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1 Piecewise Smooth Systems

\( X = (X^u, X^f) \) with vector-fields \((X^u, X^f)\) is a piecewise smooth (PWS) system. \( \Sigma = \Sigma_1 \cap \Sigma_0 : f(x, y, z) = 0 \) is the switching manifold. Locally we take \( f(x, y, z) = y \) \( \Sigma \) is divided into sliding: \( \Sigma_u \) crossing, \( \Sigma_f \) and tangencies: \( T \), see Fig. (a). On \( \Sigma_u \) we adopt the Filippov convention [2] of sliding (see Fig. (b)) to obtain a vector-field \((X_u, \Sigma_u)\).

2 Singularities

\( p \in \Sigma \) is a singularity if \( X^p(X^p(p)) = 0 \). A tangency is a fold if \( X^p(X^p(p)) \neq 0 \), being visible when \( \epsilon > 0 \) (see Fig. (a)), invisible when \( \epsilon < 0 \) (see Fig. (c)). Here \( X^p\Sigma = X^u\Sigma = X^f \Sigma \) is the Lie-derivative. A two-fold \( p \in \Sigma \) is a fold from above and below:

\[ X^u(p) = X^f(p) = 0. \]

3 Two-Folds in \( \mathbb{R}^3 \)

Proposition. [3] Generically, a two-fold \( p \) in \( \mathbb{R}^3 \) is the transverse intersection of two lines \( r : x = y = 0, z \in \mathbb{R} \) and \( l : y = z = 0, x \in \mathbb{R} \). Let \( \bar{r}, \bar{l} \) be the largest and smallest eigenspaces of \( X^u\Sigma \), respectively. The lines \( \bar{r}, \bar{l} \) divide \( \Sigma : y = 0 \) into four separate regions:

- Stable sliding \( \Sigma^s : x \leq 0, z \leq 0 \)
- Unstable sliding \( \Sigma^u : x \geq 0, z \geq 0 \)
- Crossing downwards \( \Sigma^- : x \geq 0, z \leq 0 \)
- Crossing upwards \( \Sigma^+ : x \leq 0, z \geq 0 \)

See Fig. (d)-(f). A two-fold is:

- Visible if \( \bar{r}, \bar{l} \) both visible (Fig. (d)).
- Invisible-visible if \( \bar{r}, \bar{l} \) both invisible (Fig. (e)).
- Invisible if \( \bar{r} \) is both invisible (Fig. (f)).

Definition. A singular canard of a PWS is a trajectory of \((\Sigma_u, X^u)\) containing a continuation through the two-fold singularity \( p \).

The two-fold \( p \) is an equilibrium of the vector-field \( F_p X^u_p \) defined in \( \Sigma_u \cup (p) \cup \Sigma_f \) with \( F_p = H(x, z) \) a scalar smooth function which is positive (negative) for \( x, z < 0 \) (\( x, z > 0 \)). Then:

- Proposition. [3] Non-degenerate singular canards exist if and only if \( p \) corresponds to a node or a saddle of \( V^{\Sigma_u}_p \) and an eigenspace is contained within \( \Sigma_u \cup (p) \). See Fig. (g).

4 Regularization

- What happens to the two-fold/singular canards when we regularize the PWS system?
- Can we learn something about the PWS system by regularizing?

We consider the Sotomayor-Teixeira regularization [5]:

\[ X_\epsilon = \frac{1}{2} X^u(1 + \phi(\epsilon^{-1}y)) + \frac{1}{2} X^f(1 - \phi(\epsilon^{-1}y)), \]

with \( \epsilon < 1 \) (see Fig. (h) and (i)). Writing \( y = \epsilon \bar{y} \) we obtain a hidden slow-fast system with \( (x, \bar{y}) \) slow and \( y = \bar{y} \) fast.

Theorem. [3] \( X_0 \) has critical manifolds: \( \Sigma_0 = \bar{\Sigma}_0 \) (attracting), \( \Sigma_1 = \bar{\Sigma}_1 \) (repelling) and a non-hyperbolic line \( \bar{\rho} : x = 2, \bar{y} = \epsilon \) (see Fig. (j)) on \( \Sigma_0 \). Reduced system: Filippov sliding system.

Note that in terms of \( y = \epsilon \bar{y} \) we have \( \bar{\rho} = \rho \).

5 Blowup

To study the persistence of canards we blowup the non-hyperbolic line \( \bar{\rho} : x = 2, \bar{y} = \epsilon \) to obtain a scalar smooth function \( H(x, \bar{y}) \) following the formulation of Krupa and Szczepan. [4] We study the phase space using directional charts \( \bar{\kappa}_1 : x = 1, \bar{\kappa}_2 : x = 1, \bar{\kappa}_3 : x = 1 \). We obtain:

Theorem. [3] Singular canards \( \Rightarrow \) (Primary, maximal) Canards as transverse intersections of critical manifolds in slow manifolds \( \Sigma_0 \) and \( \Sigma_1 \), provided a certain non-resonance condition holds true. These maximal canards are O(\( \sqrt{\epsilon} \)) close to the singular canards.

Result and approach very similar to [6, 7] for folds in slow-fast systems in \( \mathbb{R}^3 \). But the geometry is very different.

6 Visible-Invisible Two-Fold

The two-fold is associated with forward and backwards non-uniqueness. By regularizing, we can pick the “right orbits”.

Theorem. Consider the visible-invisible case and suppose as in Fig. (k) that there exists a singular cycle \( \rho_0 \) satisfying certain non-degeneracy conditions, see also [1]). Then for \( \epsilon \ll 1 \) sufficiently small \( X_0 \) possesses an attracting limit cycle \( \rho_0 \) satisfying \( \rho_0 = \rho_0 + O(\sqrt{\epsilon}) \).

PWS orbit \( \rho_0 \) is therefore distinguished, as \( \rho_0 = \lim_{\epsilon \to 0} \rho_0 \), among all the orbits through \( p \). Note that these results hold true for all monotone regularization functions.

References