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Interval reliability for semi-Markov systems in discrete time

Titre: Fiabilité sur intervalle des systèmes semi-markoviens à temps discret

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Abstract: In this article, we consider a repairable system with finite state space. The interval reliability is a dependability measure, defined as the probability of the system being operational over a given finite-length time interval. We study the interval reliability of a system governed by a discrete-time semi-Markov process. The explicit formula of the interval reliability is obtained via Markov renewal equation in discrete time. Additionally, the limiting interval reliability is treated. Finally, we present some numerical results on the interval reliability, applied to a three-state reliability semi-Markov system.

Résumé : Dans cet article, nous considérons un système réparable d'espace d'état fini. La fiabilité sur intervalle est une mesure de la sûreté de fonctionnement, définie par la probabilité que le système soit opérationnel pendant un intervalle de temps de longueur donnée finie. Nous étudions la fiabilité sur intervalle d'un système décrit par un processus semi-markovien à temps discret. La formule de la fiabilité sur intervalle se présente comme la solution d'une équation de renouvellement markovien à temps discret. Nous obtenons également la fiabilité sur intervalle limite. Enfin, nous présentons quelques résultats numériques pour la fiabilité sur intervalle, appliquée à un système semi-markovien à trois états.

Keywords: discrete-time semi-Markov process, interval reliability, limiting interval reliability, dependability measures, Markov renewal equation
Mots-clés : processus semi-markovien à temps discret, fiabilité sur intervalle, fiabilité sur intervalle limite, sûreté de fonctionnement, équation de renouvellement markovien
AMS 2000 subject classifications: 60K15, 60K20, 62N05, 90B25

1. Introduction

In many application fields (e.g., nuclear and electric power systems, communication networks, biological organisms, software reliability, military systems, etc.), there is an increasing interest in evaluating the performance assessment of repairable technical systems. The evolution of a system is modeled by a stochastic process in continuous or discrete time. The state space of the system is partitioned into two disjoint subsets: the subset U of operational (or up) states and the subset D of non-operational (or down) states. Once the system is assumed to be repairable, the process is irreducible and the system alternates between U and D indefinitely. Moreover, transitions from the operational (resp. non-operational) states to the non-operational (resp. operational) states are called system failures (resp. repairs).

Formally, both continuous and discrete-time Markov models with a finite state space have been extensively used as a standard tool to describe systems evolution. An alternative approach

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is by relaxing the Markovian hypothesis and, then, applying the so-called semi-Markov models. Such models in continuous time are described in Çinlar (1975); Limnios and Oprişan (2001); Janssen and Manca (2006). For discrete-time semi-Markov systems, see Howard (1971); Mode and Sleeman (2000); Barbu and Limnios (2008) and the references therein. In this paper, we consider that the evolution of a repairable system is assumed to be modeled by a discrete-time semi-Markov process with finitely many states.

Various dependability measures have been proposed and studied in order to describe the behavior of a repairable system and its properties. The two most used and important measures are the following : the reliability $R(k)$, $k \in \mathbb{N}$, defined as the probability of the system to be operational throughout the time interval $[0,k]$, without any failure in this interval; and the pointwise availability $A(k)$, given as the probability of the system to be in an up state at time k, where system failures in $[0,k-1]$ are possible.

An alternative more general dependability measure that extends the reliability and availability functions has been proposed, called interval reliability $I\!R(k,p)$, $k,p \in \mathbb{N}$, and it is defined by the probability that a system is in operation at time k and will continue to operate over a finite time interval of length p. The first effort to study the interval reliability is reported in Barlow and Hunter (1961), where the distribution of the number of failures and repairs in a given interval is derived and a closed form integral expression is obtained for the interval reliability. Therefore, the interval reliability have been studied for a Markov system and special non-Markovian ones (see Barlow and Proschan, 1965; Birolini, 1985, 2007; Aven and Jensen, 1999; Nakagawa, 2005; Csenki, 2007). For the continuous-time semi-Markov model, the interval reliability has been introduced in Csenki (1994) and it satisfies a system of integral equations which can be solved numerically (see Csenki, 1995). In general state space case, various dependability measures are studied for both continuous and discrete semi-Markov systems in Limnios (2012), including the interval reliability.

The aim of this article is to explore the interval reliability for finite semi-Markov systems in discrete time. First, a recursive formula for computing the conditional interval reliability, given the initial state of the system, is obtained. We derive the interval reliability as the closed-form solution of a Markov renewal equation and we investigate the reliability and the availability functions as particular cases of it, i.e., $I\!R(0,\cdot) = R(\cdot)$ and $I\!R(\cdot,0) = A(\cdot)$. Afterwards, the measure of the limiting interval reliability is proposed and an explicit formula is given for it.

The paper is organized as follows: in Section 2, the framework of finite semi-Markov models is given, Section 3 depicts our main results on the measure of interval reliability and the effectiveness of these results are illustrated through a numerical application in Section 4. In the last section, some concluding remarks are given.

2. Semi-Markov setting

Let E be a finite nonempty set and $(\Omega, \mathscr{F}, \mathbb{P})$ a complete probability space. Consider the time homogeneous stochastic process $(\boldsymbol{J},\boldsymbol{S}) := (J_n, S_n)_{n\geq 0}$ on $(\Omega, \mathscr{F}, \mathbb{P})$, with state space $E \times \mathbb{N}$, called a Markov renewal chain (MRC), with semi-Markov kernel $\boldsymbol{q}(k) := (q_{ij}(k);\, i,j \in E)$, $k \in \mathbb{N}$, defined by

$$q_{ij}(k) := \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k | J_n = i), \quad n \in \mathbb{N}.$$

The process $\boldsymbol{J} := (J_n)_{n\geq 0}$ is an homogeneous Markov chain with state space E, where J_n is the system's state at the n-th jump time, with transition matrix $\boldsymbol{p} := (p_{ij};\, i,j \in E)$, where

$$p_{ij} := \mathbb{P}(J_{n+1} = j | J_n = i), \quad n \in \mathbb{N}.$$

Consider the stochastic process $\boldsymbol{S} := (S_n)_{n\geq 0}$ with state space $\mathbb{N}$, where S_n is the n-th jump time and $0 = S_0 < S_1 < \cdots < S_n < S_{n+1} < \cdots$. This implies that $\boldsymbol{S}$ is a strictly increasing sequence and instantaneous transitions are excluded, that is $q_{ij}(0) = 0$, $i,j \in E$. Moreover, transitions to the same state are not allowed, i.e., $p_{ii} = 0$, $i \in E$.

The associated semi-Markov chain (SMC) $\boldsymbol{Z} := (Z_k)_{k\geq 0}$ is an stochastic process on $(\Omega, \mathscr{F}, \mathbb{P})$ with state space E, defined as

$$Z_k := J_n, \quad S_n \leq k < S_{n+1}.$$

In addition, note that $\boldsymbol{\alpha} := (\alpha_i;\, i \in E)$ is the initial distribution of both processes $\boldsymbol{J}$ and $\boldsymbol{Z}$, that is $\alpha_i := \mathbb{P}(J_0 = i) = \mathbb{P}(Z_0 = i)$.

The process $\boldsymbol{S}$ is the sequence of the jump times of $\boldsymbol{Z}$, while $\boldsymbol{J}$ gives the successive visited states of $\boldsymbol{Z}$ and it is called the embedded Markov chain (EMC) of the SMC $\boldsymbol{Z}$, given by $J_n = Z_{S_n}$, $n \geq 0$. Also, let us denote by $\boldsymbol{X} := (X_n)_{n\geq 1}$ the sequence of sojourn times in the successively visited states with values in $\mathbb{N}^*$, defined by $X_n := S_n - S_{n-1}$.

Let $N(k) := \max\{n \geq 0 : S_n \leq k\}$ denote the number of jumps of $\boldsymbol{Z}$ in the interval $[1,k]$. The SMC $\boldsymbol{Z}$ is associated with the MRC $(\boldsymbol{J}, \boldsymbol{S})$ by

$$Z_k = J_{N(k)}, \quad k \in \mathbb{N}.$$

Let $f_{ij}(\cdot)$, $i,j \in E$, denote the conditional distribution function of the sojourn time of the SMC $\boldsymbol{Z}$ in state i when the next visited state is $j \neq i$. To be specific,

$$f_{ij}(k) := \mathbb{P}(X_{n+1} = k | J_n = i, J_{n+1} = j), \quad k \in \mathbb{N}.$$

Note that $q_{ij}(\cdot) = p_{ij} f_{ij}(\cdot)$, $i,j \in E$. Let us denote by $h_i(\cdot)$ and $H_i(\cdot)$, $i \in E$, the distribution function and cumulative distribution function, respectively, of the sojourn time in any state i, which are given as functions of the semi-Markov kernel as follows:

$$h_i(k) := \mathbb{P}(X_{n+1} = k | J_n = i) = \sum_{j\in E} q_{ij}(k), \quad k \in \mathbb{N},$$

and

$$H_i(k) := \mathbb{P}(X_{n+1} \leq k | J_n = i) = \sum_{j\in E} \sum_{l=0}^{k} q_{ij}(l), \quad k \in \mathbb{N}.$$

Let $\bar{H}_i(\cdot)$ be the survival function in state $i \in E$, which is defined as

$$\bar{H}_i(k) := \mathbb{P}(X_{n+1} > k | J_n = i) = 1 - H_i(k), \quad k \in \mathbb{N}.$$

Let $\boldsymbol{P}(\cdot) := (P_{ij}(\cdot);\, i,j \in E)$ be the transition function of the SMC $\boldsymbol{Z}$ defined by

$$P_{ij}(k) := \mathbb{P}(Z_k = j | Z_0 = i), \quad k \in \mathbb{N}.$$

The stationnary distribution $\boldsymbol{\pi} := (\pi_i;\, i \in E)$ of the SMC is defined, when it exists, by

$$\pi_i := \lim_{k\to\infty} P_{li}(k), \quad l \in E.$$

Let us denote the column vector $\boldsymbol{m} := (m_i; i \in E)^\top$, where m_i is the mean sojourn time of the SMC $\boldsymbol{Z}$ in any state $i \in E$, defined by

$$m_i := \mathbb{E}[S_1|J_0 = i] = \sum_{k\geq 0} \bar{H}_i(k).$$

In the sequel, we suppose the following assumptions :

A1 the EMC $\boldsymbol{J}$ is irreducible, with stationary distribution $\boldsymbol{\nu} := (\nu_i;\, i \in E)$;

A2 the EMC $\boldsymbol{J}$ is aperiodic;

A3 the mean sojourn times m_i are finite for any state $i \in E$.

Once the assumptions A1-A2 are satisfied, the EMC $\boldsymbol{J}$ is ergodic.

Furthermore, define the mean recurrence time of $\boldsymbol{Z}$ in state i as $\mu_{ii} := \mathbb{E}[T_i|Z_0 = i]$, where $T_i := \min\{k \geq S_1 : Z_k = i\}$. From Propositions 3.6 and 3.9 in Barbu and Limnios (2008), for a MRC $(\boldsymbol{J},\boldsymbol{S})$ that satisfies the assumptions A1-A3, we have

$$\mu_{ii} = \frac{\bar{m}}{\nu_i}, \quad i \in E, \tag{1}$$

and

$$\pi_i = \frac{m_i \nu_i}{\bar{m}}, \quad i \in E. \tag{2}$$

where $\bar{m}$ is the mean sojourn time of $\boldsymbol{Z}$, defined as $\bar{m} := \sum_{i\in E} \nu_i m_i$.

In dependability analysis of semi-Markov systems, and in general Markov renewal theory as well, an essential operation is the convolution product. Let $\boldsymbol{A}(\cdot) := (A_{ij}(\cdot);\, 1 \leq i \leq n, 1 \leq j \leq m)$ and $\boldsymbol{B}(\cdot) := (B_{ij}(\cdot);\, 1 \leq i \leq m, 1 \leq j \leq r)$ be two matrix-valued functions. The discrete-time matrix convolution product $\boldsymbol{A} * \boldsymbol{B}(\cdot)$ is the matrix-valued function $\boldsymbol{C}(\cdot) := (C_{ij}(\cdot);\, 1 \leq i \leq n, 1 \leq j \leq r)$ defined by

$$C_{ij}(k) := (A * B)_{ij}(k) := \sum_{s=1}^{m} \sum_{l=0}^{k} A_{is}(l) B_{sj}(k-l), \quad k \in \mathbb{N},$$

or, in matrix form,

$$\boldsymbol{C}(k) := \boldsymbol{A} * \boldsymbol{B}(k) := \sum_{l=0}^{k} \boldsymbol{A}(l)\boldsymbol{B}(k-l), \quad k \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, the discrete-time n-fold convolution $\boldsymbol{q}^{(n)}(\cdot) := (q_{ij}^{(n)}(\cdot);\, i,j \in E)$ of $\boldsymbol{q}(\cdot)$ by itself can be defined recursively, for any $k \in \mathbb{N}$, by

$$q_{ij}^{(0)}(k) := \begin{cases} \delta_{ij}, & k = 0, \\ 0, & k > 0, \end{cases} \quad q_{ij}^{(1)}(k) := q_{ij}(k), \quad \text{and} \quad q_{ij}^{(n+1)}(k) := (q * q^{(n)})_{ij}(k), \quad n \geq 1,$$

or, in matrix form,

$$\boldsymbol{q}^{(0)}(k) := \begin{cases} \boldsymbol{I}, & k = 0, \\ \boldsymbol{0}, & k > 0, \end{cases} \quad \boldsymbol{q}^{(1)}(k) := \boldsymbol{q}(k), \quad \text{and} \quad \boldsymbol{q}^{(n+1)}(k) := \boldsymbol{q} * \boldsymbol{q}^{(n)}(k), \quad n \geq 1,$$

where δ is the Kronecker delta symbol, $\boldsymbol{I}$ and $\boldsymbol{0}$ are the identity and the null matrices, respectively, of appropriate dimensions. For any $i, j \in E$ and any $k \in \mathbb{N}$, note that

$$q_{ij}^{(n)}(k) := \mathbb{P}(J_n = j, S_n = k | J_0 = i), \quad n \in \mathbb{N}.$$

Let $\boldsymbol{\psi}(\cdot) := (\psi_{ij}(\cdot); i, j \in E)$ be the matrix-valued function defined by

$$\psi_{ij}(k) := \sum_{n=0}^{k} q_{ij}^{(n)}(k), \quad k \in \mathbb{N},$$

called the Markov renewal function.

Let us now consider two real-valued functions $L_i(\cdot)$ and $G_i(\cdot)$, unknown and known respectively, defined on $E \times \mathbb{N}$. The Markov renewal equation (MRE) is defined as follows

$$L_i(k) = G_i(k) + \sum_{j \in E} \sum_{l=0}^{k} q_{ij}(l) L_j(k-l), \quad k \in \mathbb{N}. \tag{3}$$

The MRE (3) has a unique solution (see Barbu and Limnios, 2008, Theorem 3.1) given by

$$L_i(k) = (\psi * G)_i(k) = \sum_{j \in E} \sum_{l=0}^{k} \psi_{ij}(l) G_j(k-l), \quad k \in \mathbb{N}.$$

In vector form, define the functions $\boldsymbol{L}(\cdot) := (L_i(\cdot); i \in E)^\top$ and $\boldsymbol{G}(\cdot) := (G_i(\cdot); i \in E)^\top$ with corresponding MRE

$$\boldsymbol{L}(k) = \boldsymbol{G}(k) + \boldsymbol{q} * \boldsymbol{L}(k), \quad k \in \mathbb{N},$$

whose unique solution is given by

$$\boldsymbol{L}(k) = \boldsymbol{\psi} * \boldsymbol{G}(k), \quad k \in \mathbb{N}.$$

3. Interval reliability for semi-Markov systems

Consider the state space $E = \{1, \ldots, d\}$. Let $U = \{1, \ldots, r\}$, $0 < r < d$, and $D = \{r+1, \ldots, d\}$ be the nonempty subsets of the up and down states of the system, respectively, with $E = U \cup D$ and $U \cap D = \emptyset$. Denote by $\mathbf{1}_{d,r}$ the d-dimensional column vector whose r first elements are 1 and last $d-r$ elements are 0. As usual, we denote by $\mathbb{P}_i(\cdot)$ the conditional probability $\mathbb{P}(\cdot | J_0 = i)$, and by $\mathbb{E}_i(\cdot)$ the corresponding expectation.

3.1. Reliability and availability

First, let us define formally the reliability and availability functions. The reliability $R(k)$ of a system at time $k \in \mathbb{N}$, starting to operate at time $k = 0$, is defined as the probability that the system has operated without failure in the interval $[0, k]$, i.e.,

$$R(k) := \mathbb{P}(Z_n \in U, n \in [0, k]),$$

and, for any operational state $i \in U$, $R_i(k) := \mathbb{P}_i(Z_n \in U,\, n \in [0,k])$ is the conditional reliability at time k. The pointwise availability $A(k)$ of a system at time $k \in \mathbb{N}$ is the probability that the system is operational at time k, independently of the fact that the system has failed or not in $[0, k-1]$, that is

$$A(k) := \mathbb{P}(Z_k \in U),$$

with the conditional availability at time k, given that it starts in state $i \in E$, to be $A_i(k) := \mathbb{P}_i(Z_k \in U)$. Moreover, define the steady-state availability A_∞ as the limit of the pointwise availability, when the limit exists, as the time tends to infinity, i.e.,

$$A_\infty := \lim_{k\to\infty} A(k).$$

3.2. *Interval reliability*

Let us define the interval reliability $I\!R(k,p)$, $k,p \in \mathbb{N}$, as the probability that the system is in the set of up states U at time k and it will remain there for the next p time units, i.e.,

$$I\!R(k,p) := \mathbb{P}(Z_n \in U,\, n \in [k, k+p]).$$

Henceforth, we will use the notation Z_k^{k+p} for the vector $(Z_k, \ldots, Z_{k+p})$. In the following proposition, we derive an initial general result relating the interval reliability with the reliability and the availability.

Proposition 1. *For a discrete-time semi-Markov system, the interval reliability is bounded from above by the availability and bounded from below by the reliability, in the sense that*

$$R(k+p) \leq I\!R(k,p) \leq A(k+p), \quad k,p \in \mathbb{N}. \tag{4}$$

Proof. It is clear that, for any $k,p \in \mathbb{N}$, we have

$$\left\{Z_0^{k+p} \in U^{k+p+1}\right\} \subset \left\{Z_k^{k+p} \in U^{p+1}\right\} \subset \left\{Z_{k+p} \in U\right\},$$

which implies

$$\mathbb{P}\left(Z_0^{k+p} \in U^{k+p+1}\right) \leq \mathbb{P}\left(Z_k^{k+p} \in U^{p+1}\right) \leq \mathbb{P}\left(Z_{k+p} \in U\right).$$

From which we obtain the desired result. □

Remark 1. The system is assumed to be repairable and, therefore, the MRC to be irreducible. However, Proposition 1 holds true for any system, repairable or not. The treatment of a reliability system where a repair is not possible (i.e., a non-repairable system) is evident. For all positive integers $k \in \mathbb{N}$, the conditional reliability function $R_i(k)$ equals to the conditional availability function $A_i(k)$, for any $i \in U$, and they are both null for any $i \in D$. Indeed, the reliability and availability measures coincide, and, from Equation (4), the interval reliability concurs with them.

Denote by $I\!R_i(k,p)$, $k,p \in \mathbb{N}$, the conditional interval reliability given the event $\{Z_0 = i\}$, i.e.,

$$I\!R_i(k,p) := \mathbb{P}(Z_k^{k+p} \in U^{p+1} | Z_0 = i).$$

By the very definition of the conditional interval reliability $I\!R_i(\cdot,\cdot)$, for any $k,p \in \mathbb{N}$, we have

$$I\!R_i(0,p) = R_i(p), \quad i \in U, \tag{5}$$

and

$$I\!R_i(k,0) = A_i(k), \quad i \in E. \tag{6}$$

Considering a discrete-time semi-Markov system, the conditional reliability and availability functions at any time $k \in \mathbb{N}$, satisfy the following MREs (see Barbu and Limnios, 2008) :

$$R_i(k) = \bar{H}_i(k) + \sum_{j \in U} \sum_{l=0}^{k} q_{ij}(l) R_j(k-l), \quad i \in U, \tag{7}$$

and

$$A_i(k) = \bar{H}_i(k) \mathbf{1}_U(i) + \sum_{j \in E} \sum_{l=0}^{k} q_{ij}(l) A_j(k-l), \quad i \in E,$$

where $\mathbf{1}_U(\cdot)$ is the indicator function of the subset U.

Furthermore, we introduce the interval reliability vector $\boldsymbol{I\!R}(k,p) := (I\!R_1(k,p), \ldots, I\!R_d(k,p))^\top$, $k,p \in \mathbb{N}$. Note that, for any $k,p \in \mathbb{N}$, the interval reliability can be related with the interval reliability vector by

$$I\!R(k,p) = \sum_{i \in E} \mathbb{P}(Z_0 = i) \mathbb{P}(Z_k^{k+p} \in U^{p+1} | Z_0 = i) = \sum_{i \in E} \alpha_i I\!R_i(k,p) = \boldsymbol{\alpha} \boldsymbol{I\!R}(k,p). \tag{8}$$

Proposition 2. *Under Assumption A1, the conditional interval reliability of a discrete-time semi-Markov system in the time interval $[k, k+p] \subset \mathbb{N}$ is given as the solution of the following MRE*

$$I\!R_i(k,p) = \bar{H}_i(k+p) \mathbf{1}_U(i) + \sum_{j \in U} \sum_{l=k+1}^{k+p} q_{ij}(l) R_j(k+p-l) \mathbf{1}_U(i) + \sum_{j \in E} \sum_{l=0}^{k} q_{ij}(l) I\!R_j(k-l,p). \tag{9}$$

Proof. From the definition of the conditional interval reliability, given that the initial state of the system is $i \in E$, we can write

$$\begin{aligned} I\!R_i(k,p) =& \mathbb{P}_i\left(Z_k^{k+p} \in U^{p+1}, S_1 \le k\right) + \mathbb{P}_i\left(Z_k^{k+p} \in U^{p+1}, k < S_1 \le k+p\right) \\ &+ \mathbb{P}_i\left(Z_k^{k+p} \in U^{p+1}, S_1 > k+p\right). \end{aligned} \tag{10}$$

Then, we obtain

$$\begin{aligned} \mathbb{P}_i\left(Z_k^{k+p} \in U^{p+1}, S_1 > k+p\right) &= \mathbb{P}\left(S_1 > k+p | J_0 = i\right) \mathbf{1}_U(i) \\ &= \mathbb{P}\left(X_1 > k+p | J_0 = i\right) \mathbf{1}_U(i) \\ &= \bar{H}_i(k+p) \mathbf{1}_U(i), \end{aligned} \tag{11}$$

$$\begin{aligned}\mathbb{P}_i\left(Z_k^{k+p} \in U^{p+1}, k < S_1 \leq k+p\right) &= \mathbb{E}_i\left[\mathbb{P}_i\left(Z_k^{k+p} \in U^{p+1}, k < S_1 \leq k+p | J_1, S_1\right)\right] \mathbf{1}_U(i) \\ &= \mathbb{E}_i\left[\mathbf{1}_{\{k<S_1\leq k+p\}} \mathbb{P}_{J_1}\left(Z_0^{k+p-S_1} \in U^{k+p-S_1+1}\right)\right] \mathbf{1}_U(i) \\ &= \sum_{j\in U} \sum_{l=k+1}^{k+p} q_{ij}(l) R_j(k+p-l) \mathbf{1}_U(i) \end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathbb{P}_i\left(Z_k^{k+p} \in U^{p+1}, S_1 \leq k\right) &= \mathbb{E}_i\left[\mathbb{P}_i\left(Z_k^{k+p} \in U^{p+1}, S_1 \leq k | J_1, S_1\right)\right] \\ &= \mathbb{E}_i\left[\mathbf{1}_{\{S_1\leq k\}} \mathbb{P}_{J_1}\left(Z_{k-S_1}^{k+p-S_1} \in U^{p+1}\right)\right] \\ &= \sum_{j\in E} \sum_{l=0}^{k} q_{ij}(l) IR_j(k-l,p), \end{aligned} \tag{13}$$

since $q_{ij}(0) = 0$ for all $i, j \in E$. For any state $i \in E$ and any positive integers $k, p \in \mathbb{N}$, Equations (10) - (13) yield the MRE associated with $IR_i(k, p)$. □

Remark 2. In the statement of the Proposition 2, the system is considered repairable. Supposing a non-repairable system, from the definition of the conditional interval reliability, we have that $IR_i(k, p) = 0$, for any state $i \in D$. From Equation (9), for all $i \in U$ and all $k, p \in \mathbb{N}$, we obtain

$$IR_i(k,p) = \bar{H}_i(k+p) + \sum_{j\in U} \sum_{l=0}^{k+p} q_{ij}(l) R_j(k+p-l) = R_i(k+p).$$

In fact, Equation (9) holds true even for a non-repairable system, verifying Remark 1.

We may derive as special cases of the conditional interval reliability two dependability measures :

1. the conditional reliability $R_i(p)$ at time $p \in \mathbb{N}$, given that the system has started in an operational state $i \in U$ at time $k = 0$,

$$IR_i(0,p) = \bar{H}_i(p)\mathbf{1}_U(i) + \sum_{j\in U} \sum_{l=0}^{p} q_{ij}(l) R_j(p-l) \mathbf{1}_U(i) = R_i(p);$$

2. the conditional availability $A_i(k)$ at time $k \in \mathbb{N}$, given that the system has started in any state $i \in E$ at time $k = 0$,

$$IR_i(k,0) = \bar{H}_i(k)\mathbf{1}_U(i) + \sum_{j\in E} \sum_{l=0}^{k} q_{ij}(l) IR_j(k-l) = A_i(k).$$

As a matter of fact, Equations (5) and (6) are verified. Furthermore, for any down state $i \in D$ and any positive integer $p \in \mathbb{N}$, we have $IR_i(0, p) = 0$. Consequently, it holds true that

$$IR(0,p) = R(p), \quad p \in \mathbb{N}.$$

As far as the pointwise availability is concerned, it is apparent that

$$IR(k,0) = A(k), \quad k \in \mathbb{N}.$$

For any $i \in E$ and any $k, p \in \mathbb{N}$, let us define the following real-valued function on $\mathbb{N}$:

$$g_i^p(k) := \bar{H}_i(k+p)\mathbf{1}_U(i) + \sum_{j\in U}\sum_{l=k+1}^{k+p} q_{ij}(l)R_j(k+p-l)\mathbf{1}_U(i), \tag{14}$$

and $\boldsymbol{g}^p(\cdot) := (g_1^p(\cdot), \dots, g_r^p(\cdot), g_{r+1}^p(\cdot), \dots, g_d^p(\cdot))^\top$ the column vector of all functions $g_i^p(\cdot)$, $i \in E$, where the last $d-r$ elements of the vector are equal to zero (i.e., $g_i^p(\cdot) = 0$, for all $i \in D$). Then, the MRE (9) is written as follows

$$IR_i(k,p) = g_i^p(k) + \sum_{j\in E}\sum_{l=0}^{k} q_{ij}(l)IR_j(k-l,p). \tag{15}$$

The unique solution to the MRE (15) is given by

$$IR_i(k,p) = (\psi * g^p)_i(k) = \sum_{j\in E}\sum_{l=0}^{k} \psi_{ij}(l)g_j^p(k-l). \tag{16}$$

Proposition 3. *Under Assumption A1, the interval reliability $IR(k,p)$, $k,p \in \mathbb{N}$, of a discrete-time semi-Markov system is given by*

$$IR(k,p) = \boldsymbol{\alpha}\boldsymbol{\psi} * \boldsymbol{g}^p(k).$$

Proof. From Equations (8) and (16), we have immediately

$$IR(k,p) = \sum_{i\in E}\alpha_i IR_i(k,p) = \sum_{i\in E}\alpha_i(\psi * g^p)_i(k),$$

and the result follows. □

3.3. Limiting interval reliability

Another interesting reliability indicator related to the interval reliability may be the limiting interval reliability, defined as the limit of the interval reliability, when the limit exists, as the time k tends to infinity, i.e.,

$$IR_\infty(p) = \lim_{k\to\infty} IR(k,p), \quad p \in \mathbb{N},$$

that is the probability of the system to be operational for an finite-length time interval, as the time tends to infinity. Namely, the asymptotic interval reliability is an equilibrium point of the interval reliability.

Since it holds true $IR(k,0) = A(k)$, $k \in \mathbb{N}$, from the definition of the limiting interval reliability, we obtain

$$IR_\infty(0) = A_\infty.$$

The steady-state availability for a semi-Markov system that satisfies Assumptions A1 - A3 is given (see Barbu and Limnios, 2008, Proposition 5.3) by

$$A_\infty = \frac{1}{\boldsymbol{\nu m}} \boldsymbol{m}^\top diag(\boldsymbol{\nu}) \mathbf{1}_{d,r} = \sum_{i\in U} \pi_i.$$

We introduce some notation to help with the proof of the following lemma. Let the row vector $\boldsymbol{m} = \begin{pmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 \end{pmatrix}^\top$ denote the mean sojourn times $\boldsymbol{m}$ of the SMC $\boldsymbol{Z}$, where the subvectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ correspond to the up states U and the down states D, respectively. Let the submatrices $\boldsymbol{p}_{11}$, $\boldsymbol{p}_{12}$, $\boldsymbol{p}_{21}$ and $\boldsymbol{p}_{22}$ be the restrictions of the transition matrix $\boldsymbol{p}$ of the EMC $\boldsymbol{J}$ on $U \times U$, $U \times D$, $D \times U$ and $D \times D$, respectively, i.e.,

$$\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p}_{11} & \boldsymbol{p}_{12} \\ \boldsymbol{p}_{21} & \boldsymbol{p}_{22} \end{pmatrix}.$$

Let $T_D := \min\{n \in \mathbb{N} : Z_n \in D\}$ be the first hitting time of the subset D. An alternative definition of the reliability is $R(k) := \mathbb{P}(T_D > k)$, $k \in \mathbb{N}$, and, consequently, $R_i(k) := \mathbb{P}_i(T_D > k)$, $i \in U$, is the conditional reliability.

Replace the matrices $\boldsymbol{p}_{21}$ by the zero matrix $\mathbf{0}_{r,d-r}$ and $\boldsymbol{p}_{22}$ by the identity matrix $\boldsymbol{I}_{d-r}$. The new matrix $\tilde{\boldsymbol{p}}$, written as follows :

$$\tilde{\boldsymbol{p}} = \begin{pmatrix} \boldsymbol{p}_{11} & \boldsymbol{p}_{12} \\ \mathbf{0}_{r,d-r} & \boldsymbol{I}_{d-r} \end{pmatrix},$$

is the transition matrix of an absorbing Markov chain, which means that the subset of the operational states U is a transient set of states (i.e., a union of transient classes). In that case, the matrix $\boldsymbol{I}_r - \boldsymbol{p}_{11}$ is nonsingular (see Kemeny and Snell, 1960).

Based on the above remark, we state the following assumption:

A4 the set of operational states U is a transient set of states for $\tilde{\boldsymbol{p}}$.

Lemma 1. *Under Assumptions A2 and A4, for any $j \in E$ and any $p \in \mathbb{N}$, the following statement holds:*

$$\sum_{k\geq 0} |g_j^p(k)| < \infty.$$

Proof. First, by the definition of the real-valued function $g_j^p(k)$, for all $k, p \in \mathbb{N}$ and all $j \in E$, it is evident that $g_j^p(k) \geq 0$. Comparing Equations (7) and (14), it is clear that

$$g_j^p(k) \leq R_j(k+p).$$

Then, we obtain

$$\begin{aligned} \sum_{k\geq 0} g_j^p(k) &\leq \sum_{k\geq 0} R_j(k+p) \\ &\leq \sum_{k\geq 0} R_j(k) \\ &= \sum_{k\geq 0} \mathbb{P}_j(T_D > k) \\ &= \mathbb{E}_j[T_D]. \end{aligned} \tag{17}$$

On the other hand, we have (see Barbu et al., 2004, Proof of Proposition 15) that

$$\mathbb{E}_j[T_D] = \boldsymbol{e_j}(\boldsymbol{I}_r - \boldsymbol{p}_{11})^{-1}\boldsymbol{m}_1, \tag{18}$$

where $\boldsymbol{e_j} := (0,\dots,0,1,0,\dots,0)$ is a r-dimensional row vector of 0's with 1 in the j-th place. Under the assumptions of the lemma, the entries of the vector $\boldsymbol{m}_1$ are finite and the matrix $\boldsymbol{I}_r - \boldsymbol{p}_{11}$ is nonsingular. From Equations (17) and (18), we get the desired result. □

Proposition 4. *Under Assumptions A1 - A4, the asymptotic interval reliability $IR_\infty(p)$, $p \in \mathbb{N}$, of an discrete-time semi-Markov system over an interval of finite length p is given by*

$$IR_\infty(p) = \frac{1}{\bar{m}} \sum_{i \in U} \nu_i \sum_{k \geq 0} g_i^p(k).$$

Furthermore, for $p = 0$, we conclude that

$$IR_\infty(0) = A_\infty.$$

Proof. Using the expression of the interval reliability given in Proposition 3, for any $p \in \mathbb{N}$, we have

$$\begin{aligned} IR_\infty(p) &= \lim_{k\to\infty} IR(k,p) \\ &= \sum_{i \in E} \alpha_i \sum_{j \in U} \lim_{k\to\infty} \psi_{ij} * g_j^p(k). \end{aligned} \tag{19}$$

From Lemma 1 and the discrete-time key Markov renewal theorem (see Barbu and Limnios, 2008, Theorem 3.3), it holds true that

$$\psi_{ij} * g_j^p(k) \to \frac{1}{\mu_{jj}} \sum_{n \geq 0} g_j^p(n), \quad k \to \infty, \tag{20}$$

where $i \in E$ and $j \in U$. Thus, from Equations (19) and (20), along with Equation (1), we obtain

$$\begin{aligned} IR_\infty(p) &= \sum_{i \in E} \alpha_i \sum_{j \in U} \frac{1}{\mu_{jj}} \sum_{n \geq 0} g_j^p(n) \\ &= \frac{1}{\bar{m}} \sum_{j \in U} \nu_j \sum_{n \geq 0} g_j^p(n). \end{aligned}$$

We can derive the asymptotic availability as a particular case of the asymptotic interval reliability. In fact, for $p = 0$, using Equation (2), we get

$$\begin{aligned} IR_\infty(0) &= \frac{1}{\bar{m}} \sum_{i \in U} \nu_i \sum_{k \geq 0} \bar{H}_i(k) \\ &= \frac{1}{\bar{m}} \sum_{i \in U} \nu_i m_i \\ &= \sum_{i \in U} \pi_i, \end{aligned}$$

and the conclusion follows. □

Remark 3. In the case of a non-repairable system, the asymptotic interval reliability is zero.

4. Numerical application

In this section, we present a reliability semi-Markov model in discrete time and we obtain various numerical results concerning the interval reliability. Consider a semi-Markov system with three states, i.e., $E = \{1,2,3\}$. The first two states are assumed to be the operational ones, $U = \{1,2\}$, with $\{1\}$ to be a perfect state and $\{2\}$ a state of degradation (but still working). The last one is the failure state of the system, $D = \{3\}$. The transitions of the reliability semi-Markov model are given in the flowgraph of Figure 1.

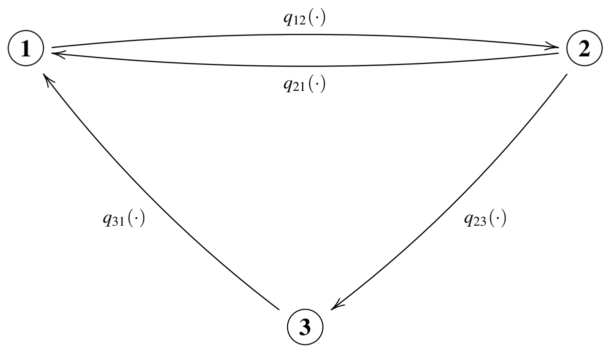

FIGURE 1. *Three-state discrete-time semi-Markov system*

The transition matrix $\boldsymbol{p}$ of the EMC $\boldsymbol{J}$ is given by

$$\boldsymbol{p} = \begin{pmatrix} 0 & 1 & 0 \\ 0.6 & 0 & 0.4 \\ 1 & 0 & 0 \end{pmatrix},$$

with the initial distribution to be $\boldsymbol{\alpha} = \begin{pmatrix} 0.9 & 0.1 & 0 \end{pmatrix}$. The standard situation for the initial distribution is $\boldsymbol{\alpha} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ but we can assume a more general case as the one considered here, i.e., the initial state of the system may differ from the perfect one.

Let X_{ij} be the conditional sojourn time of the SMC $\boldsymbol{Z}$ is state i given that the next state is j $(j \neq i)$. The conditional sojourn times are given as follows :

$$\begin{aligned} X_{12} &\sim \text{geometric}(0.15), \\ X_{21} &\sim \text{discrete Weibull}(0.9, 1.2), \\ X_{23} &\sim \text{discrete Weibull}(0.8, 1.2), \\ X_{31} &\sim \text{geometric}(0.20), \end{aligned}$$

where the probability density function of the discrete Weibull distribution with parameters (q, b) is defined in Nakagawa and Osaki (1975) (slightly modified) as

$$f(k) := \begin{cases} q^{(k-1)^b} - q^{k^b}, & \text{if } k \geq 1, \\ 0, & \text{if } k = 0. \end{cases}$$

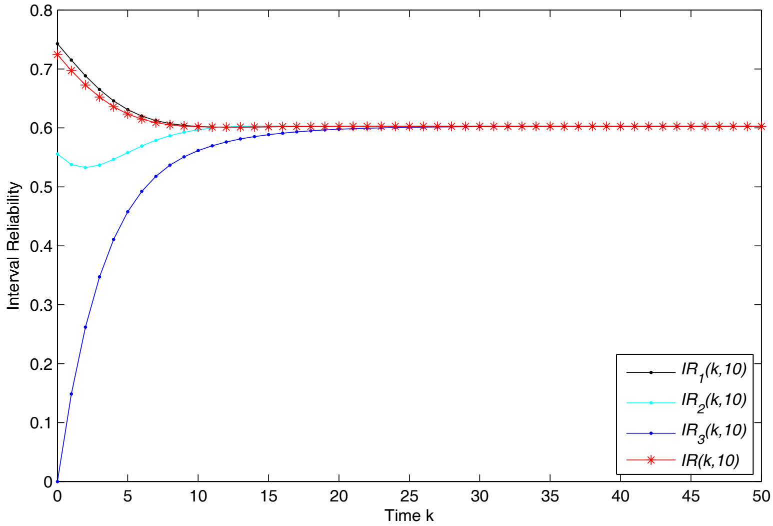

FIGURE 2. *Interval reliability plot ($p = 10$).*

The main reason of the choice of the geometric and the discrete Weibull distributions is that they are the most popular ones in reliability modeling. The Weibull distribution may have an increasing or decreasing hazard rate depending on the value of parameter b. Here, for both Weibull distributions, we have $b > 1$, i.e., the hazard rates are increasing.

The entries of the system semi-Markov kernel $\boldsymbol{q}(\cdot)$ are then given by the formula $q_{ij}(\cdot) = p_{ij} f_{ij}(\cdot)$, $i, j \in E$. It is easily seen that the Assumptions A1 - A4 are verified.

In the following figures, we present some numerical applications for the system as described above with the time in the range from 0 to 50 time units, based on the results of Propositions 2 and 3.

Figure 2 shows the interval reliability $IR(k, p)$ and the conditional interval reliabilities $IR_i(k, p)$ for $p = 10$ fixed, with k to be a variable. That is the probability of the system to be in up set U for 10 successive time units for all values of $k \in \mathbb{N}$. As far as the initial values, we have $IR_i(0, 10)$ to be equal to $R_i(10)$ for $i \in U$. Moreover, since $\alpha_3 = 0$, we have $IR(0, 10) = R(10)$. Note that, as the time k is large enough, the system is in steady state and $IR(k, 10)$ tends to the limiting interval reliability $IR(10) = 0.6026$, as given in Proposition 4.

Then, we consider the value of time k fixed to 5 and we derive the probability that the system is operational at k and will continue to be operational for any interval of length $p \in \mathbb{N}$. The interval reliability $IR(5, p)$ and the conditional interval reliabilities $IR_i(5, p)$ are given in Figure 3. For all $i \in E$, we have $IR_i(5, 0) = A_i(5)$ and therefore $IR(5, 0) = A(5)$. It comes as natural that the interval reliability tends to zero, as the value of p becomes larger.

Figure 4 depicts, on calendar time, the probability of the system to be operational at a fixed time ($k = 10$) and it will be over the following p time units, where p varies from 0 to 30, along with the reliability and availability functions. It is obvious that the bounds of Proposition 1 are verified.

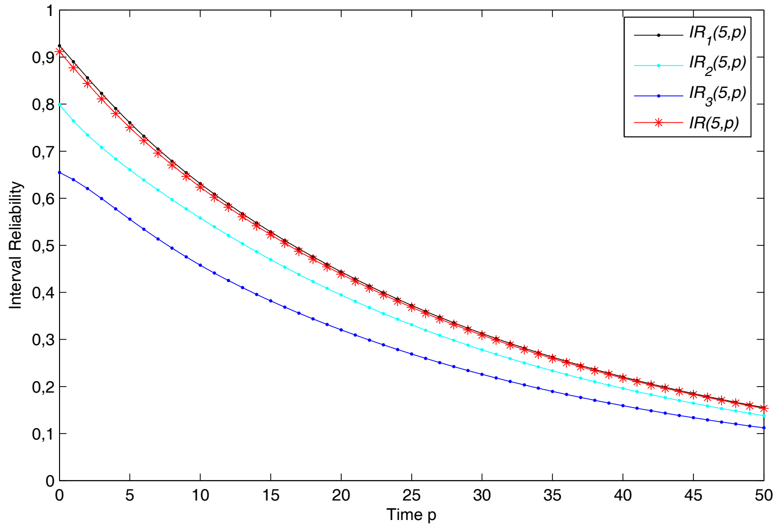

FIGURE 3. *Interval reliability plot ($k = 5$).*

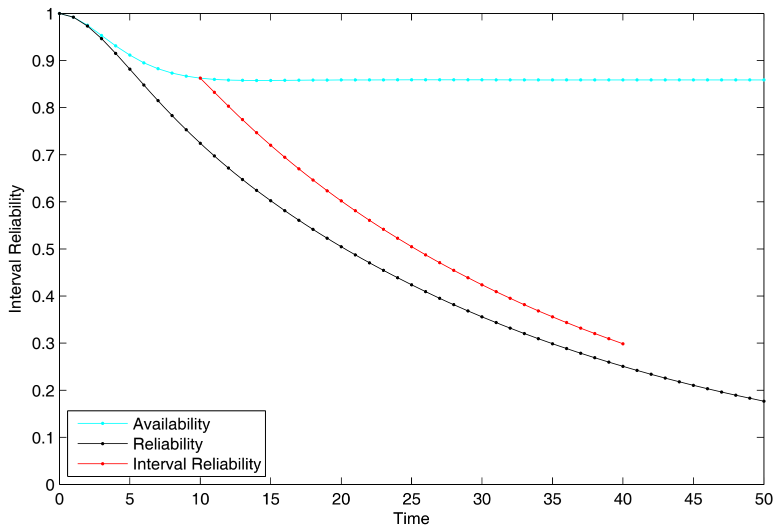

FIGURE 4. *Interval reliability plot ($k = 10$, $p = 0, 1, \ldots, 30$).*

5. Conclusions

In this paper, the interval reliability for a repairable discrete-time semi-Markov system is studied and it is related to the widely-used dependability measures of reliability and availability. In addition, the limiting interval reliability is proposed. We illustrate these results in a reliability semi-Markov model with three states.

The flexibility of interval reliability is evident, since it extends both reliability and availability.

From an application point of view, the significance of this dependability measure is great, as it deals with more useful and complicated characteristics of a system. The probability that a system operates at a time and it will continue to operate through a finite time interval is often requested in order to evaluate the performance assessment of a system. In the authors' opinion, these results can also be a good support for the calculus of the interval reliability in the continuous-time case, after some discretization of time.

Apart from the stochastic modeling, presented in this paper, it is equally important to evaluate statistically the performance of a system. The statistical estimation of the interval reliability for a discrete-time semi-Markov system and the asymptotic properties of its estimator are under investigation.

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