Uncertainty quantification of critical speed for railway vehicle dynamics

*PhD student:* D. Bigoni

*Supervisors:* A. P. Engsig-Karup, H. True, J.S. Hesthaven

1 The Technical University of Denmark, 2 Brown University
Railway vehicle dynamics - Euler’s formulation

\[
m \ddots \mathbf{x}_1 + 2D_2 \ddot{x}_1 + 2k_4 \dot{x}_1 + 2F_X (\xi_{x1}, \xi_{y1}) + 2F_X (\xi_{x2}, \xi_{y2}) = 0
\]

\[
I \ddots \mathbf{x}_2 + k_6 \ddot{x}_2 + 2ha [F_X (\xi_{x1}, \xi_{y1}) - F_X (\xi_{x2}, \xi_{y2})] +
\]

\[
+ a [F_Y (\xi_{x1}, \xi_{y1}) + F_Y (\xi_{x2}, \xi_{y2})] = 0
\]

where \(F_X\) and \(F_Y\) are the creep forces, and determine a non-linear coupling of \(x_1\) and \(x_2\). Among other components, these forces involve also the running velocity \(v\) of the vehicle, the conicity of the wheels and the wheel-rail friction.

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\(^1\)H. True and C. Kaas-Petersen 1983
Railway vehicle dynamics - Hunting

DTU Informatics, Technical University of Denmark

Uncertainty quantification of critical speed for railway vehicle dynamics
Railway vehicle dynamics - Stochastic Model

Let’s now assume that the suspension components $k_6$, $k_4$ and $D_2$ are known within a certain level of accuracy and model this by:

$$k_6 \sim \mathcal{N}(3.44 \cdot 10^6, 2.96 \cdot 10^{10})$$  \quad (\text{std. of approx. 5%})

$$k_4 \sim \mathcal{N}(9.12 \cdot 10^4, 4.15 \cdot 10^7)$$  \quad (\text{std. of approx. 7%})

$$D_2 \sim \mathcal{N}(1.46 \cdot 10^4, 1.07 \cdot 10^6)$$  \quad (\text{std. of approx. 7%})

What are the dynamics of the system under these conditions?
Uncertainty Quantification - Traditional Approaches

Analytical Methods
- Moment Equation
- Perturbation Method
  *Pros.*: recover the exact solution  
  *Cons.*: problem-dependent, cumbersome

Sampling Methods
- (MC) Monte Carlo – $O \left( N^{-1/2} \right)$
- (QMC) Quasi Monte Carlo – $O \left( (\log N)^d / \sqrt{N} \right)$
- (MCMC) Markov Chain Monte Carlo
  *Pros.*: general applicability, MC convergence independent from dimensionality $d$  
  *Cons.*: very slow convergence

Figure: Linear critical speed distribution using $10^4$ realizations for MC method.
UQ - Generalized Polynomial Chaos (gPC)²

Let \( Y \) be a r.v. with CDF \( F_Y(y) \). Use the \( N \)th-degree gPC expansion of the random parameters and the solution

\[
Y_N = \sum_{k=0}^{N} \hat{a}_k \Phi_k(Z), \quad \hat{a}_k = \frac{1}{\gamma_k} \int_{I_Z} F_Y^{-1}(F_Z(z)) \Phi_k(z) dF_Z(z)
\]

\[
u_N(t, Z) = \sum_{k=0}^{N} \hat{\nu}_k(t) \Phi_k(Z)
\]

\[
\begin{cases}
\mathbb{E} [\partial_t u_N(t, Z) \Phi_k(Z)] = \mathbb{E} [f(u_N) \Phi_k(Z)], & D \times (0, T) \\
\hat{\nu}_k(0) = \frac{1}{\gamma_k} \mathbb{E} [u(0, Z) \Phi_k(Z)], & D \times \{t = 0\}
\end{cases}
\]

\[
\mu_u(t) \approx \mathbb{E} [u_N(t, Z)] = \hat{\mu}_0(t)
\]

\[
\text{Var} [u(t, Z)] \approx \text{Var} [u_N(t, Z)] = \sum_{k=1}^{N} \gamma_k \hat{\nu}_k^2(x, t)
\]

where \( \mathbb{E} [f(Z)] = \int_{I_Z} f(z) dF_Z(z) \) and \( \{\phi_i(Z)\}_{i=0}^{N} \) are proper orthonormal basis.

---

²D.Xiu and G.Karniadakis 2004
The $N$-th order gPC expansion of the problem is given by

$$
\begin{align*}
\mathbb{E} [\partial_t u_{1,N} \phi_k] &= \mathbb{E} [u_{2,N} \phi_k] \\
\mathbb{E} [\partial_t u_{2,N} \phi_k] &= -2\mathbb{E} [D_{2,N} u_{2,N} \phi_k] - 2\mathbb{E} [k_{4,N} u_{1,N} \phi_k] \\
&\quad -2\mathbb{E} [(F_X(\xi_{x1},\xi_{y1}) + F_X(\xi_{x2},\xi_{y2})) \phi_k] \\
\mathbb{E} [\partial_t u_{3,N} \phi_k] &= \mathbb{E} [u_{4,N} \phi_k] \\
\mathbb{E} [\partial_t u_{4,N} \phi_k] &= -\mathbb{E} [k_{6,N} u_{3,N} \phi_k] - 2ha\mathbb{E} [(F_X(\xi_{x1},\xi_{y1}) - F_X(\xi_{x2},\xi_{y2})) \phi_k] \\
&\quad -a\mathbb{E} [(F_Y(\xi_{x1},\xi_{y1}) + F_Y(\xi_{x2},\xi_{y2})) \phi_k]
\end{align*}
$$

where $k$ is a multi index such that

$$
u_{i,N}(t,Z) = \sum_{|k| \leq N} \hat{u}_k(t)\Phi_k(Z), \quad i = 1, \ldots, 4
$$

We obtain a system of $K = \sum_{i=0}^{N} \binom{i + (d - 1)}{(d - 1)}$ coupled equations that can be treated using standard ODE solvers. The following table shows how this number scales:

<table>
<thead>
<tr>
<th>$N$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 1$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
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<tr>
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<td>70</td>
<td>126</td>
<td>210</td>
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<tr>
<td>$d = 5$</td>
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<td>126</td>
<td>252</td>
<td>462</td>
<td>792</td>
</tr>
</tbody>
</table>
UQ - gPC on Railway Vehicle Dynamics

(a) Mean and variance

(b) Variance

Pros: Elegant formulation, one single solution of the system, optimal accuracy

Cons: Intrusive and cumbersome to implement, non-linearities must be treated carefully, weak on time-dependent problems (but there exist improvements).
UQ - Probabilistic Collocation Methods (PCM)

Solve the deterministic ODE on a "proper" set $\Theta_M = \{Z^{(j)}\}_{j=1}^M$ of nodes in the random space:

$$\begin{cases}
\partial_t u(t, Z^{(j)}) = f(u), & D \times (0, T] \\
u(0) = u_0, & D \times \{t = 0\}
\end{cases}$$

This will give $u^{(j)} = u(t, Z^{(j)})$ solutions on which we can apply interpolation rules or projection rules. Let’s consider the discrete projection:

$$u_N(Z) = \sum_{|k| \leq N} \hat{u}_k(t) \Phi_k(Z)$$

$$\hat{u}_k(t) = \frac{1}{\gamma_k} \mathbb{E} [u(t, Z) \phi_k(Z)] = \frac{1}{\gamma_k} \int u(z) \phi_k(z) dF_Z(z)$$

where the integral can be computed by cubature rules using the "properly" selected set of nodes $\Theta_M$. Then statistics can be easily obtained:

$$\mu_u(t) \approx \mathbb{E} [u_N(t, Z)] = \hat{u}_0(t)$$

$$\text{Var} [u(t, Z)] \approx \text{Var} [u_N(t, Z)] = \sum_{|k| \leq N} \gamma_k \hat{u}_k^2(x, t)$$

Target: obtain the "best" statistics out of the smallest number of simulation!
Hermite polynomials are chosen as basis for the projection/cubature. Projection with these polynomials can be highly accurate, using proper Gauss quadrature nodes and weights, for which analytical formulas exist.

Figure: PCM convergence to highest accuracy (mean and variance).

Figure: PCM vs. Monte Carlo

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UQ - PCM for Critical Speed statistics

Let's extend the dynamical system in order to obtain a “controlled” ramping method:

\[
\begin{align*}
\dot{u}_1 &= u_2 \\
m\dot{u}_2 &= -2D_2 u_2 - 2k_4 u_1 - 2F_X(\xi_x, \xi_y) - 2F_X(\xi_x, \xi_y) \\
\dot{u}_3 &= u_4 \\
I\dot{u}_4 &= -k_6 u_3 - 2ha \left[ F_X(\xi_x, \xi_y) - F_X(\xi_x, \xi_y) \right] - \\
&\quad \quad - a \left[ F_Y(\xi_x, \xi_y) + F_Y(\xi_x, \xi_y) \right] \\
\dot{v} &= \begin{cases} 
0 & \text{if } t < t_{st} \lor \|\vec{u}\|_2 < \varepsilon_{min} \\
-\|\vec{u}\|_2 & \text{if } \|\vec{u}\|_2 < \varepsilon_{max} \\
-\varepsilon_{max} & \text{otherwise}
\end{cases}
\end{align*}
\]
Outlook - Future work

UQ on Railway vehicle dynamics
• Uncertainty quantification with Sparse Grids
• Uncertainty quantification on a realistic model
• Parameter space compression and compressed sensing

UQ on Free Water Wave Dynamics
• Parametrization of random fields

Other applications of Uncertainty Quantification