Proceedings of the ASME 2013 Rail Transportation Division Fall Technical Conference
RTDF2013
October 15-17, 2013, Altoona, Pennsylvania, USA

RTDF2013-4713

MODERN UNCERTAINTY QUANTIFICATION METHODS IN RAILROAD VEHICLE DYNAMICS

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ABSTRACT

This paper describes the results of the application of Uncertainty Quantification methods to a simple railroad vehicle dynamical example. Uncertainty Quantification methods take the probability distribution of the system parameters that stems from the parameter tolerances into account in the result. In this paper the methods are applied to a low-dimensional vehicle dynamical model composed by a two-axle truck that is connected to a car body by a lateral spring, a lateral damper and a torsional spring, all with linear characteristics.

Their characteristics are not deterministically defined, but they are defined by probability distributions. The model - but with deterministically defined parameters - was studied in [1] and [2], and this article will focus on the calculation of the critical speed of the model, when the distribution of the parameters is taken into account.

Results of the application of the traditional Monte Carlo sampling method will be compared with the results of the application of advanced Uncertainty Quantification methods [3]. The computational performance and fast convergence that result from the application of advanced Uncertainty Quantification methods is highlighted. Generalized Polynomial Chaos will be presented in the Collocation form with emphasis on the pros and cons of each of those approaches.

NOMENCLATURE

m, I	mass and inertia of the bogie
D_2, k_4, k_6	suspension parameters
$b, k_0, x_f, \alpha,$ β, δ, κ	nonlinear spring constants used to approximate the flange forces
ϕ, ψ	constants determined by the sizes of the semi axes of the contact ellipse
r_0	nominal rolling radius
λ	conicity

INTRODUCTION

In engineering, deterministic models have been extensively exploited to describe dynamical systems and their behaviors. These have proven to be useful in the design phase of the engineering products, but they always fall short in providing indications of the reliability of certain designs over others. The results obtained by one deterministic experiment describe, in practice, a very rare case that likely will never happen. However, engineers are confident that this experiment will explain most of the experiments in the vicinity of it, i.e. for small variation of parameters. Unfortunately, this assumption may lead to erroneous conclusions, in particular for realistic nonlinear dynamical systems, where small perturbations can cause dramatic changes in the dynamics. It is thus critical to find a measure for the level of knowledge of a dynamical system, in order to be able to make a reasonable risk analysis and design optimization.

Risk analysis in the railroad industry is critical for as well the increase of the safety as for targeting investments. Railroad vehicle dynamics is difficult to study even in the deterministic case, where strong nonlinearities appear in the system. A lot of phenomena develop in such dynamical systems, and the interest of the study could be focused on different parameters, such as the ride comfort or the wear of the components. This work will instead focus on ride safety when high speeds are reached and the hunting motion develops. The hunting motion is a well known phenomenon characterized by periodic as well as aperiodic lateral oscillations, due to the wheel-rail contact forces, that can appear at different speeds depending on the vehicle design. This motion can be explained and studied with notions from nonlinear dynamics [4], combined with suitable numerical methods for non-smooth dynamical systems [5]. It is well known that the behavior of the hunting motion is parameter dependent, thus good vehicle designs can increase the critical speed. This also means that suspension components need to be carefully manufactured in order to really match the demands of the customer. However, no manufactured component will ever match the simulated ones. Thus epistemic uncertainties, for which we have no evidence, and aleatoric uncertainties, for which we have a statistical description, appear in the system as a level of knowledge of the real parameters [6].

Uncertainty Quantification (UQ) tries to address the question: “assuming my partial knowledge of the design parameters, how reliable are my results?”. This work will focus on the sensitivity of the critical speed of a railroad vehicle model to the suspension parameters.

THE VEHICLE MODEL

This work will investigate the dynamics of the well known simple Cooperrider truck model [2] shown in Fig. 1. The model is composed by two conical wheel sets rigidly connected to a truck frame, that is in turn connected to a fixed car body by linear suspensions: a couple of lateral springs and dampers and one torsional spring.

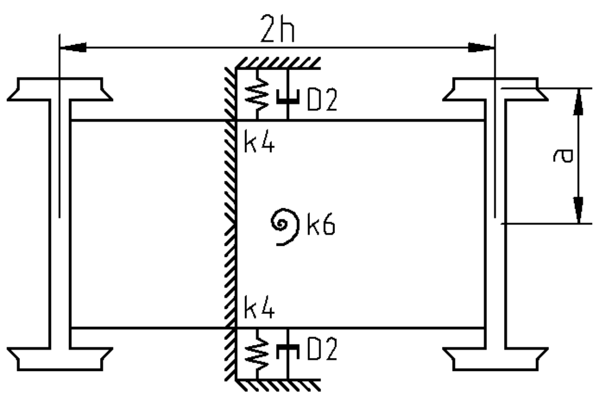

Fig. 1: Top view of the Cooperrider truck model.

The following equations govern this dynamical system [2]:

$$\begin{aligned} m\ddot{q}_1 &= -2D_2\dot{q}_1 - 2k_4 q_1 \\ &\quad - 2\left[F_x\left(\xi_{x_1},\xi_{y_1}\right) + F_x\left(\xi_{x_2},\xi_{y_2}\right)\right] \\ &\quad - F_T(q_1 + haq_2) - F_T(q_1 - haq_2), \\ I\ddot{q}_2 &= -k_6 q_2 - 2ha\left[F_x\left(\xi_{x_1},\xi_{y_1}\right) - F_x\left(\xi_{x_2},\xi_{y_2}\right)\right] \\ &\quad - 2a\left[F_y\left(\xi_{x_1},\xi_{y_1}\right) + F_y\left(\xi_{x_2},\xi_{y_2}\right)\right] \\ &\quad - ha\left[F_T(q_1 + haq_2)\right. \\ &\quad \left.- F_T(q_1 - haq_2)\right], \end{aligned} \qquad (1)$$

where D_2, k_4 and k_6 are the damping coefficient and the stiffness coefficients respectively, F_x and F_y are the lateral and longitudinal creep forces and F_T is the flange force.

The ideally stiff truck runs on a perfect straight track where the constant wheel-rail adhesion coefficient enters the system through the lateral and longitudinal creep-forces:

$$F_x\left(\xi_x,\xi_y\right) = \frac{\xi_x}{\phi}\frac{F_R\left(\xi_x,\xi_y\right)}{\xi_R\left(\xi_x,\xi_y\right)} \quad , \quad F_y\left(\xi_x,\xi_y\right) = \frac{\xi_y}{\psi}\frac{F_R\left(\xi_x,\xi_y\right)}{\xi_R\left(\xi_x,\xi_y\right)} \ ,$$

$$\xi_R\left(\xi_x,\xi_y\right) = \sqrt{\frac{\xi_x^2}{\phi^2} + \frac{\xi_y^2}{\psi^2}} \ ,$$

$$\frac{F_R\left(\xi_x,\xi_y\right)}{\mu N} = \begin{cases} u(\xi_R) - \frac{1}{3}u^2(\xi_R) + \frac{1}{27}u^3(\xi_R) & \text{for } u(\xi_R) < 3 \\ 1 & \text{for } u(\xi_R) \geq 3 \end{cases} \ ,$$

$$u(\xi_R) = \frac{G\pi ab}{\mu N}\xi_R \ ,$$

where ϕ and ψ are real numbers that are determined by the size of the semi axes of the contact ellipse, which are constant in our problem [7]. The creepages are given by:

$$\xi_{x_1} = \frac{\dot{q}_1}{v} + ha\frac{\dot{q}_2}{v} - q_2 \ , \qquad \xi_{y_1} = a\frac{\dot{q}_2}{v} + \frac{\lambda}{r_0}(q_1 + haq_2) \ ,$$

$$\xi_{x_2} = \frac{\dot{q}_1}{v} - ha\frac{\dot{q}_2}{v} - q_2 \ , \qquad \xi_{y_2} = a\frac{\dot{q}_2}{v} + \frac{\lambda}{r_0}(q_1 - haq_2) \ .$$

The flange forces are approximated by a very stiff non-linear spring with a dead band:

$$F_T(x) = \begin{cases} \exp\left(-\alpha/\left(x - x_f\right)\right) - \beta x - \kappa \ , & 0 \leq x < b \\ k_0 \cdot (x - \delta) \ , & b \leq x \\ -F_T(-x) \ , & x < 0 \end{cases} \ ,$$

The parameters used for the analysis are listed in the following:

$m = 4963\ kg$	$h = 1.5\ m$
$I = 8135\ kg \cdot m^2$	$D_2 = 29200\ N \cdot s/m$
$k_0 = 14.60 \cdot 10^6\ N/m$	$k_4 = 0.1823 \cdot 10^6\ N/m$
$k_6 = 2.710 \cdot 10^6\ N/m$	$\lambda = 0.05$
$r_0 = 0.4572\ m$	$b = 0.910685 \cdot 10^{-2}\ m$
$\phi = 0.60252$	$\psi = 0.54219$
$G\pi ab = 6.563 \cdot 10^6\ N$	$\mu N = 10^4\ N$
$\delta = 0.0091\ m$	$\alpha = 0{,}1474128791 \cdot 10^{-3}$
$\beta = 1{,}016261260$	$\kappa = 1{,}793756792$
$x_f = 0.9138788366 \cdot 10^{-2}$	$a = 0.7163\ m$

Non linear dynamics of the deterministic model

The dynamics of the deterministic model at high speed has been investigated in [2]. The existence of a subcritical Hopf-bifurcation has been detected at $v_L = 66.61$ m/s. Fig. 2 shows the bifurcation diagram of the deterministic system. The Hopf bifurcation point is obtained by observation of the stability of the trivial solution using the eigenvalues of the Jacobian of the system. The nonlinear critical speed, the fold bifurcation, characteristic in subcritical Hopf-bifurcations, is found at $v_{NL} = 62.02$m/s using a ramping method, where the speed is quasi-statically decreased, according to

$$\dot{v} = \begin{cases} 0\ , & \text{if } t < t_{st} \vee \|\vec{q}\|_2 < \epsilon_{min} \\ -\Delta\ , & \text{otherwise} \end{cases} \ . \qquad (2)$$

The stochastic model

Let us now consider suspensions that are provided by the manufacturer with a certain level of working accuracy. Due to the lack of real data regarding the probability distributions of such working accuracies, this initial study will consider Gaussian distributions to describe them:

$$\begin{aligned} k_6 &\sim \mathcal{N}\left(\mu_{k_6}, \sigma_{k_6}^2\right) , \quad \text{(std. ~ 5\%)} \\ k_4 &\sim \mathcal{N}\left(\mu_{k_4}, \sigma_{k_4}^2\right) , \quad \text{(std. ~ 7\%)} \\ D_2 &\sim \mathcal{N}\left(\mu_{D_2}, \sigma_{D_2}^2\right) . \quad \text{(std. ~ 7\%)} \end{aligned} \tag{3}$$

where the symmetry of the model is taken into consideration in the standard deviation of the parameters k_4 and D_2 that both represent two elements. The applicability and efficiency of the methods presented in the next section will not be affected by the particular choice of distribution.

Now the deterministic model is turned into a stochastic model, where the single solution represents a particular realization and probabilistic moments can be used to describe the statistics of the stochastic solution.

A straightforward way of computing the moments of the solution is to approximate the integrals as:

$$\begin{aligned} \boldsymbol{\mu_q}(t) &\approx \bar{\boldsymbol{\mu}}_{\boldsymbol{q}}(t) = \frac{1}{M}\sum_{j=1}^{M} \boldsymbol{q}\left(t, \boldsymbol{Z}^{(j)}\right) , \\ \boldsymbol{\sigma}_{\boldsymbol{q}}^2(t) &\approx \bar{\boldsymbol{\sigma}}_{\boldsymbol{q}}^2(t) = \frac{1}{M-1}\sum_{j=1}^{M} \left(\boldsymbol{q}\left(t, \boldsymbol{Z}^{(j)}\right) - \bar{\mu}_{\boldsymbol{Z}}(t)\right)^2 , \end{aligned} \tag{5}$$

where $\left\{\boldsymbol{Z}^{(j)}\right\}_{J=1}^{M}$ are realizations sampled randomly from the probability distribution of $\boldsymbol{Z}$. This is the Monte-Carlo (MC) method and it has a probabilistic error of $\mathcal{O}\left(1/\sqrt{M}\right)$.

Even though the MC methods are really robust and versatile, such a slow convergence rate is problematic, when the solution of a single realization of the system is computationally expensive. Alternative sampling methods are the Quasi Monte-Carlo methods (QMC). These can provide convergence rates of $\mathcal{O}((\log M)^d/M)$, where d is the co-dimension of the problem. They use low discrepancy sequences in order to uniformly cover the sampling domain. Without presumption of completeness, in this work only the Sobol sequence will be considered as a measure of comparison with respect to other advanced UQ methods. QMC methods are known to work better than MC methods when the integrand is sufficiently smooth, whereas they can completely fail on an integrand of unbounded variation [8]. Furthermore, randomized versions of the QMC method are available in order to improve the variance estimation of the method.

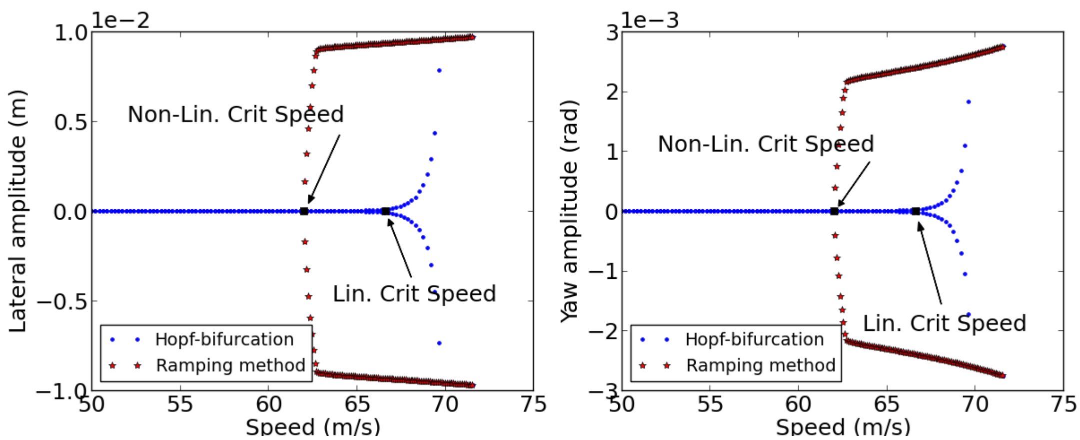

Fig. 2: Non-linear dynamics of the deterministic system. The subcritical Hopf-bifurcation is highlighted and the critical speed is determined exactly at $v_L = 66.61\ m/s$. The ramping method is then used in order to detect the non-linear critical speed at $v_{NL} = 62.02\ m/s$.

UNCERTAINTY QUANTIFICATION

The stochastic solution of the system is now represented by $\boldsymbol{q}(t, \boldsymbol{Z})$, where $\boldsymbol{Z}$ is a vector of random variables distributed according to (3). The solution is a function that spans over a three dimensional random parameter space. The dimension of the parameter space is called *the co-dimension* of the dynamical problem. In this work the focus will be restricted to the first two moments of this solution, namely the mean $\boldsymbol{E}[\boldsymbol{q}(t, \boldsymbol{Z})]$ and variance $\boldsymbol{V}[\boldsymbol{q}(t, \boldsymbol{Z})]$, but the following derivations can be used similarly for higher moments too. Mean and variance are defined as

$$\begin{aligned} \boldsymbol{\mu_q}(t) &= \boldsymbol{E}[\boldsymbol{q}(t, \boldsymbol{Z})]_{\rho_Z} = \iiint \boldsymbol{q}(t, \boldsymbol{z})\rho_{\boldsymbol{Z}}(\boldsymbol{z})d\boldsymbol{z} , \\ \boldsymbol{\sigma}_{\boldsymbol{q}}^2(t) &= \boldsymbol{V}[\boldsymbol{q}(t, \boldsymbol{Z})]_{\rho_Z} = \iiint \left(\boldsymbol{q}(t, \boldsymbol{z}) - \mu_{\boldsymbol{q}}(t)\right)^2 \rho_{\boldsymbol{Z}}(\boldsymbol{z})d\boldsymbol{z} \end{aligned} \tag{4}$$

where $\rho_{\boldsymbol{Z}}(\boldsymbol{z})$ is the probability density function of the random vector $\boldsymbol{Z}$ and the integrals are computed over its domain.

Stochastic collocation method (SCM)

Collocation methods require the residual of the governing equations to be zero at the collocation points $\left\{\boldsymbol{Z}^{(j)}\right\}_{\boldsymbol{j=1}}^{\boldsymbol{Q}}$, i.e.

$$\begin{cases} \partial_t \boldsymbol{q}\left(t, \boldsymbol{Z}^{(j)}\right) = \mathcal{L}\left(\boldsymbol{q}\left(t, \boldsymbol{Z}^{(j)}\right)\right), & (0, T] \\ \boldsymbol{q}(0) = \boldsymbol{q}_0, & t = 0 . \end{cases} \tag{6}$$

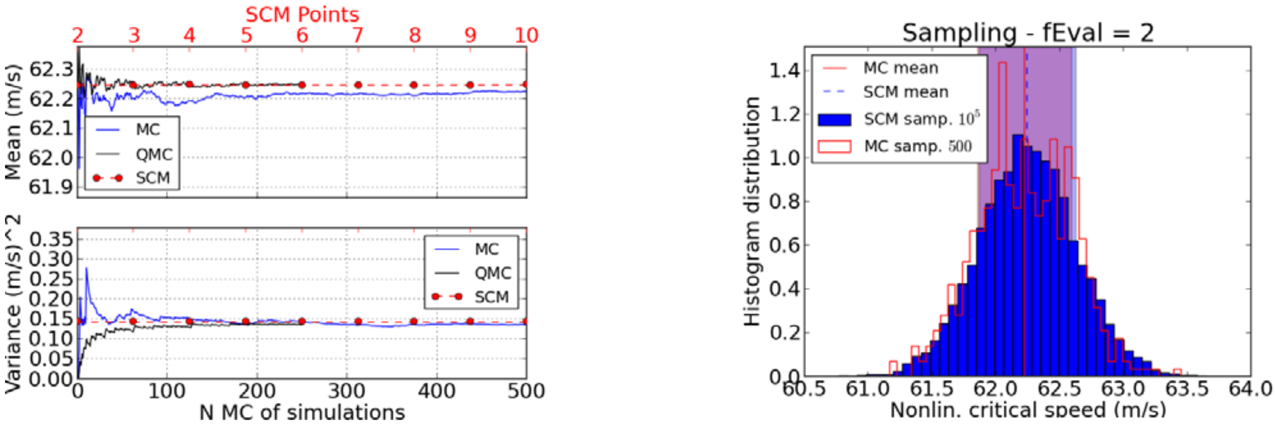

Fig. 3: SCM on the model with 1D uncertainty on parameter k_4 compared with MC and QMC. Left, estimation of mean and variance of the nonlinear critical speed. Right, histograms of NL critical speeds obtained using 500 MC simulations of model (1)-(2) and 10^5 realizations using the approximated stochastic solution (7) with only 2 function evaluations. The standard deviation is shown as a shaded confidence interval, blue for SCM and red for MC. The two confidence intervals are overlapping almost exactly.

Then an approximation $\boldsymbol{w}(t,\boldsymbol{Z})$ of $\boldsymbol{q}(t,\boldsymbol{Z})$ is found as an expansion in a set of Hermite polynomials, which are suitable for approximations of the Gauss distribution functions:

$$\boldsymbol{w}_N(t,\boldsymbol{Z}) = \sum_{|k|\leq N} \widehat{\boldsymbol{w}}_k(t)\, \mathcal{H}_k(\boldsymbol{Z}) \ ,$$

$$\widehat{\boldsymbol{q}}_k = \frac{1}{\gamma_k}\iiint \boldsymbol{q}(t,\boldsymbol{z})\, \mathcal{H}_k(\boldsymbol{z})\rho_{\boldsymbol{Z}}(\boldsymbol{z})d\boldsymbol{z} \approx \widehat{\boldsymbol{w}}_k \qquad (7)$$

$$= \frac{1}{\gamma_k}\sum_{j=1}^{Q} \boldsymbol{q}\left(t,\boldsymbol{z}^{(j)}\right) \mathcal{H}_k\left(\boldsymbol{z}^{(j)}\right)\alpha^{(j)} \ ,$$

where we used a cubature rule with points and weights $\left\{\boldsymbol{z}^{(j)},\alpha^{(j)}\right\}_{j=1}^{Q}$. The points $\left\{\boldsymbol{z}^{(j)}\right\}_{j=1}^{Q}$ are the set of parameter values for which deterministic solutions must be computed. Cubature rules with different accuracy levels and sparsity exist. In this work simple tensor product structured Gauss cubature rules will be used. These are the most accurate but scale with $\mathcal{O}(m^d)$, where m is the number of points in one dimension and d is the co-dimension. The fast growth of the number of collocation points with the dimensionality goes under the name of "the curse of dimensionality" and can be addressed using more efficient cubature rules such as Smolyak sparse grids [9].

UNCERTAINTY QUANTIFICATION IN RAILROAD VEHICLE DYNAMICS

Uncertainty quantification is recently gaining much attention from many engineering fields and in vehicle dynamics there are already some contributions on the topic. In [10] a railroad vehicle dynamic problem with uncertainty on the suspension parameters was investigated using MC method coupled with techniques from Design of Experiments.

Here SCM will be applied to the simple Cooperrider truck [2] in order to study its behavior with uncertainties, and the results will be compared to the ones obtained by the MC and QMC methods.

These methods belong to the class of non-intrusive methods for Uncertainty Quantification. This means that they only require a deterministic method to compute the quantity of interest (QoI) for different parameters. In this work this is the ramping method to detect the critical speed.

The focus of this work is on the determination of the nonlinear critical speed with uncertainties, so the investigation of the stochastic dynamics with respect to time will be disregarded here. Fig. 3 shows the SCM method applied to the model with 1D uncertainty on parameter k_4, for the determination of the first two moments of the nonlinear critical speed. The estimation done by the SCM is already satisfactory at low order and little is gained by increasing it. This means that the few first terms of the expansion (7) are sufficient in approximating the nonlinear critical speed distribution.

Fig. 4 shows the SCM method applied to the same problem with 1D uncertainty on the torsional spring stiffness k_6. Again the first few terms in expansion (7) are sufficient in order to give a good approximation of the nonlinear critical speed distribution. It is worth noting that the torsional spring stiffness k_6 has an higher influence on the critical speed than k_4.

Fig. 5 shows the SCM method on the problem with uncertainty on parameters k_6, k_4 and D_2. Again, a low-order SCM approximation is sufficient to get the most accurate solution.

In the figures 3-5, left, we have compared the convergence of the SCM method with that of the MC method. Therefore the number of evaluations was prescribed. It is also of interest to compare the computation time of the methods expressed by the CPU time. For the comparison we used the calculated mean values of the critical speed as the basis for the comparison. For the SCM method the iteration process was ended when the second decimal remained constant. The mean values in the MC and QMC methods change however a good deal as shown in the figures 3-5, left. Therefore, for the comparison a window with 20 iterative values, which is glided over the number of iterations was used. When the second decimal of the average of

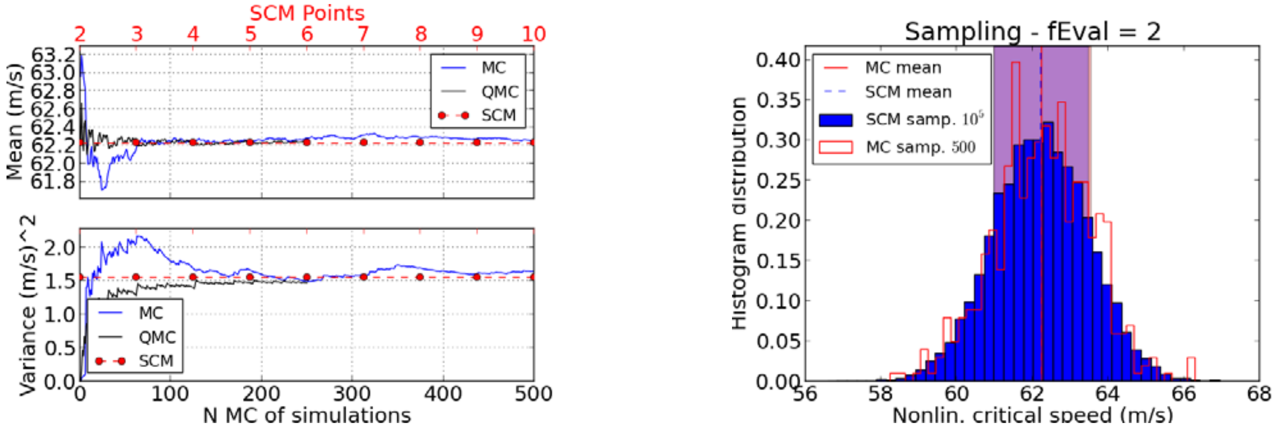

Fig. 4: SCM on 1D uncertainty on parameter k_6 compared with MC and QMC. Left, estimation of mean and variance of the non-linear critical speed. Right, histograms of NL critical speeds obtained using 500 MC simulations of model (1)-(2) and 10^5 realizations using the approximated stochastic solution (7) with only 2 function evaluations. The standard deviation is shown as a shaded confidence interval, blue for SCM and red for MC. The two confidence intervals are overlapping almost exactly.

the iterated values in the window remained constant, when the widow was pushed one more step, then the iterations were stopped, and the CPU time was stored.

Table 1 shows the final results obtained with the chosen accuracy, using the three methods, Monte-Carlo (MC), Quasi-Monte-Carlo (QMC) and Stochastic Collocation (SCM). We can observe that the variances in the multi-dimensional cases are almost equal to the sum of the single-dimensional cases. This means that the effect of the nonlinear interactions between the three elements of the suspension is small with the variances chosen in this problem.

CONCLUSIONS

Manufacturing tolerances have been introduced into the dynamical investigations of vehicles. A new method, the Stochastic Collocation Method (SCM) is applied as a tool for "Uncertainty Quantification", and the accuracy and computational effort is compared with that of Monte-Carlo (MC) and Quasi-Monte-Carlo (QMC) methods. The "Uncertainty Quantification" methods are applied to the estimate of the calculated critical speed of a railroad vehicle model. The critical speed is delivered as a mean value with variance. The results show that under the condition of the same accuracy the convergence rate of the SCM outperforms the rates of as well the MC as the QMC methods. Table 1 shows that the CPU time and thus the computational effort by application of the SCM is much smaller than the computational effort by application of the MC or QMC methods. By all three methods the total computational effort is larger than the effort by a deterministic computation, because the same dynamical system must be solved repeatedly only with different parameter values. Under these conditions it is however possible to reduce the total *elapsed time* significantly by straightforward application of *parallel computing*. The dynamics of the vehicle model is calculated in the process, but the results are not shown here due to the limited space. A very simple model was chosen in order to demonstrate the superiority of SCM over the MC

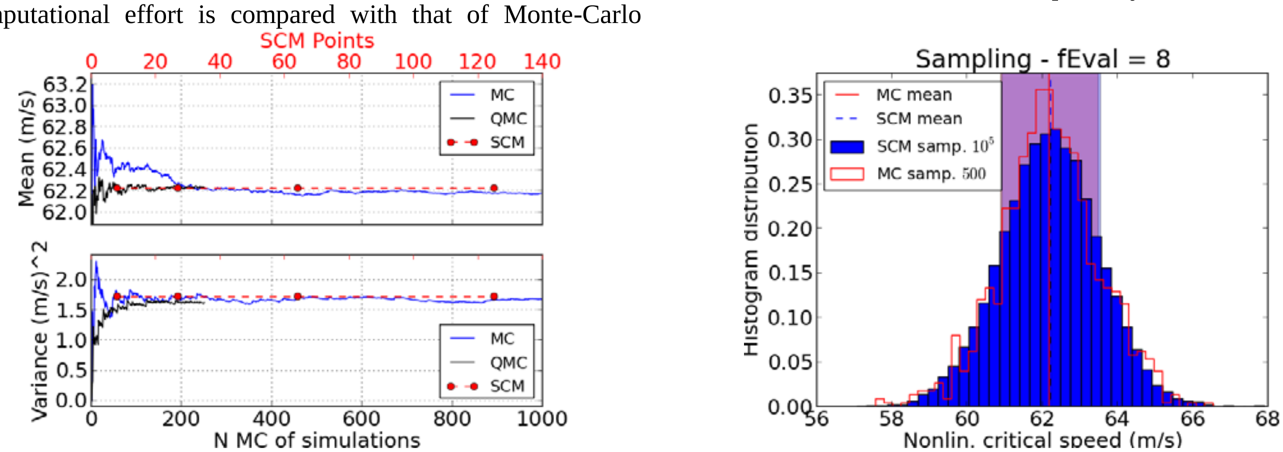

Fig. 5: SCM on 3D uncertainty compared with MC and QMC. Left, estimation of the mean and variance of the non-linear critical speed. Right, histograms of nonlinear critical speeds.

	MC				QMC				SCM			
	μ	σ^2	**#fE**	**CPUt**	μ	σ^2	**#fE**	**CPUt**	μ	σ^2	**#fE**	**CPUt**
k_6	62,26	1,64	169	∼24h	62,24	1,47	152	∼21 h	62,23	1,55	2	∼10m
k_4	62,23	0,14	17	∼2,5h	62,25	0,14	22	∼3 h	62,25	0,14	2	∼11m
D_2	62,23	0,02	9	∼1 h	62,25	0,02	4	∼30m	62,25	0,03	2	∼11m
k_6, k_4	62,22	1,53	148	∼21 h	62,22	1,62	152	∼22 h	62,28	1,69	4	∼36m
k_6, D_2	62,18	1,72	216	∼30 h	62,24	1,50	142	∼20 h	62,28	1,57	4	∼37m
k_4, D_2	62,25	0,17	25	∼3,5h	62,25	0,16	25	∼3,5h	62,30	0,17	4	∼35m
k_6, k_4, D_2	62,18	1,68	221	∼32 h	62,23	1,63	154	∼22 h	62,23	1,72	8	∼1 h

Table 1: Estimated mean and variance of the nonlinear critical speed using MC, QMC and SCM. The three methods are compared in terms of number of function evaluations (#fE) and computation time (CPUt).

and QMC methods. By using the same distributions for the characteristics of the two lateral springs and dampers the effect of the loss of symmetry in a real vehicle was not investigated here. SCM can be 100 times faster than MC for low co-dimensional problems, but for high co-dimensional problems SCM methods suffer from the “curse of dimensionality”. The computational effort of the SCM grows very fast with the number of independent parameters. In a realistic vehicle model that number easily surpasses 20. Therefore the work continues with an investigation of the application of statistical methods that may reduce the computational effort by singling out the parameters that have the most important influence on the wanted result of the dynamical problem. Some early results are shown in [11].

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