FVM for Nonlinear Soil Stress analysis involving Pore Pressure Coupling

Tian Tang\(^1\), Ole Hededal\(^1\), Philip Cardiff\(^2\), Johan Roenby\(^3\)

\(^1\)Technical University of Denmark, \(^2\)University College Dublin, \(^3\)DHI

8\(^{th}\) OpenFOAM\(^\text{®}\) Workshop, Jeju, 11\(^{th}\) – 14\(^{th}\) June 2013
Outline

1. Introduction
   - Motivation
   - Objectives

2. Soil Solver Development using OpenFOAM
   - Mathematical model
   - Discretization & Solution procedure
   - Test cases

3. Collaborative works in UCD OpenFOAM group
   - More soil features
   - Convergence acceleration

4. Summary & Future works

Abbreviations
Outline

1. **Introduction**
   - Motivation
   - Objectives

2. **Soil Solver Development using OpenFOAM**
   - Mathematical model
   - Discretization & Solution procedure
   - Test cases

3. **Collaborative works in UCD OpenFOAM group**
   - More soil features
   - Convergence acceleration

4. **Summary & Future works**
Why we started with this research topic?

- Offshore structure and foundation failures due to seabed instability (liquefaction) are observed.
- Integrated numerical modelling of seabed-wave-structure interaction is demanding.
- OpenFOAM as an open source FVM library facilitates the customer solver developments.

Figure: An illustration of wave-seabed-structure interaction
Our Goals

- Current goal: Development of an efficient soil solver with plastic soil deformation and pore pressure coupling.
- Future goal: Multiphysics modeling by combining the developed soil solver with existing fluid and structure solvers.

**Figure:** The multiphysics solver structure using OpenFOAM
Outline

1. Introduction
   - Motivation
   - Objectives

2. Soil Solver Development using OpenFOAM
   - Mathematical model
   - Discretization & Solution procedure
   - Test cases

3. Collaborative works in UCD OpenFOAM group
   - More soil features
   - Convergence acceleration

4. Summary & Future works
Soil Mathematical Model

Governing Equations:

- Total momentum balance for the soil mixture (steady-state)

\[ \nabla \cdot \boldsymbol{\sigma} - \nabla p = 0 \]

- Storage equation for pore fluid flow

\[ \frac{n}{K'} \frac{\partial p}{\partial t} - \frac{k}{\gamma_w} \nabla^2 p + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}) = 0 \]

\( \boldsymbol{\sigma}: \) soil effective stress, \( p: \) pore fluid pressure,
\( \mathbf{u}: \) soil skeleton displacement,
\( n, K': \) soil porosity and pore fluid bulk modulus,
\( k, \gamma_w: \) soil permeability and water specific weight.
Soil Mathematical Model

Governing Equations:

- Total momentum balance for the soil mixture (steady-state)

\[ \nabla \cdot \sigma - \nabla p = 0 \]

- Storage equation for pore fluid flow

\[ \frac{n}{K'} \frac{\partial p}{\partial t} - \frac{k}{\gamma_w} \nabla^2 p + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) = 0 \]

nonlinear constitutive relation & displacement-pressure coupling

\( \sigma \): soil effective stress, \( p \): pore fluid pressure,
\( \mathbf{u} \): soil skeleton displacement,
\( n, K' \): soil porosity and pore fluid bulk modulus,
\( k, \gamma_w \): soil permeability and water specific weight.
Soil Mathematical Model

Constitutive relations:

- **Linear elasticity**
  \[ d\sigma = C^e : (d\varepsilon - d\varepsilon^p), \quad d\varepsilon = \frac{1}{2} \left\{ \nabla (du) + [\nabla (du)]^T \right\} \]

- **Mohr-Coulomb perfect plasticity**
  
  Yield surface \( f = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin \varphi - 2c \cos \varphi \)

  Plastic potential \( g = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin \psi \)

  Flow rule \( d\varepsilon^p = d\Lambda \cdot \frac{\partial g}{\partial \sigma}, \quad d\Lambda = \left( \frac{\partial f}{\partial \sigma} \right)^T C^e d\varepsilon \left( \frac{\partial f}{\partial \sigma} \right)^T C^e \frac{\partial g}{\partial \sigma} \)

  \( \sigma_1, \sigma_3 \): maximum and minimum principal stress, 
  \( \varphi, c, \psi \): soil friction angle, cohesion, and dilation angle, 
  \( C^e \): linear elastic stiffness tensor.
Discretization & Solution procedure

- Cell-centered finite volume discretization
- Global solution procedure:
  1. Partitioned (segregated) approach

fvScalarMatrix pEqn
(
    fvm::ddt(p) == fvm::laplacian(Dp, p) - fvc::div(fvc::ddt(Dp2,U))
);

fvVectorMatrix dUEqn
(
    fvm::laplacian(2.0*mu + lambda, dU, "laplacian(dU)"")
    ==
    - divDsigmaExp
    + fvc::div(2.0*mu*(mesh.Sf() & fvc::interpolate(dEpsP)))
    + fvc::div(lambda*(mesh.Sf() & I*fvc::interpolate(tr(dEpsP))))
    + fvc::grad(dp)
);

2. Fixed Point iteration + Underrelaxation

**Figure**: The iterative solution strategy of `nonLinearBiotFoam` in OpenFOAM
**Discretization & Solution procedure**

- **Local stress update**

**Stress return algorithm**

**INPUT:** $d\mathbf{u}$, displacement increments
- $\sigma^A$, initial/old-time stress

1. Compute the elastic trial stress $\sigma^B$ by:
   \[
   \sigma^B = \sigma^A + \{\mu \nabla (d\mathbf{u}) + \mu [\nabla (d\mathbf{u})]^T + \lambda \text{tr}[(d\mathbf{u})]\}
   \]

2. Transform $\sigma^B$ into principal space as $\sigma^B_{\text{prin}}$. Store the principal directions.

3. Evaluate the yield function $f(\sigma^B_{\text{prin}})$:
   - if $f < 0$, EXIT, $\sigma^C = \sigma^B$, $d\varepsilon^P = 0$
   - if $f \geq 0$, CONTINUE

4. Determine the right stress return type.
   - Obtain the principal plastic corrector stress $\sigma^C_{\text{prin}}$.

5. Reuse the preserved principal directions and transform $\sigma^C_{\text{prin}}$ back to $\sigma^C$

6. Calculate the plastic strain increment $d\varepsilon^P$

**OUTPUT:** $\sigma^C$, $d\varepsilon^P$

---

**Figure:** Illustration of return mapping
Test cases

- Elastic consolidation test
- Drained triaxial compression test
- Elasto-plastic consolidation test
Test cases: Elastic consolidation test

A saturated soil column subjected to a surface step loading:

\[ p = 0, \quad \frac{\partial p}{\partial n} = 0, \quad u_x = 0, \quad T_y = 0 \]

\[ \frac{\partial p}{\partial n} = 0, \quad u_x = 0, \quad u_y = 0 \]

**Figure:** Case definition

**Figure:** Pore pressure distribution along the depth after different consolidation time

- **Soil parameters:**
  \[ E = 3.0 \times 10^7 N/m^2 \]
  \[ \nu = 0.2 \]
  \[ k = 0.0001 m/s \]
  \[ n = 0.3 \]
  \[ \gamma_w = 1.0 \times 10^4 N/m^2 \]
  \[ K_w = 2.1 \times 10^9 N/m^2 \cdot s, \quad S_r = 0.99 \]
  \[ h = 30 m \]
Test cases: Drained triaxial compression test

A drained soil cube compressed by constant strain rate:

![Solution domain and boundary conditions](image)

Soil parameters:
- \( E = 9.5 \times 10^3 \) N/m², \( \nu = 0.185 \)
- \( \phi = 40^\circ, c = 5 \times 10^5 \) N/m², \( \psi = 10^\circ \)
- \( L = 0.1 \) m

**Figure:** Simulated elastic perfect-plastic soil response
Test cases: Elasto-plastic consolidation test

A layer of saturated soil loaded by a strip footing with different loading rate:

Figure: A sketch of the case geometry (left); OpenFOAM mesh and boundary conditions (right)
Test cases: Elasto-plastic consolidation test

A layer of saturated soil loaded by a strip footing with different loading rate:

Figure: FEM results by Small et al. (left); FVM results by nonLinearBiotFoam (right).
Test cases: Elasto-plastic consolidation test

Animations: Pore pressure variation + Yielding zone

(displacement exaggerated by a factor of 10)
Modeling more soil features: Large deformation

- Linear elastic stress-strain + Nonlinear strain-displacement relation (Total Lagrangian format):

\[ \delta S = 2\mu \delta E + \lambda \text{tr}(\delta E) I \]

\[ \delta E = \frac{1}{2} \left[ \nabla (\delta u) + (\nabla (\delta u))^T + (\nabla (\delta u)) \cdot \nabla^T + \nabla u \cdot (\nabla (\delta u))^T + \nabla (\delta u) \cdot \nabla (\delta u)^T \right] \]

- All the nonlinear terms treated explicit + Fixed point iteration:

\[ \left( 2\mu + \lambda \right) \nabla^2 (\delta u) + \nabla \cdot \left\{ -(\mu + \lambda) \nabla (\delta u) + \mu [ (\nabla (\delta u))^T + (\nabla (\delta u)) \cdot (\nabla^T) + (\nabla u) \cdot (\nabla (\delta u))^T + (\nabla (\delta u)) \cdot (\nabla (\delta u))^T] + \text{tr}(\delta E) \right\} \]

\[ + \nabla \cdot \left[ \delta S \cdot \nabla u + (S + \delta S) \cdot \nabla (\delta u) \right] = 0 \]

- Extend to small strain large displacement elastoplastic solver (similar solution procedure to small strain small displacement EP solver)
Modeling more soil features: Large deformation

- Verification: a simple tension test

*Figure:* a) Resulting stress-displacement relationships from `soilEpTLFoam` prediction; b) Resulting force-displacement relationships from `soilEpTLFoam` prediction
Soil deposits are inherently anisotropic due to the process of sedimentation followed by predominantly one-dimensional consolidation. Soil anisotropy is used in reference to soil structure, soil strength, and soil permeability changes with direction of measurement.

Assumption: cross-anisotropic soil

\[
\mathbf{C}^e = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C_{44}
\end{pmatrix}
\]

\[
\mathbf{k} = \begin{pmatrix}
k_1 & 0 & 0 \\
0 & k_1 & 0 \\
0 & 0 & k_3
\end{pmatrix}
\]

\(\mathbf{C}^e\): cross-anisotropic elastic stiffness tensor, \(\mathbf{k}\): permeability coefficient tensor.
Modeling more soil features: Anisotropy

- Implementation:

\[
\nabla \cdot (K \cdot \nabla (du)) + \nabla \cdot (C : d\varepsilon) - \nabla \cdot (K \cdot \nabla (du)) - \nabla p = 0
\]

\[
\frac{n}{K'} \frac{\partial p}{\partial t} - \frac{1}{\gamma_w} \nabla \cdot (k \cdot \nabla p) + \frac{\partial}{\partial t} (\nabla \cdot u) = 0
\]

- Tested by a case of standing wave induced cross-anisotropic seabed response.
Modeling more soil features: Anisotropy

Figure: Standing-wave induced anisotropic and isotropic soil response. Isotropy (right):

$$E = 10^7 \text{Pa}, \nu = 0.3,$$
$$E_z = 10^7 \text{Pa}, \nu_{xx} = \nu_{zx} = 0.3, n = m = 0.6, k_x = k_z = 10^{-4} \text{m/s}.$$
Convergence consideration

Fixed point iteration method in FVM has both advantages and drawbacks:

- No need to form and update the Jacobian matrix. 😊

- Create diagonally dominant sparse matrices ideally suited for iterative solver. 😊

- No convergence for some highly nonlinear and strong coupling problems. If combined with fixed underrelaxation, slow convergence. 😞

Seek for adaptive underrelaxation → Aitken’s method
Convergence consideration

Fixed point iteration method in FVM has both advantages and drawbacks:

- No need to form and update the Jacobian matrix. 😊
- Create diagonally dominant sparse matrices ideally suited for iterative solver. 😊
- No convergence for some highly nonlinear and strong coupling problems. If combined with fixed underrelaxation, slow convergence. 😞

Seek for adaptive underrelaxation → Aitken’s method
Convergence consideration: Aitken’s method

Aitken’s method is most useful for accelerating a linear convergent sequence:

\[ x^{i+1} = \tilde{x}^{i+1} + \theta^{i+1} r^i \]

\[ r^i = \tilde{x}^{i+1} - x^i \]

\[ \theta^{i+1} = -\theta^i \frac{r^{i-1} (r^i - r^{i-1})}{(r^i - r^{i-1}) (r^i - r^{i-1})} \]

where, \( x \) is the solving variable (or variable vector). The tilde sign denotes the solved value before underrelaxation.

Apply to a large deformation elastoplastic simple tension test case:

<table>
<thead>
<tr>
<th></th>
<th>Number of outer iterations (plastic step)</th>
<th>total CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed under-relaxation</td>
<td>618</td>
<td>38.34</td>
</tr>
<tr>
<td>Aitken’s relaxation</td>
<td>264</td>
<td>15.79</td>
</tr>
</tbody>
</table>
Outline

1 Introduction
   • Motivation
   • Objectives

2 Soil Solver Development using OpenFOAM
   • Mathematical model
   • Discretization & Solution procedure
   • Test cases

3 Collaborative works in UCD OpenFOAM group
   • More soil features
   • Convergence acceleration

4 Summary & Future works
Summary:

- FVM soil solver has been developed using OpenFOAM, it has the following features:
  - elasto-plastic soil deformations
  - pore pressure coupling
- The nonlinearity and coupling in the equations are tackled by partitioned approach and fixed point iteration.
- Large deformation and soil anisotropy added to the soil solver.
- Aitken’s method is applied for convergence acceleration.

Next step:

- Implementation of advanced soil solver based on critical state and cyclic plasticity.
- Further convergence improvement.
References:


Thank you for your attention!