A hybrid model for optimal decisions within personal finance and retirement

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case (A) - optimal investment, consumption and life insurance, [Richard, 1975]
case (B) - optimal investment with optimal annuities
Optimization technology

stochastic optimal control - explicit solutions

✔ ideal framework - produce an optimal policy that is easy to understand and implement

✗ explicit solution may not exist

✗ can’t deal with details

stochastic (linear) programming (SLP)

✔ general purpose decision model with an objective function that can take a wide variety of forms

✔ can address realistic considerations, such as transaction costs

✔ can deal with details

✗ problem size grows quickly as a function of number of periods and scenarios

✗ challenge to select a representative set of scenarios for the model
Hybrid model

- first years decisions - multi-stage stochastic linear programming (SLP)
- decisions for the long steady period - stochastic optimal control (dynamic programming)
Dynamic programming I

Wealth dynamics

\[ dX_t = \left( r + \pi_t(\alpha - r) \right) X_t dt + \pi_t \sigma X_t dW_t + l_t dt - c_t dt - \mu^*_t l_t dt, \]
\[ X_0 = x_0. \]

Maximize expected utility of consumption and bequest

\[ V(t, x) = \sup_{\pi, c, l \in Q[t, \tilde{T}]} E_{t,x} \left[ \int_t^{\tilde{T}} e^{-\int_t^s \mu_t d\tau} \left( u(s, c) + \mu_s U(s, X_s + l_s) \right) ds \right], \]

with the utility functions:

\[ u(c, t) = \frac{1}{\gamma} w^{1-\gamma(t)} c^\gamma = \frac{1}{\gamma} e^{-\rho t} c^\gamma, \quad U(x) = \frac{1}{\gamma} v^{1-\gamma(t)} x^\gamma = \frac{1}{\gamma} \lambda^{-\gamma} e^{-\rho t} x^\gamma, \]

1 - \gamma - risk aversion, \( \rho \) - impatience factor, \( \lambda \) - weight on bequest, \( \mu_t \) - mortality rate, \( \mu^*_t \) - pricing mortality rate.
Dynamic programming II

Solution (optimal value function)

\[ V(t, x) = \frac{1}{\gamma} f^{1-\gamma}(t) (x + g(t))^\gamma, \]

\[ f(t) = \int_t^{\tilde{T}} e^{-\frac{1}{1-\gamma} \int_t^s (\mu_\tau - \gamma (\mu^*_\tau + \varphi)) d\tau} \left[ w(s) + \left( \frac{\mu_s}{(\mu^*_s)^\gamma} \right)^{1/(1-\gamma)} \nu(s) \right] ds, \]

\[ g(t) = \int_t^{\tilde{T}} e^{\int_t^s (r + \mu^*_\tau) d\tau} l(s) ds. \]

The optimal controls

\[ \pi_t^* = \frac{\alpha - r}{\sigma^2(1-\gamma)} \frac{X_t + g(t)}{X_t}, \quad c_t^* = \frac{w(t)}{f(t)} (X_t + g(t)), \]

\[ l_t^* = \left( \frac{\mu_t}{\mu^*_t} \right)^{1/(1-\gamma)} \frac{\nu(t)}{f(t)} (X_t + g(t)) - X_t. \]
SLP model - objective

\[
\sum_{T} s_{\text{SLP}} - \sum_{n \nu} \left( s_{\nu} \right) \cdot e^{-\int_{s_{0}}^{T} \mu \tau d\tau} \left[ u(s, \tilde{C}_{n \nu}(s)) + \mu s U(s, \sum_{i=1}^{N} \tilde{X}_{i n \nu}(s) + \tilde{I}_{n \nu}(s)) \right] + \sum_{n \nu} \left( T_{\text{SLP}} \right) \cdot e^{-\int_{T_{\text{SLP}} s_{0}}^{T_{\text{SLP}}} \mu \tau d\tau} \cdot V(T_{\text{SLP}}, \sum_{i=1}^{N} \tilde{X}_{i n \nu}(T_{\text{SLP}})) \rightarrow \max
\]
SLP model - objective

\[
\sum_{s=t_0}^{T_{SLP}-1} \sum_{\nu(s)} P r_{\nu(s)} \cdot e^{-\int_{t_0}^{s} \mu_{\tau} d\tau} \left[ u(s, \tilde{C}_{\nu(s)}) + \mu_s U(s, \sum_{i=1}^{N} \tilde{X}_{i\nu(s)} + I_{\nu(s)}) \right] \\
+ \sum_{\nu(T_{SLP})} P r_{\nu(T_{SLP})} \cdot e^{-\int_{t_0}^{T_{SLP}} \mu_{\tau} d\tau} \cdot V \left( T_{SLP}, \sum_{i=1}^{N} \tilde{X}_{i\nu(T_{SLP})} \right) \rightarrow \text{max}
\]
SLP model - objective

\[ \sum_{s=t_0}^{T_{SLP}-1} \sum_{n_{\nu(s)}} P r_{n_{\nu(s)}} \cdot e^{-\int_{t_0}^{s} \mu_\tau d\tau} \left[ u(s, \tilde{C}_{n_{\nu(s)}}) + \mu_s U(s, \sum_{i=1}^{N} \tilde{X}^i_{n_{\nu(s)}} + \tilde{l}_{n_{\nu(s)}}) \right] \\
+ \sum_{n_{\nu(T_{SLP})}} P r_{n_{\nu(T_{SLP})}} \cdot e^{-\int_{t_0}^{T_{SLP}} \mu_\tau d\tau} \cdot V \left( T_{SLP}, \sum_{i=1}^{N} \tilde{X}^i_{n_{\nu(T_{SLP})}} \right) \rightarrow \text{max} \]

- Obs! linearize the objective

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Constraints

**budget equation**, \( t = t_0, \ldots, T_{SLP} - 1 \) and \( \nu(t) = 1, \ldots, K_t \),

\[
\sum_{i=1}^{N} \tilde{P}^i_{n\nu(t)} + \tilde{C}_{n\nu(t)} + \mu^* \tilde{I}^i_{n\nu(t)} = x_0 \mathbb{1}\{t=t_0\} + \sum_{i=1}^{N} \tilde{S}^i_{n\nu(t)} + l_t, \tag{1}
\]

**asset inventory balance**, \( t = t_0, \ldots, T_{SLP} \), \( \nu(t) = 1, \ldots, K_t \), \( i = 1, \ldots, N \):

\[
\tilde{X}^i_{n\nu(t)} = (1 + R^i_{n\nu(t)}) \tilde{X}^i_{n\nu(t-1)} \mathbb{1}\{t>t_0\} + \tilde{P}^i_{n\nu(t)} \mathbb{1}\{t<T_{SLP}\} - \tilde{S}^i_{n\nu(t)} \mathbb{1}\{t<T_{SLP}\}, \tag{2}
\]

**non-negativity**, \( t = t_0, \ldots, T_{SLP} - 1 \), \( \nu(t) = 1, \ldots, K_t \):

\[
\tilde{C}_{n\nu(t)} > 0, \quad \sum_{i=1}^{N} \tilde{X}^i_{n\nu(t)} + \tilde{I}^i_{n\nu(t)} > 0, \quad \tilde{P}^i_{n\nu(t)} \geq 0, \quad \tilde{S}^i_{n\nu(t)} \geq 0, \tag{3}
\]

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Results I - case (A)

- payments: \( l_t = 27,000 \) EUR, \( x_0 = 60,000 \) EUR,
- market: \( N = 2, r = 0.02, \alpha = 0.04, \sigma = 0.2, \)
- utility function: \( \gamma = -3, \rho = 0.04, \lambda = 10, \)
- age_0 = 45, age_T = 65,
- life uncertainty: \( \theta = 0.0, \beta = 4.59364, \delta = 0.05032, \)
- scenario tree: \( T_{SLP} = 8, bf = 3, \) number of trees = 10,
- linearization: \( m = 40, \)
- \( bp_1^c = 0.7E[c_{T_{SLP}}^*], bp_m^c = 2E[c_{T_{SLP}}^*], \)
- \( bp_1^{x+g} = 0.5E[X_{T_{SLP}}^* + g_{T_{SLP}}], bp_m^{x+g} = 2E[X_{T_{SLP}}^* + g_{T_{SLP}}], \)
- \( bp_1^{x+ins} = 0.5E[X_{T_{SLP}}^* + I_{T_{SLP}}^*], bp_m^{x+ins} = 1.5E[X_{T_{SLP}}^* + I_{T_{SLP}}^*], \)
Modified constraints

**budget equation with transaction costs**, \( t = t_0, \ldots, T_{SLP} - 1, \nu(t) = 1, \ldots, K_t \)

\[
\sum_{i=1}^{N} \tilde{P}_{n_{\nu}(t)}^i (1 + q^i) + \tilde{C}_{n_{\nu}(t)} + \mu^* \tilde{l}_{n_{\nu}(t)} = x_0 \mathbb{1}_{\{t = t_0\}} + \sum_{i=1}^{N} \tilde{S}_{n_{\nu}(t)}^i (1 - q^i) + l_t,
\]  

(1')

**asset inventory balance with taxes on capital gains**, \( t = t_0, \ldots, T_{SLP}, \nu(t) = 1, \ldots, K_t, \)

\( i = 1, \ldots, N : \)

\[
\tilde{X}_{n_{\nu}(t)}^i = (1 + \text{net} R_{n_{\nu}(t)}^i) \tilde{X}_{n_{\nu}(t-1)}^i \mathbb{1}_{\{t > t_0\}} + \tilde{P}_{n_{\nu}(t)}^i \mathbb{1}_{\{t < T_{SLP}\}} - \tilde{S}_{n_{\nu}(t)}^i \mathbb{1}_{\{t < T_{SLP}\}},
\]

(2')

**limits on portfolio composition**, \( t = t_0, \ldots, T_{SLP}, \nu(t) = 1, \ldots, K_t : \)

\[
\tilde{X}_{n_{\nu}(t)}^i \geq d_i \sum_{i=1}^{N} \tilde{X}_{n_{\nu}(t)}^i, \quad \tilde{X}_{n_{\nu}(t)}^i \leq u_i \sum_{i=1}^{N} \tilde{X}_{n_{\nu}(t)}^i.
\]  

(4)
Results II - case (A) with modifications

Optimal investment: a) original constraints, b) limit on portfolio composition, $u = 100\%$, c) transaction costs, $q = 0.5\%$, d) taxes on capital gains, $\tau = 20\%$. 
Bibliography

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