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ON DESIGN OF ANISOTROPY DISTRIBUTIONS, APPLYING LAMINA FORMULAS FOR 2D RESULT VISUALIZATION

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Summary: In rotational transformation of constitutive matrices, some practical quantities are often termed invariants, but the invariance relates to an unchanged reference direction. Rotating this reference direction, the practical quantities do change and this point is clarified with derived rotational transformation for these practical quantities. The research background for optimal anisotropic constitutive matrices is shortly presented. Then design results are applied in a 2D visualization of optimized constitutive matrices, that are distributed in a finite element (FE) model where each element has a specific reference direction. The visualized distributions of physical quantities are; stiffest material direction, material stiffest longitudinal constitutive component, level of anisotropy, absolute or relative shear stiffness and orthotropy test.

1. INTRODUCTION

In free material optimization (FMO), the components of the constitutive matrices are optimized and they change in the space of a finite element (FE) model, i.e., they are distributed. The constraints for the non-dimensional description of these matrices are; symmetry, positive definite and normalized to unit trace. The optimized constitutive matrices should be visualized, but this is not an easy task and different techniques are applied in the literature. From the authors point-of-view the visualization should be related to the most important physical quantities, and for 2D problems the traditional lamina analysis is found valuable.

Analysis and optimization may be performed without rotational transformations in a common coordinate system with the x-direction as reference. However, the visualizations of the optimized results involve rotational transformation of material behavior, i.e., of the constitutive matrices. For each element in a FE model, the direction of stiffest material direction is taken as reference direction with stiffest direction defined as the direction of largest longitudinal components in an optimal constitutive matrix, here termed $(\alpha_{1111})_\theta$ with θ being the angle counter-clockwise from the common x-direction to a direction termed the θ-direction.

The traditional lamina formulas are well suited for localizing θ for a specific element. With $\theta, (\alpha_{1111})_\theta$ determined for all elements the available further physical information is calculated, applying practical parameters $(\alpha_2, \alpha_3, \alpha_6, \alpha_7)_\theta$ as evaluated for element e in the specific reference direction θ_e. In the present note the non-dimensional, normalized practical quantities are given notation α, as alternative to the often preferred notation Q for corresponding dimensional quantities. The note shows that the name invariant is not a god choice. The practical parameters depend on the reference direction and the relations to the common x-direction are derived.

Although written in relation to 2D constitutive matrices, the approach is also valid for 2D structural stiffness matrices $[S]$, 2D structural flexibility matrices $[F]$, and 2D strength matrices in stress space $[H]$ or in strain space $[G]$. Also laminate stiffness sub-matrices and laminate flexibility sub-matrices may be analyzed similarly.

The main readers in mind are researchers with interest in laminate formulation, but it is found necessary to give a short introduction in Section 2. to optimal constitutive matrices, before the application of laminate formulation is detailed in Section 3. Especially the discussion on "invariant" parameters should be noted. Finally, fields for constitutive matrices are exemplified with a suggested visualization for the optimal constitutive design obtained in Pedersen and Pedersen (2015).

2. OPTIMAL DESIGN OF CONSTITUTIVE MATRICES

In recent research simple formula for design of constitutive matrices are obtained, related to different static as well as to eigenfrequency optimal design problems. It is shown that for quite different design objectives, the elastic energy density plays a major role and the results are expressed directly by the current strains, with unit matrix norms and separated from the amount of material.

2.1 Separation from the amount of material

The distribution of material in a continuum is separated into two steps: firstly how much material to be used in a reference volume V_e? and secondly how this local material should be used to obtain an optimal local constitutive matrix? By this separation a clear measure of the total amount of material is possible.

The total amount of material volume V is constrained and this constraint is assumed to be active, i.e., all material is assumed to be used. This assumption is essential for the obtained optimality criteria. With ρ_e as local, non-dimensional design parameters for density, this is written

$$\begin{aligned}
&\sum_e \rho_e V_e = V \ \text{ with size limits} \\
&0 < \rho_{min} \leq \rho_e \leq \rho_{max} \leq 1 \ \text{ and the major constraint is written} \\
&g = \sum_e \rho_e V_e - V = 0 \ \ \Rightarrow \ \ \partial g / \partial \rho_e = V_e
\end{aligned} \tag{1}$$

In Pedersen and Pedersen (2015) the theory and procedures for iterative optimization to obtain the densities ρ_e are presented and will not be further commented in the present paper.

The separated local (element e) constitutive matrix $[C_e]$ is

$$[C_e] = \rho_e E_0 [\widetilde{C}_e] = \rho_e E_0 \begin{bmatrix} (\widetilde{C}_{1111})_e & (\widetilde{C}_{1122})_e & \sqrt{2}(\widetilde{C}_{1112})_e \\ (\widetilde{C}_{1122})_e & (\widetilde{C}_{2222})_e & \sqrt{2}(\widetilde{C}_{2212})_e \\ \sqrt{2}(\widetilde{C}_{1112})_e & \sqrt{2}(\widetilde{C}_{2212})_e & 2(\widetilde{C}_{1212})_e \end{bmatrix} \tag{2}$$

where E_0 is a fixed value of modulus, ρ_e a current local, non-dimensional density and $[\widetilde{C}_e]$ is a non-dimensional matrix, normalized to unit trace as well as to unit Frobenius norm. The discussion of this matrix is of primary interest.

2.2 Constraint for the non-dimensional constitutive components

The constitutive matrices are constrained to be symmetric and positive semi-definite and furthermore normalized such that the Frobenius norm $F_e = F([\widetilde{C}_e])$ is equal to 1 for all elements, here stated in terms of the squared norm F_e^2

$$h_e = F_e^2 - 1 = 0 \tag{3}$$

With a design objective Φ and only the constraint (3), the necessary condition for optimality is proportionality between the gradients of the objective and the gradients of the constraint

$$\frac{\partial\Phi}{\partial(\widetilde{C}_{ijkl})_e} = \lambda\frac{\partial h_e}{\partial(\widetilde{C}_{ijkl})_e} \tag{4}$$

where for 2D problems $(\widetilde{C}_{ijkl})_e$ is one of the six independent components of the constitutive matrix and λ is a common factor for all six of these components, related to a specific constitutive matrix.

With F_e^2 defined as the sum of the squared components of the matrix $[\widetilde{C}_e]$ in (2)

$$F_e^2 = (\widetilde{C}^2_{1111})_e + (\widetilde{C}^2_{2222})_e + 4(\widetilde{C}^2_{1212})_e + 2(\widetilde{C}^2_{1212})_e + 4(\widetilde{C}^2_{1112})_e + 4(\widetilde{C}^2_{2212})_e \tag{5}$$

the gradients of the constraint $h_e = F_e^2 - 1 = 0$ are directly

$$\frac{\partial h_e}{\partial(\widetilde{C}_{1111})_e} = 2(\widetilde{C}_{1111})_e, \quad \frac{\partial h_e}{\partial(\widetilde{C}_{2222})_e} = 2(\widetilde{C}_{2222})_e, \quad \frac{\partial h_e}{\partial(\widetilde{C}_{1212})_e} = 8(\widetilde{C}_{1212})_e,$$
$$\frac{\partial h_e}{\partial(\widetilde{C}_{1122})_e} = 4(\widetilde{C}_{1122})_e, \quad \frac{\partial h_e}{\partial(\widetilde{C}_{1112})_e} = 8(\widetilde{C}_{1112})_e, \quad \frac{\partial h_e}{\partial(\widetilde{C}_{2212})_e} = 8(\widetilde{C}_{2212})_e \tag{6}$$

The gradients of the objective, i.e. the left hand side of (4) for specific optimization objectives are derived below.

2.3 Compliance or total elastic energy as objective

Compliance is, for design independent loads, equal to the total elastic energy U (twice the total strain energy) and a gradient of U, say with respect to the constitutive components $(\widetilde{C}_{ijkl})_e$, can be determined in a fixed strain field (fixed displacements field)

$$\frac{\partial U}{\partial(\widetilde{C}_{ijkl})_e} = -(\frac{\partial U}{\partial(\widetilde{C}_{ijkl})_e})_{fixed\ strain} = -V_e\rho_e E_0(\frac{\partial \widetilde{u}_e}{\partial(\widetilde{C}_{ijkl})_e})_{fixed\ strain} \tag{7}$$

where V_e is the volume in which we have constant strains $\{\epsilon\}_e$ and the constant constitutive matrix $[\widetilde{C}]_e$. Expanding the non-dimensional matrix product $\widetilde{u}_e = \{\epsilon\}_e^T[\widetilde{C}]_e\{\epsilon\}_e$ with $\{\epsilon\}_e^T = \{\epsilon_{11}, \epsilon_{22}, \sqrt{2}\epsilon_{12}\}_e$ give

$$\begin{aligned}\widetilde{u} =& (\widetilde{C}_{1111})_e(\epsilon^2_{11})_e + (\widetilde{C}_{2222})_e(\epsilon^2_{22})_e + 4(\widetilde{C}_{1212})_e(\epsilon^2_{12})_e + \\ & 2(\widetilde{C}_{1122})_e(\epsilon_{11})_e(\epsilon_{22})_e + 4(\widetilde{C}_{1112})_e(\epsilon_{11})_e(\epsilon_{12})_e + 4(\widetilde{C}_{2212})_e(\epsilon_{22})_e(\epsilon_{12})_e\end{aligned} \tag{8}$$

and the gradients are

$$\begin{aligned}
\frac{\partial U}{\partial(\widetilde{C}_{1111})_e} &= \rho_e V_e E_0 (\epsilon_{11})_e (\epsilon_{11})_e, \quad & \frac{\partial U}{\partial(\widetilde{C}_{2222})_e} &= \rho_e V_e E_0 (\epsilon_{22})_e (\epsilon_{22})_e, \\
\frac{\partial U}{\partial(\widetilde{C}_{1212})_e} &= 4\rho_e V_e E_0 (\epsilon_{12})_e (\epsilon_{12})_e, \quad & \frac{\partial U}{\partial(\widetilde{C}_{1122})_e} &= 2\rho_e V_e E_0 (\epsilon_{11})_e (\epsilon_{22})_e, \\
\frac{\partial U}{\partial(\widetilde{C}_{1112})_e} &= 4\rho_e V_e E_0 (\epsilon_{11})_e (\epsilon_{12})_e, \quad & \frac{\partial U}{\partial(\widetilde{C}_{2212})_e} &= 4\rho_e V_e E_0 (\epsilon_{22})_e (\epsilon_{12})_e
\end{aligned} \tag{9}$$

2.4 Multiple load cases and resulting optimality criterion for compliance optimizations

With multiple load cases, all design independent, numbered $n = 1, 2, ...N$, the gradients (9) holds for each load case. The corresponding strains $(\epsilon_{11})_n$, $(\epsilon_{22})_n$ and $(\epsilon_{12})_n$ are all determined in the same coordinate system. Therefore, the simple optimization of minimizing a linear combination of compliance's expressed in the energies U_n is

$$\text{Minimizing} \quad U = \sum_n \eta_n U_n \quad \text{for} \quad h_e = F_e^2 - 1 = 0 \tag{10}$$

for given weight factors η_n, say with $\sum_n \eta_n = 1$.

The design for the multiple load case, that satisfies the optimality criterion is

$$(\widetilde{C}_{ijkl})_e = \lambda \sum_n \eta_n ((\epsilon_{ij})_e (\epsilon_{kl})_e)_n \tag{11}$$

a simple optimal design result. The case of a single load case is further simplified

$$(\widetilde{C}_{ijkl})_e = \lambda (\epsilon_{ij})_e (\epsilon_{kl})_e \tag{12}$$

as seen directly by inserting (6) and (9) in (4).

2.5 Gradients and resulting optimality criterion for single eigenfrequency optimization

The local gradient of the Rayleigh quotient with respect to the components of the local constitutive matrix is simple when the mass distribution is unchanged (kinetic energies T and T_e unchanged), here with hat notation as an alternative to extended index of fixed displacements

$$\frac{\partial \omega^2}{\partial (\widetilde{C}_{ijkl})_e} = \frac{\partial (U/T)}{\partial (\widetilde{C}_{ijkl})_e} = \frac{\widehat{\partial (U/T)}}{\partial (\widetilde{C}_{ijkl})_e}) = \frac{\widehat{\partial (U_e/T_e)}}{\partial (\widetilde{C}_{ijkl})_e} = \frac{1}{T_e}\frac{\widehat{\partial U_e}}{\partial (\widetilde{C}_{ijkl})_e} = \frac{\rho_e V_e E_0}{T_e}\frac{\widehat{\partial \widetilde{u}_e}}{\partial (\widetilde{C}_{ijkl})_e}$$

with fixed strains in $\widetilde{u}_e = \{\epsilon\}_e^T[\widetilde{C}]_e\{\epsilon\}_e$ (13)

From the final relation in (13) then follows

$$\begin{aligned}
\frac{\partial \omega^2}{\partial (\widetilde{C}_{1111})_e} &= \frac{\rho_e V_e E_0}{T_e}(\epsilon_{11})_e(\epsilon_{11})_e, \quad \frac{\partial \omega^2}{\partial (\widetilde{C}_{2222})_e} = \frac{\rho_e V_e E_0}{T_e}(\epsilon_{22})_e(\epsilon_{22})_e, \\
\frac{\partial \omega^2}{\partial (\widetilde{C}_{1212})_e} &= 4\frac{\rho_e V_e E_0}{T_e}(\epsilon_{12})_e(\epsilon_{12})_e, \quad \frac{\partial \omega^2}{\partial (\widetilde{C}_{1122})_e} = 2\frac{\rho_e V_e E_0}{T_e}(\epsilon_{11})_e(\epsilon_{22})_e, \\
\frac{\partial \omega^2}{\partial (\widetilde{C}_{1112})_e} &= 4\frac{\rho_e V_e E_0}{T_e}(\epsilon_{11})_e(\epsilon_{12})_e, \quad \frac{\partial \omega^2}{\partial (\widetilde{C}_{2212})_e} = 4\frac{\rho_e V_e E_0}{T_e}(\epsilon_{22})_e(\epsilon_{12})_e
\end{aligned} \tag{14}$$

that except for a factor is identical to (9).

Comparing with (11) and (12) it is seen that the optimality criterion for the discussed different 2D plane problems is for all of them satisfied for

$$(\widetilde{C}_{ijkl})_e = \left(\epsilon_{ij}\epsilon_{kl}/(\epsilon_{11}^2 + \epsilon_{22}^2 + 2\epsilon_{12}^2)\right)_e \tag{15}$$

now written with the appropriate normalization.

2.6 Proof of unit norms

The result (15) shows that $[\widetilde{C}_e] = \{\alpha\}\{\alpha\}^T$ is described by such a dyadic product. Then by definitions of trace and Frobenius norms follows, that the values of trace and Frobenius norms are always equal and $[\widetilde{C}_e]$ is semi-positive definite.

$$\begin{aligned}
&\text{trace}[\widetilde{C}_e] = \text{Frobenius}[\widetilde{C}_e] = \{\alpha\}^T\{\alpha\} \\
&\text{where } \{\alpha\}^T\{\alpha\} > 0 \text{ for } \{\alpha\} \neq \{0\}
\end{aligned} \tag{16}$$

Omitting the index e for element we proceed the discussion of the obtained constitutive matrix as described directly by the corresponding strain state. Although a constitutive matrix is not necessary obtainable as a dyadic product, this will be the case for the optimal constitutive matrix, where the important result in 2D plane problems with normalization to unit norms is

$$[\widetilde{C}] = \{\alpha\}\{\alpha\}^T \text{ with } \{\alpha\}^T = \{\epsilon_{11} \;\; \epsilon_{22} \;\; \sqrt{2}\epsilon_{12}\}/\sqrt{\epsilon_{11}^2 + \epsilon_{22}^2 + 2\epsilon_{12}^2} \quad (17)$$

That the optimal constitutive matrix of unit norms in 2D is described by only three parameters (the strain components) limits the possibilities for a matrix with normally up to 6 independent parameters. An example is that an isotropic $[\widetilde{C}]$ is only possible with zero Poisson's ratio and for this case $\{\alpha\}^T = \{1\ 1\ 1\}/\sqrt{3}$.

Numerically the rate of change of the constitutive matrices are in each redesign limited by a non-dimensional step parameter $0 \leq \beta \leq 1$ similar to the design approach for strength optimization in Pedersen and Pedersen (2013) where $\beta = 0.5$ and $\beta = 0.1$ were used, i.e.,

$$[\widetilde{C}]_{\text{new}} = \beta[\widetilde{C}]_{\text{from (17)}} + (1-\beta)[\widetilde{C}]_{\text{old}} \quad (18)$$

The design approach is initiated with $[\widetilde{C}]_0 = [I]/3$, i.e., zero Poisson's ratio isotropic material, positive definite, non-dimensional and normalized. It is concluded that for a given strain state the optimized non-dimensional constitutive constitutive matrix is known with unit trace and Frobenius norm. Note, that with initial positive definite $[\widetilde{C}]$ it will for $\beta < 1$ stay positive definite through the redesign iterations. Numerical value $\beta = 0.2$ is applied for the visualized example in Section 4.and even with this rather low β value fast convergence is obtained.

3. VISUALIZATION OF FIELD OF CONSTITUTIVE MATRICES

Visualization of fields of 3×3, symmetric, positive definite constitutive matrices of unit norms is based on formulations from laminate theory. Practical parameters that often are termed invariants are valuable, but there seems to be a need for discussion of the property "invariant".

3.1 Use of laminate formula

For anisotropic material the anisotropy should be visualized, but without going into all details of the six 2D components. A 2D material non-dimensional constitutive matrix $[\widetilde{C}]$ is given in a global x, y coordinate system with the x-direction as the reference direction by

$$[\widetilde{C}] = \begin{bmatrix} \alpha_{1111} & \alpha_{1122} & \sqrt{2}\alpha_{1112} \\ \alpha_{1122} & \alpha_{2222} & \sqrt{2}\alpha_{2212} \\ \sqrt{2}\alpha_{1112} & \sqrt{2}\alpha_{2212} & 2\alpha_{1212} \end{bmatrix}_x \tag{19}$$

with the assumed condition that $[\widetilde{C}]$ is positive definite and the trace of the positive diagonal elements is normalized to unity, i.e.,

$$\alpha_{1111} + \alpha_{2222} + 2\alpha_{1212} = 1 \tag{20}$$

These conditions then hold in any rotated coordinate system. A physical description of the constitutive matrix is of major interest, so the direction of largest longitudinal material stiffness must be located.

According to laminate theory α_{1111} as a function of rotation, termed $(\alpha_{1111})_\theta$, is given by the six components in the x reference coordinate system, here chosen in a form linear in trigonometric factors,

$$\begin{aligned}(\alpha_{1111})_\theta =& (\alpha_{1111} + \alpha_{2222})_x/2 + (\alpha_2)_x \cos(2\theta) - (\alpha_3)_x(1 - \cos(4\theta)) + \\ & (\alpha_6)_x 2\sin(2\theta) + (\alpha_7)_x \sin(4\theta)\end{aligned} \tag{21}$$

where the practical parameters are defined by

$$\begin{aligned}(\alpha_2)_x &= (\alpha_{1111} - \alpha_{2222})_x/2 \\ (\alpha_3)_x &= (\alpha_{1111} + \alpha_{2222} - 2(\alpha_{1122} + 2(\alpha_{1212}))_x/8 \\ (\alpha_6)_x &= (\alpha_{1112} + \alpha_{2212})_x/2 \\ (\alpha_7)_x &= (\alpha_{1112} - \alpha_{2212})_x/2\end{aligned} \tag{22}$$

For orthotropic materials $\alpha_6 = \alpha_7 = 0$ in specific directions, but for the free material this will not always be the case, so we analyze the more general case. Several extremum solutions for $(\alpha_{1111})_\theta$ may exist in the interval $0 \le \theta < \pi$. To locate the maximum of $(\alpha_{1111})_\theta$, the function (21) is numerically evaluated at a number of θ values (here chosen with increments $\Delta\theta = \pi/1800$). This can be done for each elements and θ_e is then the angle for the largest value $(\alpha_{1111})_{\theta e}$. The values of $(\alpha_{1111})_\theta$ has an upper bound of 1 and a lower bound of 1/3. This follows from the trace being 1, and having positive eigenvalues in this interval. This then also follows for the non-dimensional longitudinal stiffness. For high values of $(\alpha_{1111})_\theta$ a single fiber direction is approached and for lower values of $(\alpha_{1111})_\theta$ an isotropic material with zero Poisson's ratio material is approached.

Similar to (21) the remaining constitutive components with the θ-direction as reference direction may then be evaluated by

$$\begin{aligned}
(\alpha_{2222})_\theta =&(\alpha_{1111}+\alpha_{2222})_x/2-(\alpha_2)_x\cos(2\theta)-(\alpha_3)_x(1-\cos(4\theta))-\\
&(\alpha_6)_x 2\sin(2\theta)+(\alpha_7)_x\sin(4\theta)\\
(\alpha_{1122})_\theta =&(\alpha_{1122})_x+(\alpha_3)_x(1-\cos(4\theta))-(\alpha_7)_x\sin(4\theta)\\
(\alpha_{1212})_\theta =&(\alpha_{1212})_x+(\alpha_3)_x(1-\cos(4\theta))-(\alpha_7)_x\sin(4\theta)\\
(\alpha_{1112})_\theta =&(\alpha_2)_x\sin(2\theta)x/2-(\alpha_3)_x\sin(4\theta))+(\alpha_6)_x\cos(2\theta)+(\alpha_7)_x\cos(4\theta)\\
(\alpha_{2212})_\theta =&(\alpha_2)_x\sin(2\theta)x/2+(\alpha_3)_x\sin(4\theta))+(\alpha_6)_x\cos(2\theta)-(\alpha_7)_x\cos(4\theta)
\end{aligned} \tag{23}$$

all this well known from laminate theory.

3.2 Discussion on "invariant" parameters

The definitions of $(\alpha_2)_\theta, (\alpha_3)_\theta, (\alpha_6)_\theta), (\alpha_7)_\theta$ with reference to a specific θ-direction are defined by

$$\begin{aligned}
(\alpha_2)_\theta &= (\alpha_{1111}-\alpha_{2222})_\theta/2\\
(\alpha_3)_\theta &= (\alpha_{1111}+\alpha_{2222}-2(\alpha_{1122}+2(\alpha_{1212}))_\theta/8\\
(\alpha_6)_\theta &= (\alpha_{1112}+\alpha_{2212})_\theta/2\\
(\alpha_7)_\theta &= (\alpha_{1112}-\alpha_{2212})_\theta/2
\end{aligned} \tag{24}$$

and their numerical values may be different from the parameters in (22). The following relations are derived by inserting (21) and (23) in (24)

$$\begin{aligned}
(\alpha_2)_\theta &= (\alpha_2)_x\cos(2\theta)+(\alpha_6)_x 2\sin(2\theta) && (=(\alpha_2)_x \text{ for } \theta=0 \text{ and } \pi)\\
(\alpha_3)_\theta &= (\alpha_3)_x\cos(4\theta)+(\alpha_7)_x\sin(4\theta) && (=(\alpha_3)_x \text{ for } \theta=0 \text{ and } \pi)\\
(\alpha_6)_\theta &= (\alpha_6)_x\cos(2\theta)-(\alpha_2)_x\sin(2\theta)/2 && (=(\alpha_6)_x \text{ for } \theta=0 \text{ and } \pi)\\
(\alpha_7)_\theta &= (\alpha_7)_x\cos(4\theta)-(\alpha_3)_x\sin(4\theta) && (=(\alpha_7)_x \text{ for } \theta=0 \text{ and } \pi)
\end{aligned} \tag{25}$$

Material orthotropy imply zero of the following parameter combinations

$$\begin{aligned}
z_x &= (\alpha_7)_x(\alpha_2)_x^2-4(\alpha_7)_x(\alpha_6)_x^2-4(\alpha_6)_x(\alpha_3)_x(\alpha_2)_x\\
z_\theta &= (\alpha_7)_\theta(\alpha_2)_\theta^2-4(\alpha_7)_\theta(\alpha_6)_\theta^2-4(\alpha_6)_\theta(\alpha_3)_\theta(\alpha_2)_\theta
\end{aligned} \tag{26}$$

If z_x is zero, then the material is orthotropic and also z_θ is zero, because the condition (26) holds in any coordinate system. The derived functions(25) fulfills this, by setting $(\alpha_6)_x = (\alpha_7)_x = 0$.

The conclusion from the present analysis is that the parameters (24) as well as (22) should be termed practical parameters instead of invariant parameters.

3.3 Important anisotropy quantities

It is suggested for the constitutive matrices of an optimized design to present the following five distributions

- Largest longitudinal stiffness by $(\alpha_{1111})_{\theta e}$ for all elements e in a color plot, noting the limits $1/3 \leq (\alpha_{1111})_{\theta e} \leq 1$ with 1/3 for isotropy with zero Poisson's ratio and with 1 for unidirectional fiber.
- Direction of largest longitudinal stiffness θ_e for all elements e by directional lines, noting the limits $0 \leq \theta_e \leq \pi$. May be combined with the color plot above.
- Level of anisotropy by $2(\alpha_2)_{\theta e}$ = $(\alpha_{1111})_{\theta e} - (\alpha_{2222})_{\theta e}$ for all elements e in a color plot, noting the limits 0 and 1 with 1 for high level of anisotropy and 0 for symmetry.
- Relative importance of shear stiffness by $8(\alpha_3)_{\theta e}$ for all elements e in a color plot. High shear stiffness corresponds to negative values of $8(\alpha_3)_{\theta e}$, i.e. $4\alpha_{1212} > \alpha_{1111} + \alpha_{2222} - 2\alpha_{1122}$, as seen in (24). Alternatively, α_{1212} may be directly visualized.
- Test for material orthotropy by z_e for all elements e in a color plot. Only places with $z_e = 0$ have orthotropic material. The color plot relates to a scaled, squared test quantity with a lower limit to identity orthotropy.

3.4 Other matrices with similar rotational transformations

The present note written in relation to 2D constitutive matrices, is also valid for 2D structural stiffness matrices $[S]$, 2D structural flexibility matrices $[F]$, and 2D strength matrices in stress space $[H]$ or in strain space $[G]$. Also laminate stiffness sub-matrices and laminate flexibility sub-matrices may be analyzed similarly.

4. VISUALIZATION EXAMPLE FROM OPTIMAL ANISOTROPY

In Pedersen and Pedersen (2015) a cantilever (with fixed material at the tip) is optimized to maximize the first eigenfrequency. Without specifying here the details of analysis and optimization by iterative redesign, we visualize in Figure 1 the obtained constitutive matrices, as suggested above in Section 3.3

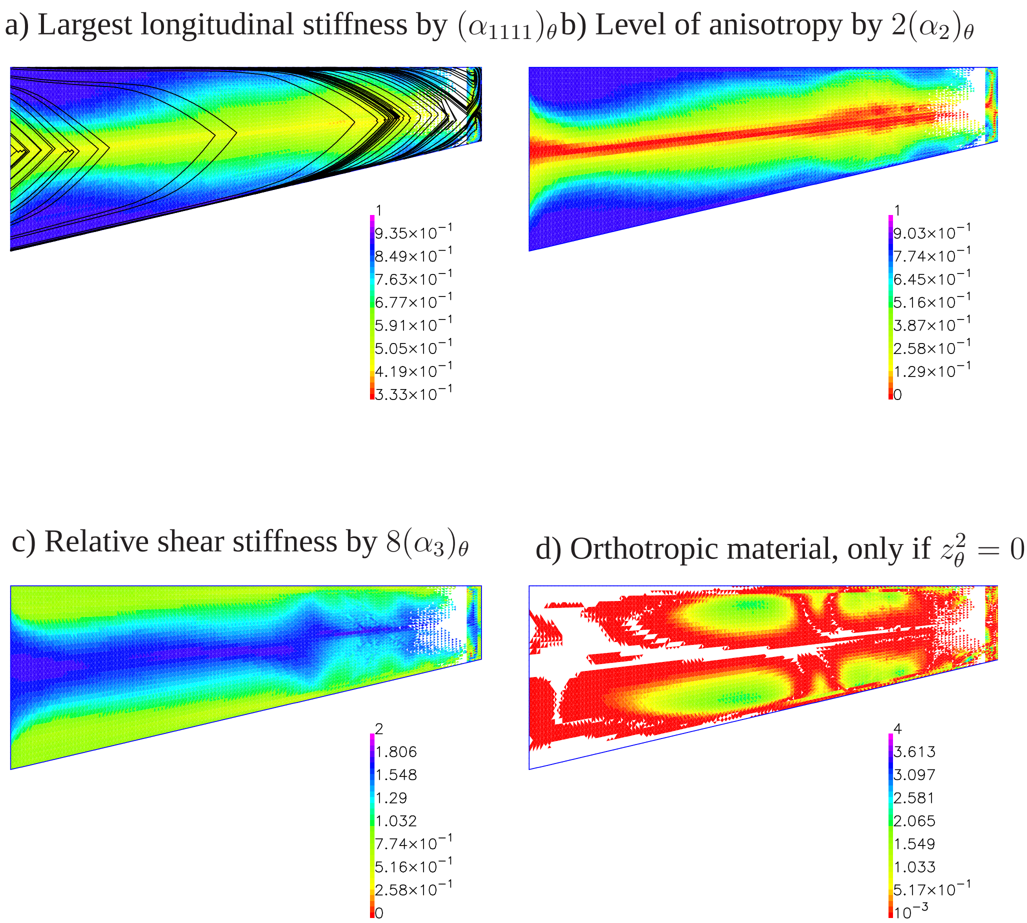

Figure 1: Visualization of distributions for constitutive matrices added direction of largest longitudinal stiffness in Figure 1a. The white spots near the tip of fixed material are places of minimum material density, being the same in all Figures 1a-d. The further white areas in Figure 1d contains materials classified as orthotropic.

Note, that the quantities in Figures 1a-c are measured in individual rotated coordinate systems that are visualized by the directions in Figure 1a.

For largest longitudinal stiffness in Figure 1a we see blue color (close to 0.85) at the upper and lower boundaries, yellow color (close to 0.45) at the "beam axis" and green color between these zones. All this as expected in relation to the most simple bending eigenmode. For direction of largest longitudinal stiffness added in Figure 1a, the 45 degrees at the "beam axis" and parallel to the upper and lower boundaries also agree with simple bending of a cantilever.

The distribution of level of anisotropy is visualized by $2(\alpha_2)_\theta$ in Figure 1b and show small relative values of $(\alpha_{2222})_\theta$ by blue color close to upper and lower boundaries, i.e., high level of anisotropy. Close to symmetry $(\alpha_{2222})_\theta \simeq (\alpha_{1111})_\theta$ by red color close to the "beam axis".

The relative importance of shear stiffness is visualized by distribution of $8(\alpha_3)_\theta$ in Figure 1c. For high relative shear stiffness this quantity will be negative. For the present case only positive values are found.

The distribution of possible orthotropy is visualized by a scaled z_θ^2 in Figure 1d, where the zero limit is set to 0.001. The white areas (away from the tip) are thus areas of material orthotropy, without showing the directions of orthotropy.

5. Conclusion

Visualization of results from optimal design may not be too complicated in traditional size, shape or topology design, but in free material optimization (FMO) constitutive matrices in the continuum or structural space are part of the obtained design. A visualization of such distribution of matrices for 2D problems with 6 different matrix components is demonstrated.

From laminate analysis, the formulation for rotational transformation is applied and is found useful. Practical parameters that usual are stated as invariants are an important part of this formulation, but the notion invariants needs to be discussed, because it only relates to a specific reference direction. The visualized distributions of physical quantities are; stiffest material direction, material stiffest longitudinal constitutive component, level of anisotropy, absolute or relative shear stiffness and orthotropy test.

Optimal design of material distribution is often effectively obtained by design iterations based on a stated optimality criterion. It is recently found that the optimal constitutive matrices (the anisotropy) is simply related to the actual strain field(s). Since this is not well known, it is chosen to shortly describe the theory behind this result as an introductory to the visualization aspects.

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