Multi-material topology optimization of truss structures

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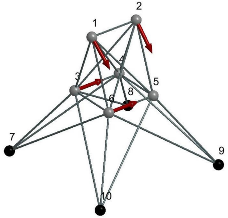
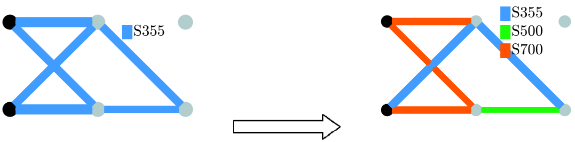
- Consider a truss structure with a given set of loads and boundary conditions.



- The **minimum weight/cost truss topology optimization** problem
  - the constraints are limits on stress, displacement, etc.
  - the design variable is **usually only the cross-sectional area** of the bars.
  - **often single candidate material** is considered.

- This study,
  - considers the selection of materials concurrently with the cross-sectional areas.
  - minimizes the cost of the structures.
  - suggests the possibility of cost efficient multi-material truss structures.



Parameters/ **single material**

- n bars with fixed lengths $l_1, \ldots, l_n$
- the candidate material has
  - Young's modulus E
  - density ρ
  - cost per unit mass c
- m discrete cross-sectional areas $\{a^1, ..., a^m\}$

The minimum cost optimization problem:

$$\begin{array}{ll} \text{minimize} & c\rho \sum_{i=1}^{n} l_i a_i \\ \text{subject to} & \boldsymbol{K}(\boldsymbol{a})\boldsymbol{u} = \boldsymbol{f} \\ & \boldsymbol{u}^{\min} \leq \boldsymbol{u} \leq \boldsymbol{u}^{\max} \\ & \underline{\sigma}^i \leq \sigma_i \leq \bar{\sigma}^i, \ i: \ a_i > 0 \\ & a_i \in \{a^1, ..., a^m\}, \ i = 1, \ldots, n \\ & \boldsymbol{u} \in \mathbb{R}^d \end{array}$$

where $\boldsymbol{K}(\boldsymbol{a}) = \sum_{i=1}^{n} a_i \boldsymbol{K}_i$ with $\boldsymbol{K}_i = \frac{E}{l_i} \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i^T$.

Parameters/ **multiple materials**

- n bars with fixed lengths $l_1, \ldots, l_n$
- p candidate materials with
  - Young's modulus $\{E^1, ..., E^p\}$
  - densities $\{\rho^1, ..., \rho^p\}$
  - costs per unit mass $\{c^1, ..., c^p\}$
- m discrete cross-sectional areas $\{a^1, ..., a^m\}$

We formulate the minimum cost optimization problem as Disjunctive Programming (DJ)[1]

$$
\begin{array}{ll}
\text{minimize} & z = f(x) + \sum_{k \in K} c_k \\
\text{subject to} & r(x) \leq 0 \\
& \bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \leq 0 \\ c_k = \gamma_{ik} \end{bmatrix}, k \in K \\
& \Omega(Y) = True \\
& x \in \mathbb{R}^n,\ c_k \in \mathbb{R}^1,\ Y_{jk} \in \{True, False\}
\end{array}
$$

[1] Raman, R. and I.E. Grossmann, Modelling and Computational Techniques for Logic Based Integer Programming,vol 18, 1994, Computers and Chemical Engineering, pp 563

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The minimum cost optimization problem formulated as DJ :

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\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{i=1}^{n} c_i l_i a_i \rho_i \\
\text{subject to} & \boldsymbol{K}(\boldsymbol{a})\boldsymbol{u} = \boldsymbol{f} \\
& \boldsymbol{u}^{\min} \leq \boldsymbol{u} \leq \boldsymbol{u}^{\max} \\
& \underline{\sigma}^i \leq \sigma_i \leq \bar{\sigma}^i, \ i: \ a_i > 0 \\
& \bigvee_{k=1}^{p} \begin{bmatrix} E_i = E^k \\ \rho_i = \rho^k \\ c_i = c^k \\ \underline{\sigma}^i = \underline{\sigma}^{ik} \\ \bar{\sigma}^i = \bar{\sigma}^{ik} \end{bmatrix}, i = 1, \ldots, n \\
& a_i \in \{a^1, ..., a^m\}, \ i = 1, \ldots, n \\
& \boldsymbol{u} \in \mathbb{R}^d
\end{array}
$$

where $\boldsymbol{K}(\boldsymbol{a}) = \sum_{i=1}^{n} a_i \boldsymbol{K}_i$ with $\boldsymbol{K}_i = \frac{E_i}{l_i} \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i^T, \ E_i \in \{E^1, ..., E^p\}$.

- We follow the big-M approach to reformulate the DJ as as Mixed Integer Linear Programming (MILP).

DJ

$$\begin{array}{ll} \text{minimize} & z = f(x) + \sum_{k \in K} c_k \\ \text{subject to} & r(x) \leq 0 \\ & \bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \leq 0 \\ c_k = \gamma_{ik} \end{bmatrix}, k \in K \\ & \Omega(Y) = True \\ & x \in \mathbb{R}^n,\ c_k \in \mathbb{R}^1,\ Y_{jk} \in \{True, False\} \end{array}$$

MILP

$$\begin{array}{ll} \text{minimize} & z = f(x) + \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} \\ \text{subject to} & r(x) \leq 0 \\ & g_{jk} \leq M_{jk}(1 - \lambda_{jk}),\ j \in J_k, k \in K \\ & \sum_{j \in J_k} \lambda_{jk} = 1,\ k \in K \\ & A\lambda \leq a \\ & x \geq 0, \lambda_{jk} \in \{0, 1\} \end{array}$$

- Introduce a binary variable $x_{ijk} \in \{0, 1\}$ such that

$$x_{ijk} = \begin{cases} 1, & \text{if } a_i = a^j \text{ and } E_i = E^k \\ 0, & \text{otherwise.} \end{cases}$$

- We force the binary variable to satisfy the relation

$$\sum_{k=1}^{p} \sum_{j=1}^{m_k} x_{ijk} = 1, \quad i = 1, \dots, n.$$

The minimum cost MILP:

$$\begin{aligned}
\underset{\boldsymbol{x}\in\mathbb{R}^{mnp},\boldsymbol{u}\in\mathbb{R}^{d},\boldsymbol{q}\in\mathbb{R}^{mnp}}{\text{minimize}} \quad & \sum_{i=1}^{n} l_i \sum_{j=1}^{m} a^j \sum_{k=1}^{p} x_{ijk} c^k \rho^k \\
\text{subject to} \quad & \sum_{i=1}^{n} \boldsymbol{\gamma}_i \sum_{j=1}^{m} \sum_{k=1}^{p} q_{ijk} = \boldsymbol{f} \\
& \boldsymbol{u}^{\min} \leq \boldsymbol{u} \leq \boldsymbol{u}^{\max} \\
& x_{ijk} a^j \underline{\sigma}^{ik} \leq q_{ijk} \leq x_{ijk} a^j \bar{\sigma}^{ik}, \quad \forall (i,k) \text{ and } j = 1,\ldots,m \\
& (1 - x_{ijk}) a^j \underline{\sigma}^{ik}_{\min} \leq a^j \frac{E^k}{l_i} \boldsymbol{\gamma}_i^T \boldsymbol{u} - q_{ijk} \leq (1 - x_{ijk}) a^j \bar{\sigma}^{ik}_{\max}, \quad \forall (i,k) \\
& \qquad \text{and } j = 1,\ldots,m \\
& \textstyle\sum_{k=1}^{p} \sum_{j=1}^{m_k} x_{ijk} = 1, \quad i = 1,\ldots,n \\
& x \in \{0,1\}^{mnp}.
\end{aligned}$$

Numerical experiments

- We solve the MILP problem using IBM ILOG CPLEX.

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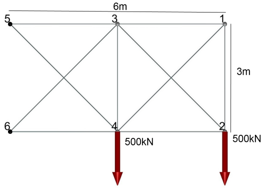
- The 2D 10-bar problem, $|u|^{\max} = 55$mm.



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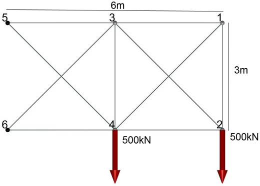
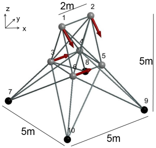


- The 3D 25-bar problem, $|u|^{\max} = 40$mm, 8 group of members.



Node	Loads (kN) f_x	f_y	f_z
1	80	-800	-400
2	0	-800	-400
3	4	0.0	0.0
6	4	0.0	0.0

Member groups	End nodes
1	(1,2)
2	(1,4),(1,5),(2,3),(2,6)
3	(1,3),(1,6),(2,4),(2,5)
4	(3,6),(4,5)
5	(3,4),(5,6)
6	(3,10),(4,9),(5,8),(6,7)
7	(3,8),(4,7),(5,10),(6,9)
8	(3,7),(4,8),(5,9),(6,10)

- Material: steel of Young's modulus $E = 210$GPa, density $\rho = 7850$ kg/m^3 that is available in
  - three grades, denoted by, S355, S500 and S700, with yield strengths 355MPa, 500MPa, and 700MPa and relative costs per unit kilogram c, $1.44c$ and $1.63c$.[2]
  - the cross-sectional areas (0,8,10,12,14,16,18,20,22,24,26,28,30,32 cm^2).

[2] T. Tiainen, T. Mela, K. Jokinen, and M. Heinisuo. High strength steel in tubular trusses. Technical report, Tampere University of Technology, Faculty of Business and Built Environment, Finland.

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  - the cross-sectional areas (0,8,10,12,14,16,18,20,22,24,26,28,30,32 cm^2).
- Seven possible combinations, i.e.,
  - single material
    S355
    S500
    S700
  - multiple materials
    S355/S500
    S355/S700
    S500/S700
    S355/S500/S700

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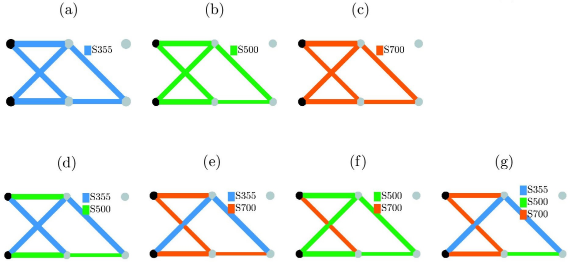
- The 2D 10-bar truss problem



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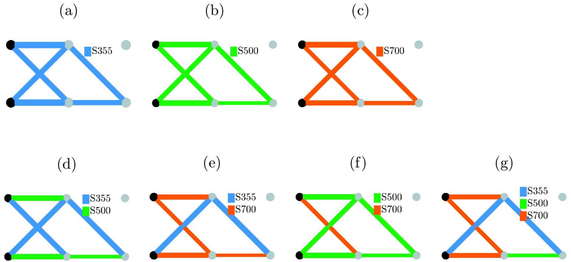
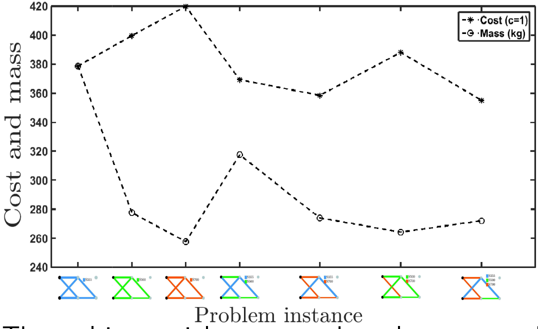
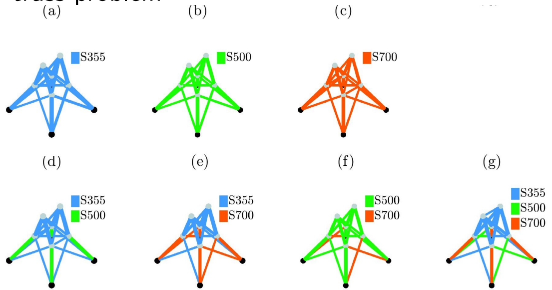


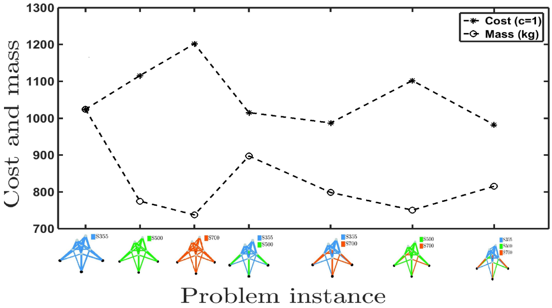


Problem	mass(kg)	cost(c)
S355	378.81	378.81
S500	277.61	399.76
S700	257.63	419.94
S355/S500	317.58	369.39
S355/S700	274.05	358.57
S500/S700	264.29	388.17
S355/S500/S700	272.09	355.11

- The multi-material structures have lower costs than that of the (corresponding) single-material structures.

- The 3D 25-bar truss problem





Problem	mass(kg)	cost(c)
S355	1024.14	1024.14
S500	774.80	1115.72
S700	737.50	1202.12
S355/S500	897.52	1015.30
S355/S700	798.35	987.18
S500/S700	750.35	1102.25
S355/S500/S700	815.06	982.16

- The multi-material structures have lower costs than that of the (corresponding) single-material structures.

Conclusion

- A general minimum cost truss topology optimization problem with discrete design variables is alternatively formulated as
  - disjunctive programming and
  - mixed integer linear programming.
- The partial use of expensive stronger materials can benefit in the overall reduction of the cost of multi-material structures.

Future work

- Reformulate the objective function to consider more representative cost of the structures (Material and other costs).
- Extend the study to more complex problem instances.

Thank you for your attention!