For example, in a jellium with $r_s = 2.0$ two points separated by 5 Å give a renormalized interaction $v'(r_s = 2)[|\mathbf{r} - \mathbf{r}'|] = 0.97v(\mathbf{r} - \mathbf{r}')$ and the magnitude of the Coulomb part of the kernel is a factor of 30 larger than $f_{xc}^{\text{ALDA}}$.

The renormalized ALDA functional has been implemented in the DFT code GPAW, which uses the projector augmented wave (PAW) method. The response function is calculated in a plane wave basis set as described in Ref. 29. The coupling constant integration is evaluated using eight Gauss-Legendre points and the frequency integration is performed with 16 Gauss-Legendre points with the highest point situated at 800 eV. Since the kernel Eq. (5) is only invariant under simultaneous lattice translation in $\mathbf{r}$ and $\mathbf{r}'$, its plane wave representation takes the form

$$f_{xc}^{ALDA}(q) = \frac{1}{V} \int_V \frac{d\mathbf{r}}{V} \int_V \frac{d\mathbf{r}' e^{iG \cdot \mathbf{r}} \tilde{f}(q; \mathbf{r}, \mathbf{r}') e^{iG \cdot \mathbf{r}'},}$$

FIG. 1. (Color online) Fourier transform of the coupling constant averaged correlation hole for the homogeneous electrons gas. Left: $r_s = 1$. Right: $r_s = 10$.

FIG. 2. (Color online) Correlation energy per electron of the homogeneous electron gas evaluated with different approximations for $f_{xc}$.
Here we have introduced the lattice point difference \( \mathbf{r}_{ij} = \mathbf{R}_i - \mathbf{R}_j \) and the number of sampled unit cells \( N \) (\( k \) points). \( \tilde{f}(\mathbf{q}; \mathbf{r}, \mathbf{r}') \) is periodic in both \( \mathbf{r} \) and \( \mathbf{r}' \) and \( f^\text{rALDA}G(\mathbf{q}) \) should be converged by sampling a sufficient number of nearest neighbor unit cells. While the response function is calculated within the full PAW framework, it is not trivial to obtain the PAW corrections for a nonlocal functional and we use the bare \( \text{ALDA}_X \) kernel to calculate contributions to the \( \text{rALDA} \) kernel from the augmentation spheres.\(^{29}\)

As a first test of the functional for \textit{ab initio} applications, we have calculated the correlation energy of the valence electrons of bulk Na. We do not have a number for the exact value of the correlation energy, but due to the delocalized nature of the valence electrons it is expected that the result should be close to the correlation energy of the homogeneous electron gas at the average valence density of Na. This is supported by the close agreement between the RPA correlation energy of Na and the homogeneous electron gas.\(^{30}\) We found the \( \text{rALDA} \) calculations to be converged when two nearest unit cells were included. The result is shown in Fig. 3 as a function of plane wave cutoff energy along with the RPA and \( \text{ALDA}_X \) results. As expected, RPA underestimates the correlation energy while \( \text{ALDA}_X \) overestimates it. Again, one should note the slow convergence of the \( \text{ALDA}_X \) calculation originating from the \( q \)-independent kernel. For plane wave implementations, an additional problem is posed by the divergences of \( f^\text{rALDA}_X \) at small densities. A particularly nice feature of the kernel (5) is that the small density divergence of \( \text{ALDA}_X \) is regulated. For example, for small \( r = |\mathbf{r} - \mathbf{r}'| \) one obtains

\[
\tilde{f}^\text{rALDA}_X(r) = 4n f^\text{rALDA}_X[n],
\]

whereas \( \text{ALDA} \) diverges.

![Image](image-url)  
FIG. 3. (Color online) Correlation energy per valence electron in Na evaluated with RPA, ALDA, and \( \text{rALDA} \). The dashed lines show the values obtained with the functionals for the homogeneous electron gas using the average valence density of Na.

Table I. Correlation energies of H, H\(_2\), and He evaluated with different functionals. Exact values are taken from Ref. 31. All numbers are in kcal/mol.

<table>
<thead>
<tr>
<th></th>
<th>LDA</th>
<th>PBE</th>
<th>RPA</th>
<th>( \text{ALDA}_X )</th>
<th>( \text{rALDA} )</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>-14</td>
<td>-2</td>
<td>-13</td>
<td>6</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>H(_2)</td>
<td>-59</td>
<td>-28</td>
<td>-51</td>
<td>-16</td>
<td>-28</td>
<td>-26</td>
</tr>
<tr>
<td>He</td>
<td>-70</td>
<td>-41</td>
<td>-19</td>
<td>-27</td>
<td>-27</td>
<td>-26</td>
</tr>
</tbody>
</table>

The accuracy of molecular atomization energies by RPA is comparable to that of PBE; however, total correlation energies are typically severely underestimated. ALDA, on the other hand, tends to overestimate total correlation energies by approximately the same amount. This is clearly seen for homogeneous systems displayed in Figs. 2 and 3 and the trend is also observed for inhomogeneous systems. In Table II we show a few examples of atomic and molecular correlation energies calculated with the \( \text{rALDA} \) functional and compared with LDA, PBE, RPA, and ALDA results. The ACDF correlation energies were calculated in a 6 \( \times \) 6 \( \times \) 7 Å unit cell. The RPA and \( \text{rALDA} \) results were calculated at increasing cutoffs up to 400 eV and extrapolated to infinity.

The \( \text{rALDA}_X \) results were extrapolated from 1000 eV, but are still not well converged with respect to cutoff and represent a lower bound on the absolute \( \text{ALDA}_X \) correlation energies. It is clear that the \( \text{rALDA} \) functional performs much better than both RPA and ALDA.

As a simple test of the functional, we have calculated the correlation energy of diatomic molecules. The \( \text{ALDA} \) values are taken from Ref. 2 and experimental values (corrected for zero point vibrational energies) are taken from Ref. 32. Results in parentheses are from Ref. 6. All numbers are in kcal/mol. The bottom line shows the mean absolute error for this small test set.

<table>
<thead>
<tr>
<th></th>
<th>LDA</th>
<th>PBE</th>
<th>RPA@LDA</th>
<th>RPA@PBE</th>
<th>ALDA</th>
<th>( \text{rALDA} )</th>
<th>Expt.</th>
</tr>
</thead>
<tbody>
<tr>
<td>N(_2)</td>
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<td>-244</td>
<td>-225</td>
<td>-225</td>
<td>-229</td>
<td>-226</td>
<td>-228</td>
</tr>
<tr>
<td>O(_2)</td>
<td>-174</td>
<td>-144</td>
<td>-103</td>
<td>-112</td>
<td>-115</td>
<td>-118</td>
<td>-120</td>
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<td>CO</td>
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<td>-269</td>
<td>-234</td>
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<td>-122</td>
<td>-130</td>
<td>-157</td>
<td>-136</td>
<td>-141</td>
</tr>
<tr>
<td>H(_2)O</td>
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<td>-234</td>
<td>-218</td>
<td>-222</td>
<td>-249</td>
<td>-225</td>
<td>-233</td>
</tr>
</tbody>
</table>

Table II. Atomization energies of diatomic molecules. The MAE values are taken from Ref. 2 and experimental values (corrected for zero point vibrational energies) are taken from Ref. 32. Results in parentheses are from Ref. 6. All numbers are in kcal/mol. The bottom line shows the mean absolute error for this small test set.
used, a consistent choice is the ground state functional from which the kernel was derived. In the present case of ALDA and rALDA we thus only consider calculations on top of the LDA ground state. For these molecules the rALDA kernel is seen to underbind by a few kcal/mol (F₂ excepted) but is superior to the RPA and ALDA results.

The additional computational cost of calculating the kernel is insignificant compared to evaluating the noninteracting response function and inverting the Dyson equation. For a pure exchange kernel, it is possible to perform the coupling constant integration analytically, however, it involves an inversion of the noninteracting response function, which may become near singular at particular frequencies. The numerical coupling constant integration thus represents an additional computational cost compared to RPA calculations.

In summary, we have presented a parameter-free exchange kernel for total correlation energy calculations within the ACDF formalism. The kernel largely cancels the self-correlation energy of RPA and seems to perform better than both RPA and ALDA for molecular atomization energies as well as for simple metals. Although more benchmarking is needed, these preliminary results indicate that the rALDA functional is clearly superior to RPA. In contrast to RPA, the functional has the very good feature that it provides a consistent choice of input orbitals beyond the Hartree approximation. Finally, it will be straightforward to extend the kernel to include ALDA correlation, which might be expected to improve results further, but we will leave this to future work. In fact, the renormalization method naturally generalizes to all semilocal adiabatic approximations, which all suffer from the same pathological behavior in their pair-distribution functions, and the present work just represents a single example of an entire class of renormalized adiabatic exchange-correlation kernels.

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