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A simple stationary semi-analytical wake model

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Abstract (max. 2000 char.): We present an idealized simple, but fast, semi-analytical algorithm for computation of *stationary* wind farm wind fields with a possible potential within a multi-fidelity strategy for wind farm topology optimization.

Basically, the model considers wakes as *linear perturbations* on the ambient non-uniform mean wind field, although the modelling of the individual stationary wake flow fields includes non-linear terms. The simulation of the individual wake contributions are based on an analytical solution of the thin shear layer approximation of the NS equations. The wake flow fields are assumed *rotationally symmetric*, and the rotor inflow fields are consistently assumed uniform.

Expansion of *stationary* wake fields is believed to be significantly affected by meandering of wake deficits as e.g. described by the Dynamic Wake Meandering model. In the present context, this effect is approximately accounted for by imposing suitable empirical downstream boundary conditions on the wake expansion that depend on the rotor thrust and the ambient turbulence conditions, respectively. For downstream distances beyond approximately 10 rotor diameters (at which distance the calibrated wake expansion boundary conditions are imposed), the present formulation of wake expansion is believed to underestimate wake expansion, because the analytical wake formulation dictates the wake expansion to behave as $x^{1/3}$ with downstream distance, whereas wake expansion as primary controlled by wake meandering develops approximately linearly with the downstream distance.

The link from a non-uniform wind farm wind field, consisting of *linear perturbations* on the ambient non-uniform mean wind field, to a fictitious uniform wake generating inflow field is established using two different averaging approaches – a linear and a non-linear. With each of these approached, a parabolic system are described, which is initiated by first considering the most upwind located turbines and subsequently successively solved in the downstream direction. Algorithms for the resulting wind farm flow fields are proposed, and it is shown that in the limit of very large downstream distances, the simulated field is shown to recover to the undisturbed ambient wind field upstream the wind farm.

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Preface

This report constitutes part of the Work Package 1 (WP1) reporting associated with the EU project "Next Generation Design Tool for Optimization of Wind Farm Topology and Operation" with the acronym TOPFARM. WP1 concerns description the internal wind farm wind field/climate as affected by wakes generated by the wind farm turbines.

1 Introduction

A wind turbine *wake* is characterized by a mean wind speed decrease (i.e. a wake deficit) and an increased turbulence level behind the turbine. Due to upstream emitted wakes, the inflow wind conditions for wind farm turbines is therefore substantially modified compared to the ambient wind field that apply for stand-alone wind turbines. This drastic change of the environmental wind field conditions has implications not only for the *power production* of a wind farm but also, and equally important, for the *loading* conditions of the individual turbines within the farm.

To achieve the maximum economic output from a wind farm, the layout and control of the wind farm has to be optimized on a rational background. The TOPFARM project addresses *optimization of wind farm topology and control strategies based on aero-elastic modeling of loads as well as of power production*. Crucial factors in this connection are the overall (ambient) wind climate at the wind farm site, the position of the individual wind turbines, the wind turbine characteristics, the wind turbine control/operation strategy for wind turbines interacting through wakes, and possible a priory defined constraints imposed on the wind farm topology.

To enable both *loading*- and *production* aspects to be included in a realistic and coherent framework in the optimization procedure without posing outrageous requirements on computer resources, a major challenge is to develop fast approximate sub-models and yet preserve the essential physics of the problem. As for the detailed simulation of the wind farm wind field, the Dynamic Wake Meandering (DWM) model [1], [2], [3] offers the requested features. The core of this model is a *split of scales* in the wake flow field, with large scales being responsible for stochastic *wake meandering*, and small scales being responsible for wake *attenuation* and *expansion* in the meandering frame of reference as caused by turbulent mixing. While preserving the *in-stationary* character of the wake flow field, the DWM approach is nevertheless orders of magnitude faster than comparable full non-linear in-stationary 3D CDF based models, even when these are simplified by approximating the turbines using an Actuator Line (ACL) approach [4], [5].

However, even with the computational speed gained from the DWM approach, the turnaround time for the complete wind farm optimization may be unsatisfactory, and supplementing strategies for computer savings in turn investigated. There is (at least) two philosophies with particular interest in this respect [6] – the use of *structured optimization grids*, and the use of *variable fidelity* modeling. The present report relates to the latter of these. Variable fidelity modeling is an approach often seen in applied aeronautics. It means e.g. that an *approximate and fast* model is used for a vast majority of the parameter evaluations needed for a design study, while a more *detailed and accurate* model is used in regions of specific interest. It is, however, important that the *low fidelity* describes the same physics as the *high fidelity* model, but in a coarser and more approximate way. It is straight forward to generalize the method to more fidelity levels.

For the TOPFARM application, the highest fidelity level will be constituted by the DWM approach. A more crude approach, thus constituting a lower fidelity approach, could be based on a *stationary* description of the internal wind farm wind field (i.e. a statistical description of the mean flow features). Although this approach eliminates the possibility for a detailed explicit description of the wind turbine loading, it is believed to offer a fair approximation of the wind farm production. In addition, when applied in the micro siting optimization context, it is also believed implicitly to reflect at least elements of unfavorable turbine loading, because optimization of the wind farm production relates to minimizing the mutual wake shadowing effects, which in turn relates to the wind farm flow conditions that are responsible for increased wind farm wind turbine fatigue loading.

The focus in the present report is on a simple stationary semi-analytical wake model being a candidate for a lower fidelity wind farm wind climate model. Being on closed form, the model requires a minimum of computational resources. The model is constituted by a stationary model for solitary wakes along with a simple engineering recipe for subsequent wake superposition.

2 Single wake model

We simplify the problem by considering the wake as a perturbation on a mean flow, which besides conventional shear may include "wake shear" contributions from upstream emitted wakes. The *apparent mean flow* thus develop downstream in the wind farm both due to conventional shear (i.e. due to e.g. roughness changes) and due to wake contributions, which expand in space and therefore attenuate with increasing downstream distance from the wake emitting turbines. This procedure implicitly implies a *linearity assumption* allowing for linear superposition of the relevant mean flow and the wake contributions, although not as consistent/stringent as the linear perturbation based approach suggested by Ott in [7].

Noticing the free jet character of wind turbine wakes, with gradients of mean flow quantities being much bigger in the radial direction (denoted by r) than in the axial along wind direction (denoted by x), leads to a wake model formulation based on the *thin shear layer approximation* of the Navier-Stokes (NS) equations. With this basis, further simplification is gained from a dimension reduction (from three to two) resulting from an assumption of rotational symmetry of the wake deficit. The proposed dimension reduction is motivated by the fact that it opens for an asymptotic closed form solution [8], and is partly justified by the detailed CFD based investigations performed in [9] (demonstrating that the organized flow structures around the individual wind turbine blades are azimuth averaged within 1 rotor diameter (1D) downstream) and in [10] (showing that the wake deficit, as resolved in a meandering frame of reference, is rather rotationally symmetric even in *sheared inflow*).

With the assumption of rotational symmetry follows in turn implicitly the need for a uniform or rotationally symmetric inflow field for the wake deficit prediction – in practice though a uniform inflow field. Different philosophies for suitable/consistent estimation of such pseudo-uniform inflow fields are described in Section 3. Note, that the pseudo-uniform inflow fields are only used for the wake deficit prediction, and they contribute otherwise not to the description of the wind farm flow field – i.e. the resulting flow field are obtained by superposition of the non-uniform mean flow and the predicted wake deficits.

Neglecting viscous terms and presuming homogeneous and incompressible fluid conditions (whereby the leading equations become invariant with respect to the fluid mass density) the downstream development of the individual wake deficit is finally governed by the following system of equations

$$\frac{\partial\left(\Delta\overline{U}r\right)}{\partial x}+\frac{\partial\left(\Delta\overline{V}r\right)}{\partial r}=0\ , \tag{1}$$

$$\begin{aligned}\left(\overline{U}+\Delta\overline{U}\right)\frac{\partial\Delta\overline{U}}{\partial x}+\Delta\overline{V}\,\frac{\partial\Delta\overline{U}}{\partial r}&=-\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r\langle uv\rangle\right)-\frac{1}{\rho}\frac{\partial p}{\partial x}\\&\cong-\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(l^2 r\left(\frac{\partial\Delta\overline{U}}{\partial x}\right)^2\right),\end{aligned} \tag{2}$$

with $\overline{U}$ denoting the pseudo-uniform inflow velocity, l denoting Prandtl's mixing length, p denoting the pressure, brackets denoting ensemble averaging, $\Delta\overline{U}$ and $\Delta\overline{V}$ being the wake perturbations of the inflow in the axial- and radial directions, respectively, and (u,v) being the corresponding/associated turbulence components. For the closed form asymptotic solution of equations (1) and (2), the involved terms are classified according to their magnitude using a wake deficit similarity approach [8], and as a result it is recognized that the pressure term will not contribute to neither the first order solution, nor the second order solution.

With the proposed approach, the energy dissipation caused by the Reynold's stresses are approximately included by adopting Prandtl's mixing length hypothesis to account for the wake self induced turbulence, however, discharging the contribution from the atmospheric turbulence. Although not included directly in the system of equations, the diffusion effects caused by the atmospheric turbulence are nevertheless implicitly accounted for by an ad hoc adjustment of the boundary conditions for the problem (cf. equation (9)).

Due to the linearity of the divergence operator, the continuity equation is automatically fulfilled for the sum of two fields each obeying this equation, and thus consequently for the sum of the non-uniform mean wind field and the wake perturbation. This is contrary to the momentum equation. Here the replacement of the mean wind field with the pseudo-uniform inflow field for the wake perturbation prediction, leads to a resulting flow field that does not exactly satisfy the momentum balance.

Adopting the *similarity assumption* for the shape of the wake deficit, the first and second order solutions are obtained as closed form solutions of two second order ordinary differential equations. This solution is asymptotically correct in the far field. The associated two arbitrary constants are determined by imposing suitable boundary conditions – i.e. by equating the wake extension partly to the value at the rotor plane after (assumed instantaneous) pressure recovery (using a Bernoulli equation based approximation) and partly to the *mean* value in a fixed frame of reference 9.6D downstream as defined from analysis of full scale experiments, respectively. The latter boundary condition explains the wording "semi-analytical" in the title of this report. The wake $\Delta\overline{U}$ component is of primary importance and, although obtained, we will not use the wake $\Delta\overline{V}$ component in the present context.

The wake extension, expressed in terms of the wake radius, R_w, and the wake deficit in the mean wind direction, $\Delta\overline{U}$, expressed as the sum of a 1. order contribution and a 2. order contribution, are thus finally determined according to the following expressions [11], [9]:

$$R_w(x) = \left(\frac{105c_1^2}{2\pi}\right)^{1/5}\left(C_T A(x+x_0)\right)^{1/3} \quad , \tag{3}$$

and

$$\Delta\overline{U}(x,r) = \Delta\overline{U}_1(x,r) + \Delta\overline{U}_2(x,r) \quad , \tag{4}$$

where x and r denote axial and radial coordinate directions, and $\Delta\overline{U}_1$ and $\Delta\overline{U}_2$ are the first order- and the second order wake deficit contributions, respectively. A is the rotor area, C_T denotes the rotor drag coefficient, and c_1 and x_0 are parameters defined by

$$x_0 = \frac{9.6D}{\left(\frac{2R_{9.6}}{kD}\right)^3 - 1} \quad , \tag{5}$$

$$c_1 = \left(\frac{kD}{2}\right)^{5/2}\left(\frac{105}{2\pi}\right)^{-1/2}\left(C_T A x_0\right)^{-5/6} \quad , \tag{6}$$

where D is the diameter of the wake generating rotor, and

$$k = \sqrt{\frac{m+1}{2}} \quad , \tag{7}$$

$$m = \frac{1}{\sqrt{1-C_T}} \quad . \tag{8}$$

The wake radius at 9.6D downstream distance, $R_{9.6}$, is (empirically) approximated from the following expression

$$R_{9.6} = a_1\, exp\left(a_2 C_T^2 + a_3 C_T + a_4\right)\left(b_1 I_a + 1\right)D \quad , \tag{9}$$

with I_a being the ambient turbulence intensity, and the empirically determined coefficients given by

$$\begin{aligned} a_1 &= 0.435449861 \quad , \\ a_2 &= 0.797853685 \quad , \\ a_3 &= -0.124807893 \quad , \\ a_4 &= 0.136821858 \quad , \\ b_1 &= 15.6298 \quad . \end{aligned} \tag{10}$$

The first order contribution to the wake deficit is expressed as

$$\begin{aligned} \Delta\overline{U}_1(x,r) = &-\frac{\overline{U}}{9}\left(C_T A (x+x_0)^{-2}\right)^{1/3} \\ &\times\left[r^{3/2}\left(3c_1^2 C_T A(x+x_0)\right)^{-1/2} - \left(\frac{35}{2\pi}\right)^{3/10}\left(3c_1^2\right)^{-1/5}\right]^2, \end{aligned} \tag{11}$$

whereas the second order contribution is given by

$$\Delta\overline{U}_2(x,r)=\overline{U}\left(C_T A(x+x_0)^{-2}\right)^{2/3}\sum_{i=0}^{4} d_i z(x,r)^i \quad , \tag{12}$$

with

$$z(x,r)=r^{3/2}\left(C_T A(x+x_0)\right)^{-1/2}\left(\frac{35}{2\pi}\right)^{-3/10}\left(3c_1^2\right)^{-3/10} , \tag{13}$$

and

$$\begin{aligned} d_0 &= \frac{4}{81}\left[\left(\frac{35}{2\pi}\right)^{1/5}\left(3c_1^2\right)^{-2/15}\right]^6 \\ &\times\left(-1-3\left(4-12\left(6+27\left(-4+\frac{48}{40}\right)\frac{1}{19}\right)\frac{1}{4}\right)\frac{1}{5}\right)\frac{1}{8}, \end{aligned} \tag{14}$$

$$\begin{aligned} d_1 &= \frac{4}{81}\left[\left(\frac{35}{2\pi}\right)^{1/5}\left(3c_1^2\right)^{-2/15}\right]^6 \\ &\times\left(4-12\left(6+27\left(-4+\frac{48}{40}\right)\frac{1}{19}\right)\frac{1}{4}\right)\frac{1}{5} \quad , \end{aligned} \tag{15}$$

$$\begin{aligned} d_2 &= \frac{4}{81}\left[\left(\frac{35}{2\pi}\right)^{1/5}\left(3c_1^2\right)^{-2/15}\right]^6 \\ &\times\left(6+27\left(-4+\frac{48}{40}\right)\frac{1}{19}\right)\frac{1}{4} \quad , \end{aligned} \tag{16}$$

$$d_3 = \frac{4}{81}\left[\left(\frac{35}{2\pi}\right)^{1/5}\left(3c_1^2\right)^{-2/15}\right]^6\left(-4+\frac{48}{40}\right)\frac{1}{19} \quad , \tag{17}$$

$$d_4 = \frac{4}{81}\left[\left(\frac{35}{2\pi}\right)^{1/5}\left(3c_1^2\right)^{-2/15}\right]^6 \frac{1}{40} \quad , \tag{18}$$

Considering the approximate character of the proposed stationary wake deficit prediction, only the first order model is considered to be of practical importance. The second order model may thus be neglected for the present purpose, whereby the wake deficit prediction simplify to the expression given in equation (11). Referring to equations (5) and (6), it is noted that the imposed boundary conditions basically relate to the definition of the origin of the along wind coordinate, and to the magnitude of the turbulent mixing (i.e. the Reynolds stresses) expressed in terms of the mixing length.

3 Wake superposition

Even for simple wind farm topologies, the wind field inside the wind farm is usually characterized by multiple wake situations for part of the time. Various methods for “superposition” of individual wakes are reported in the literature [12], [13]. Most of these are of a purely empirical character without any direct physical interpretation. In the present report, we will, as stated in Section 2, consider wakes as a *linear perturbation* on the ambient mean wind field. This has, as already mentioned, implications for the exact fulfilment of the momentum equation. However, for the *stationary* approach considered in the present report the wake expansion is amplified compared to the more physical correct *unsteady* situation, where only a modest expansion of the free shear (i.e. wake) flow is taken place in the meandering frame of reference [14], [3]. This in turn implies a corresponding increased attenuation of the of the wake deficit magnitude compared the unsteady wake case, which partly motivates the selected linear approach that in the limit of infinitesimal wake magnitudes is correct [15].

Before formulating the proposed wake superposition algorithm, we will return to a definition of the pseudo-uniform mean inflow of the individual wake-generating rotors. First, we will rewrite equation (11) to emphasize an analogy between the parametric dependence of wake deficit on the inflow wind speed and the rotor thrust, respectively. By isolating C_T in expression (6) for the wake deficit, the wake deficit expression is reformulated as

$$\Delta\overline{U}_1(x,r) = -\frac{\overline{U}C_T}{9}\left(A(x+x_0)^{-2}\right)^{1/3}\left[r^{3/2}\left(3\tilde{c}_1^{\,2}A(x+x_0)\right)^{-1/2} - \left(\frac{35}{2\pi}\right)^{3/10}\left(3\tilde{c}_1^{\,2}\right)^{-1/5}\right]^2, \quad (19)$$

with

$$\tilde{c}_1 = c_1 C_T^{5/6} = \left(\frac{kD}{2}\right)^{5/2}\left(\frac{105}{2\pi}\right)^{-1/2}\left(Ax_0\right)^{-5/6} \quad . \quad (20)$$

From equation (19) it is clear that an uncertainty in the mean inflow velocity has the same consequence for the wake deficit magnitude as an uncertainty in the rotor thrust coefficient. Therefore two different approaches for definition of the pseudo-uniform rotor field, associated with a *physical* inflow field with non-uniform spatial characteristics, are proposed

The *first approach* relates to the geometric average of the inflow wind speed over the rotor area expressed by

$$\overline{U} = \frac{1}{A}\int_A U\,dA \quad , \quad (21)$$

where U denotes the non-uniform inflow field. This is the logical choice if only the inflow wind speed was to be taken into consideration. However, as stated above, the rotor thrust coefficient is equally important.

Due to difficulties in obtaining the rotor trust from measurements, the rotor thrust coefficient is often obtained from simulations. In these simulations the mean inflow wind field is often taken as uniform or with very moderate shear, thus essentially closely approximating the situation of a uniform inflow field.

However, for the internal wind farm wind field the deviations from a uniform inflow situation (i.e. the deviations between the physical mean inflow field and a uniform reference wind speed at the spatial points of the rotor) may be significantly larger over the rotor area than assumed in the computations defining the rotor thrust curve. Because the kinetic energy of the inflow scales with the flow velocity cubed, the energy available over the rotor area is larger for sheared flows than for uniform flows with identical rotor averaged mean wind speeds. This in principle call on rotor thrust coefficients computed for the particular type of inflow shear considered and referenced to e.g. the rotor averaged mean wind speed. This is not a simple task [18], and for the present purpose a crude heuristic approach is therefore suggested.

Basically, we assume that the requested thrust properties can be derived from the traditionally computed thrust curves by a suitable adjustment of the *reference wind speed*. This is the focal point of the *second approach*. Adopting the Blade Element Momentum (BEM) algorithm [16], the rotor flow is presumed to be confined to independent stationary and laminar filamentous flow regimes directed along the flow stream lines. Excluding recirculation phenomena, the infinitesimal thrust contribution, dT, from one of these filaments is given by

$$dT = \frac{1}{2}\rho C_T U^2 dA \ , \tag{22}$$

where C_T is the rotor thrust coefficient corresponding to the sought reference wind speed applying the conventional rotor thrust curve. The aggregated rotor thrust measure is thus given by

$$T = \frac{1}{2}\rho C_T \int_A U^2 dA \equiv \frac{1}{2}\rho C_T \overline{U}^2 A \ , \tag{23}$$

whereby a suitable reference wind speed for the rotor thrust coefficient may be obtained as

$$\overline{U} = \sqrt{\frac{1}{A}\int_A U^2 dA} \ . \tag{24}$$

We note that the pseudo uniform reference wind speed defined by (24) is equivalent to the average wind speed defined in [17], however, motivated differently.

Referring to the presumed linear perturbation approach, we imagine a *parabolic system*, where initially the pseudo-uniform wind field, associated with the undisturbed ambient mean wind speed profile, is defined in terms of either equation (21) or equation (24). Subsequently, the relevant wake affected pseudo-uniform wind fields (i.e. the pseudo-fields at the relevant rotor positions) are determined in an iterative manner, starting with from the most upstream located rotors and progressing graduately downstream. These pseudo-fields are only used for computation of the individual (non-uniform) wakes. At a given downstream position the contributing upstream generated wake effects are superimposed linearly on the undisturbed ambient wind field to obtain the *non-uniform* wind farm wind field emulating the stationary physical wind farm wind field. We will describe the pseudo-uniform wind fields, and the derived wind farm wind fields, originating from respectively the linear averaging operator (cf. equation (21)) and the non-linear averaging operator (cf. equation (24)) separately in the following.

3.1 Wind farm wind field – linear approach

With the geometric average defined by equation (21) being a linear operator, we can calculate separately the contributions to the wind farm wind field originating from the undisturbed ambient field and the relevant individual wakes, respectively. This is also the case for the involved pseudo-uniform inflow fields, which is a considerable simplification.

Ambient contribution

The undisturbed ambient wind field is assumed horizontal homogeneous and given in terms of a vertical Cartesian coordinate, *z*, with origin at the ground surface, as

$$U = f(z) = \frac{u_*}{\kappa} Ln(z / z_0) , \tag{25}$$

where u_* is the friction velocity, κ is von Karman's constant (~ 0.4) and z_0 is the terrain roughness length. The mean of the undisturbed inflow field over the rotor area is thus equivalent to a weighted vertical averaging of the wind shear profile, with the weighting defined by the shape of the rotor area. For convenience, we will formulate the average wind field over the rotor in polar coordinates.

$$\begin{aligned} \overline{U} &= \frac{1}{A}\int_A f(z)dx\,dz = \frac{1}{A}\int_A r\ f(z_c + r\sin\theta)dr\,d\theta \\ &= \frac{1}{A}\int_0^R \int_0^{2\pi} r\frac{u_*}{\kappa} Ln([z_c + r\sin\theta]/z_0)dr\,d\theta\ , \end{aligned} \tag{26}$$

with z_c being the turbine hub height and *R* the rotor diameter. As a closed form solution of this integral is not available, and we will therefore solve expression (26) numerically, which moreover offers the flexibility to replace the horizontal mean wind speed profile with another formulation than the logarithmic.

A simple and fast numerical algorithm for evaluation of the integral defined in equation (26) is offered by the Gauss integration approach. Assuming that the *f*-function, in both the radial direction and the azimuthal direction, is well approximated by a (2N-1)'th order polynomial, the requested integral is evaluated applying a N point Gauss integration[1]. To this end, we introduce the following variable transformations

$$\begin{aligned} \tilde{r} &= \frac{2r}{R} - 1\ , \\ \tilde{\theta} &= \frac{\theta}{\pi} - 1\ . \end{aligned} \tag{27}$$

These transformations are introduced in order to bring the integral defined in equation (26) on the standard form for Gauss integration; i.e. with the independent variables in

[1] A N'th order Gauss integration integrates exactly an (2N-1)'th order polynomial.

the range between -1 and 1. Introducing transformations (27) into equation (26), we obtain

$$\overline{U} = \frac{1}{A}\int_{-1}^{1}\int_{-1}^{1} f\left(z_c + \frac{R(\tilde{r}+1)}{2} sin\left(\pi\left[\tilde{\theta}+1\right]\right)\right)\frac{\pi R^2(\tilde{r}+1)}{4} d\tilde{r}\, d\tilde{\theta} \ , \tag{28}$$

which by N point Gauss integration may be approximated by

$$\overline{U} = \frac{\pi R^2}{4A}\sum_{j=1}^{N}\sum_{i=1}^{N} w_j w_i f\left(z_c + \frac{R(\tilde{r}_i+1)}{2} sin\left(\pi\left[\tilde{\theta}_j+1\right]\right)\right)(\tilde{r}_i+1) \ , \tag{29}$$

with w_i and w_j being the Gauss weights. In the actual case we consider the integral (26) as well approximated by a forth order Gauss integration with the Gauss weights and the associated values of the integration variables given by

$$\begin{aligned}
&\tilde{r}_1 = \tilde{\theta}_1 = -0.339981043584856 \ , w_1 = 0.652145154862546 \ ; \\
&\tilde{r}_2 = \tilde{\theta}_2 = -0.861136311594053 \ , w_2 = 0.347854845137454 \ ; \\
&\tilde{r}_3 = \tilde{\theta}_3 = 0.339981043584856 \ , w_3 = 0.652145154862546 \ ; \\
&\tilde{r}_4 = \tilde{\theta}_4 = 0.861136311594053 \ , w_4 = 0.347854845137454 \ .
\end{aligned} \tag{30}$$

Having determined the contribution originating from the ambient wind field using equations (29) and (30), we turn to the contributions from the individual wake deficits.

Wake contribution

For one particular wake deficit, the situation is sketched in Figure 1.

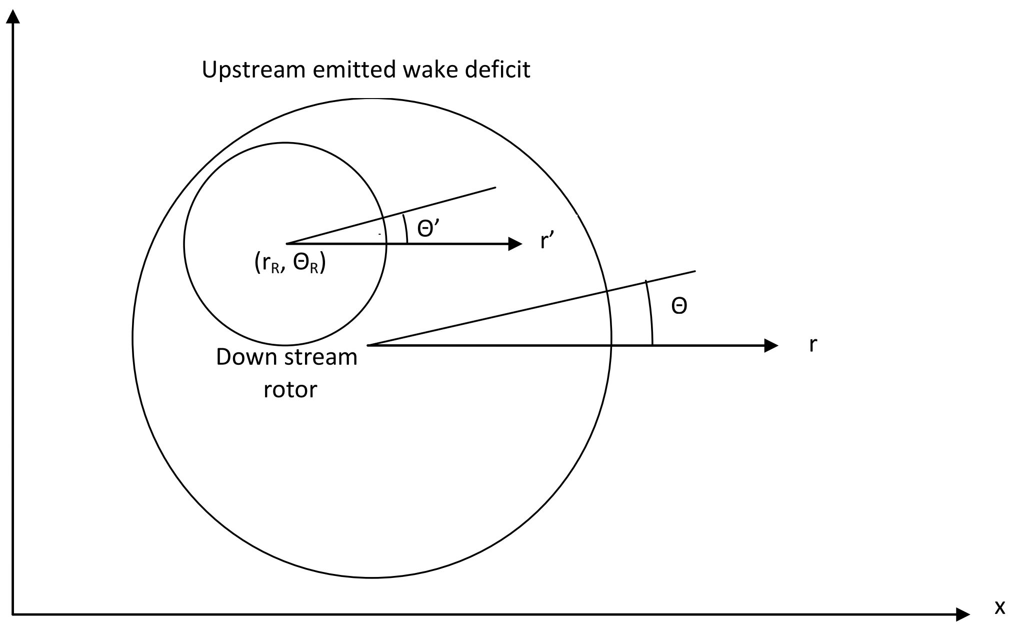

Figure 1 Coordinate systems related to one particular wake deficit contribution to the pseudo-uniform inflow field.

For a given downstream position we denote the stationary wind speed surface, describing the i'th upstream emitted wake deficit imposed on the ambient flow field, by $\Delta\overline{U}_i(r_i,\theta_i)$ in a polar coordinate system centered at the associated wake "center of gravity". This function is defined for $\theta \in [0; 2\pi]$ and for $r \in [0; \infty[$, and in general it does not possess rotational symmetry. However, in the present context we have confined ourselves to rotationally symmetric wake deficits (cf. equation (11) in Section 2), whereby the following simplification is obtained

$$\Delta\overline{U}_i(r_i, \theta_i) = \Delta\overline{U}_i(r_i) \,. \tag{31}$$

Because the extension of the upstream emitted wake deficit in practice is limited by $R_{w,i}$ we further have

$$\Delta\overline{U}_i(r_i) \equiv 0 \quad \text{for} \quad r_i \geq R_{w,i} \;. \tag{32}$$

We now consider a downstream rotor and introduce a rotor polar coordinate system (r',θ') centered at the rotor center of gravity. The transformation between the i'th wake coordinate system and the rotor coordinate system is given by

$$\begin{aligned} r_i \cos\theta_i &= r_{R,i} \cos\theta_{R,i} + r' \cos\theta' \\ r_i \sin\theta_i &= r_{R,i} \sin\theta_{R,i} + r' \sin\theta' \;, \end{aligned} \tag{33}$$

where $r_{R,i}$ and $\theta_{R,i}$ are constants that denote the coordinates of the downstream rotor expressed in the wake coordinate system.

With the restriction introduced in equation (31) the magnitude of the wake deficit depends on the radial coordinate in the wake coordinate system only. The radial coordinate in the wake coordinate system, r_i, of a point with coordinates (r',θ') in the rotor coordinate system is easily derived from equation (33) as

$$r_i = \sqrt{r_{R,i}^2 + r'^2 + 2r_{R,i}r' \cos(\theta_{R,i} - \theta')} \,. \tag{34}$$

Introducing equation (34) into equations (31) and (32), the i'th wake deficit surface, as expressed in the rotor frame of reference, may be defined by

$$g_i(r', \theta') \equiv \Delta\overline{U}_i\left(\sqrt{r_{R,i}^2 + r'^2 + 2r_{R,i}r' \cos(\theta_{R,i} - \theta')}\right) , \tag{35}$$

with

$$g_i(r', \theta') = 0 \quad \text{for} \quad \sqrt{r_{R,i}^2 + r'^2 + 2r_{R,i}r' \cos(\theta_{R,i} - \theta')} \geq R_{w,i} \;. \tag{36}$$

We note, that in case $r_{R,i} \geq R_{w,i} + R$ the i'th wake deficit will not contribute to the pseudo-uniform inflow field associated with the rotor in question.

In line with the procedure applied for the undisturbed ambient wind field, we will use Gauss integration to evaluate the contribution, associated with the i'th wake deficit, to the pseudo-uniform inflow field for the rotor in question, $\overline{U}_i$. Assuming that the g_i-functions, in both the radial direction and the azimuthal direction, are well approximated by a (2N-1)'th order polynomial, the requested integral is evaluated applying a N point Gauss integration. Using definition (35) and the variable transformation defined by equation (27) we obtain

$$\overline{U}_i = \frac{1}{A}\int_A g_i(r',\theta')r'\,dr'\,d\theta'$$

$$= \frac{1}{A}\int_{-1}^{1}\int_{-1}^{1}\Delta\overline{U}_i\left(\sqrt{r_{R,i}^2 + \frac{R^2(\tilde{r}+1)^2}{4} + r_{R,i}R(\tilde{r}+1)cos\left(\theta_{R,i} - \pi\left[\tilde{\theta}+1\right]\right)}\right)\frac{\pi R^2(\tilde{r}+1)}{4}d\tilde{r}d\tilde{\theta} \quad , \tag{37}$$

which by N point Gauss integration may be approximated by

$$\overline{U}_i = \frac{\pi R^2}{4A}\sum_{j=1}^{N}\sum_{k=1}^{N} w_j w_k \Delta\overline{U}_i\left(\sqrt{r_{R,i}^2 + \frac{R^2(\tilde{r}_k+1)^2}{4} + r_{R,i}R(\tilde{r}_k+1)cos\left(\theta_{R,i} - \pi\left[\tilde{\theta}_j+1\right]\right)}\right)(\tilde{r}_k+1) \quad , \tag{38}$$

with w_j and w_k being the Gauss weights. For the present application, we consider the kernel of integral (37) as well approximated by a forth order Gauss integration with the Gauss weights and the associated values of the integration variables given in equation (30).

Utilizing the linearity of the geometric averaging operator, defined by equation (21), the pseudo-uniform inflow field to rotor number "*m*", $\overline{U}_{(m)}$, is finally given by

$$\overline{U}_{(m)} = \overline{U} - \sum_{\substack{i=1 \\ r_{R,i} \leq R_{w,i}+R}}^{M} \overline{U}_i \quad , \tag{39}$$

where M denotes the number of wake generating rotors upstream of rotor number "*m*".

The linear perturbation approach thus finally yield the *non-uniform stationary wind farm wind field* as a linear superposition of the ambient undisturbed wind field, U, and the individual stationary wake contributions, $\Delta\overline{U}_i$, as determined from equation (11). We note, that with the proposed approach the wind farm wind field recovers to the undisturbed ambient wind field for sufficiently long downstream distances, because each of the individual wake deficit contributions, according to equation (11), approach zero when the downstream coordinate, x, approach infinity.

For *below rated mean wind speeds* a simple approximation may apply. In this wind speed regime, the thrust coefficient of a conventional wind turbine may often be satisfactory approximated as $C_T = \alpha/\overline{U}$, where α is some constant. Referring to equation (19) this means that the *m*'th wake deficit, $\Delta\overline{U}_m$, is approximately invariant with respect to the particular inflow mean wind speed to the *m*'th wind turbine, $\overline{U}_m$. Therefore, within this mean wind speed regime, we have the following approximation $\Delta\overline{U}_m \approx \Delta\overline{U}_1$, i.e. that the *m*'th wake deficit can be approximated by the wake deficit associated with *ambient* wind speed conditions, and thus consequently computation of $\overline{U}_m$ from equation (39) may be omitted.

3.2 Wind farm wind field – non-linear approach

The linearity of the averaging operator defined by equation (21) allows for a decomposition of a particular pseudo-uniform wind field in contributions from the undisturbed ambient wind field and from the individual upstream emitted wake deficits, respectively. For the *non-linear* averaging operator, defined by equation (24), such decomposition is not possible, and the *aggregated* wind inflow field imposed on each rotor consequently has to be considered. This in turn means that the rotational symmetry of the individual stationary wake deficits cannot be utilized in the averaging of the involved wake contributions.

Adopting the linear perturbation approach and a recursive parabolic scheme, starting with the most upwind located rotor, the *non-uniform* stationary wind farm inflow field, at the location of rotor number "*m*", is determined as a sum of contributions from the undisturbed ambient mean wind field and the relevant upwind emitted stationary wake deficits, respectively. Expressed in the polar rotor frame of reference, as defined in equation (33), this field is given by

$$U_{(m)}(r',\theta') = f\left(z_c + r'\sin\theta'\right) + \sum_{i=1}^{M} g_i(r',\theta')$$
$$= \frac{u_*}{\kappa} Ln\left(\left[z_c + r'\sin\theta'\right]/ z_0\right) + \sum_{\substack{i=1 \\ r_{R,i} \le R_{w,i}+R}}^{M} \Delta\overline{U}_i\left(\sqrt{r_{R,i}^2 + r'^2 + 2r_{R,i}r'\cos\left(\theta_{R,i} - \theta'\right)}\right) , \tag{40}$$

where M denotes the number of wake generating rotors upstream of rotor number "*m*", and the remaining nomenclature is inhered from Section 3.1.

Applying the averaging operator proposed in equation (24), the pseudo-uniform inflow field to be imposed on rotor number "*m*" is determined from

$$\overline{U}_{(m)} = \sqrt{\frac{1}{A}\int_A U_{(m)}^2(r',\theta')dA}$$
$$= \sqrt{\frac{1}{A}\int_0^R\int_0^{2\pi} r' U_{(m)}^2(r',\theta')dr'\,d\theta'} \; . \tag{41}$$

With the present approach, equation (41) is the analogy to equation (39) for the linear averaging operator. In analogy with the linear averaging approach, we will apply a numerical scheme for the successive solution of (41) based on Gauss integration. Using the transformation defined in equation (27), the integral expressed in equation (41) is rewritten as

$$\overline{U}_{(m)} = \sqrt{\frac{1}{A}\int_{-1}^{1}\int_{-1}^{1} U_{(m)}^2\left(\frac{R(\tilde{r}+1)}{2}, \pi\left(\tilde{\theta}+1\right)\right)\frac{R^2(\tilde{r}+1)\pi}{4}\,d\tilde{r}\,d\tilde{\theta}} \; , \tag{42}$$

and thus approximated by

$$\overline{U}_{(m)} = \frac{\sqrt{\pi}\,R}{2\sqrt{A}}\sum_{j=1}^{N}\sum_{k=1}^{N} w_j w_k\left(\tilde{r}_k + 1\right)U_{(m)}^2\left(\frac{R\left(\tilde{r}_k+1\right)}{2}, \pi\left(\tilde{\theta}_j + 1\right)\right) \; . \tag{43}$$

In the actual case we consider the integral (42) as well approximated by a forth order Gauss integration with the Gauss weights and the associated values of the integration variables given specified in equation (30).

Completing the described recursive scheme, the *non-uniform stationary wind farm wind field* is determined from the linear superposition expressed in equation (40) for an arbitrary downstream rotor position. The rotor position does not have to be within the wind farm. For large downstream distances the second term in equation (40) is seen to vanish, and the stationary wind farm wind field thus recovers asymptotically to the undisturbed ambient mean wind field.

4 Conclusions

We present an idealized simple, but fast, semi-analytical algorithm for computation of *stationary* wind farm wind fields with a possible potential within a multi-fidelity strategy for wind farm topology optimization.

Basically, the model considers wakes as *linear perturbations* on the ambient non-uniform mean wind field, although the modelling of the individual stationary wake flow fields includes non-linear terms. The simulation of the individual wake contributions are based on an analytical solution of the thin shear layer approximation of the NS equations. The wake flow fields are assumed *rotationally symmetric*, and the rotor inflow fields are consistently assumed uniform. This is of cause an idealization, because of both *turbulence* and the *mean wind shear* resulting from the surface friction and upstream emitted wakes. However, detailed CFD simulations have identified wake flow fields to be reasonable rotational symmetric even under sheared inflow conditions, except in flow regimes near the surface.

Expansion of *stationary* wake fields are believed to be significantly affected by meandering of comparable “narrow” free shear wake deficit fields as described by the DWM model [1]. In the present context, this effect is approximately accounted for by imposing suitable empirical downstream boundary conditions on the wake expansion that depend on the rotor thrust and the ambient turbulence conditions, respectively. The algorithm does, however, not take into account possible modification of the downstream inflow turbulence, but as only the large scale turbulent eddies contributes significantly to wake meandering, and as these scales are known to be somewhat unaffected by the presence of the wind farm, the modification of the turbulence field inside the wind farm is not considered important for the expansion of the individual wakes. The relatively significant downstream expansion of the individual wake deficits is believed to some degree to justify the linear perturbation assumption, because of the associated downstream attenuation of the wake deficits. However, for downstream distances beyond approximately 10 rotor diameters (at which distance the calibrated wake expansion boundary conditions are imposed), the present formulation of wake expansion is believed to underestimate wake expansion. This is because the analytical wake formulation dictates the wake expansion to behave as $x^{1/3}$ with downstream distance, whereas wake expansion as primary controlled by wake meandering develops approximately linearly with the downstream distance.

To establish the link from a non-uniform wind farm wind field, consisting of *linear perturbations* on the ambient non-uniform mean wind field, to a uniform wake generating inflow field, two approaches for estimation of suitable *pseudo-uniform* inflow fields are proposed – a linear and a non-linear operator. Note in this respect, that zero yaw error has implicitly been assumed in these operators, and further that the possibility of flow recirculation has been neglected for the non-linear operator. With each of these approached, a parabolic system are described, which is initiated by first considering the most upwind located turbines and subsequently successively solved in the downstream direction. Algorithms for the resulting wind farm flow fields are proposed, and in the limit of very large downstream distances, the simulated field is shown to recover to the undisturbed ambient wind field upstream the wind farm.

Finally, we note that possible reflection phenomena, resulting from “collision” of stationary wake fields with the ground are neglected.

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5 References

[1] Larsen, G.C.; Madsen, H.Aa.; Thomsen, K. and Larsen, T.J. (2008). Wake meandering **-** a pragmatic approach. *Wind Energy*, **11**, 377-395.

[2] Larsen, G.C.; Madsen, H.Aa.; Larsen, T.J. and Troldborg, N. (2008). Wake modeling and simulation. Risø-R-1653(EN).

[3] Madsen, H.Aa.; Larsen, G.C.; Larsen, T.J. and Troldborg, N. (2009). Calibration and validation of the Dynamic Wake Meandering model implemented in the aeroelastic code HAWC2. Submitted for publication in Journal of Solar Energy Engineering.

[4] Troldborg, N. (2008). Actuator Line Modeling of Wind Turbine Wakes, PhD Dissertation, Technical University of Denmark.

[5] Troldborg, N.; Sørensen, J.N. and Mikkelsen, R. (2009). Numerical Simulations of Wake Characteristics of a Wind Turbine in uniform Inflow. Accepted for publication in Wind Energy.

[6] Buhl, T.; Larsen, T.J. and Carlén, I. (2009). A method to formulate a feasible optimization problem taking into account not only energy production but also aeroelastic loads, control strategies and installation costs. Risø-R-1705(EN), 18 p.

[7] Ott, S.; Sogachev, A.; Mann, J.; Jørgensen, H.E. and Frandsen, S.T. (2009). Applying Flow Models of Different Complexity for Estimation of Turbine Wakes. EWEC2009; European Wind Energy Conference, Marseille.

[8] Swain, L.M. (1929). On the turbulent wake behind a body of revolution. Proc. Roy. Soc. London A 125, 647-659.

[9] Larsen, G.C.; Madsen, H.Aa. and Sørensen, N.N. (2003). Mean wake deficit in the near field. In: Proceedings CD-ROM. CD 2. European wind energy conference and exhibition 2003 (EWEC 2003), Madrid, 16-19 June, 6p.

[10] Troldborg, N.; Sørensen, J.N. and Mikkelsen, R. (2009). Numerical Simulations of Wakes Operating in Sheared and Turbulent Inflow. Proc. EWEC 2009, European Wind Energy Conference, Marseille.

[11] Larsen, G.C. (1988). A Simple Wake Calculation Procedure. Risø-M-2760.

[12] Djerf, E. and Mattson, H. (2000). Evaluation of the Software program WindFarm and Comparisons with Measured Data from Alsvik, FFA TN-2000-30, http://www.vindenergi.foi.se/Rapporter/TN2000_30_WindFarm.pdf.

[13] Schepers, G. (2003). ENDOW: Validation and Improvement of ECN's wake model, ECN-C-03-034.

[14] Larsen, G.C.; Madsen, H.Aa.; Larsen, T.J. and Troldborg, N. (2008). Wake modeling and simulation. Risø-R-1653(EN) (2008) 28 p.

[15] Larsen, G.C. (2009). From solitary wakes to wind farm wind fields – a simple engineering approach. Risø-R-1727(EN) (2009)

[16] Hansen, M.O.L. (2008). Aerodynamics of Wind Turbines. Second Edition, Earthscan, London.

[17] Lange, B.; Waldl, H.-P.; Guerrero, A.G. and Heinemann, D. (2003). Modelling of Offshore Wind Turbine Wakes with the Wind Farm Program FLaP. Wind Energy, **6**, 87-104.

[18] Zahle, F. and Sørensen, N.N. (2010). Upwind Navier-Stokes Rotor Flow Simulations with ground proximity and shear. TORQUE 2010: The Science of Making Torque from Wind, June 28-30, Crete, Greece. 255-263.

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