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Slow-fast analysis of Earthquake Faulting

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1 Model

Burridge-Knopoff Rate and State model of Earthquake Faulting, Fig. 1 (Erickson et al. 2008).

$$\begin{aligned} \dot{x}_2 &= -e^{z_2}[x_2 + (1+\alpha)z_2], \\ \dot{y}_2 &= e^{z_2} - 1, \\ \varepsilon\dot{z}_2 &= -e^{-z_2}\left(y_2 + \frac{x_2+z_2}{\xi}\right), \end{aligned} \qquad (1)$$

with $\varepsilon \in [10^{-24}, 10^{-8}]$, $\alpha > \xi$, e^{z_2}: velocity.
Numerically **very stiff** (Fig. 1 for $\varepsilon = 10^{-2}$), can't be solved for $\varepsilon \in [10^{-24}, 10^{-8}]$.

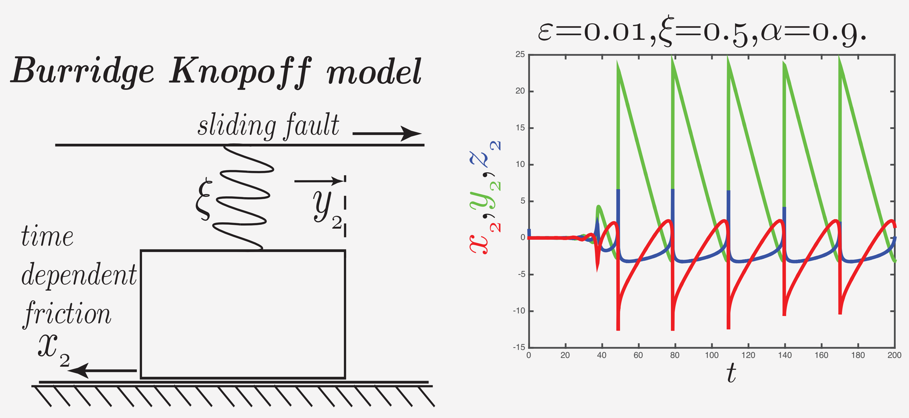

Figure 1: Left: Model of eq. 1. Right: Time solution of eq. 1.

We perform slow-fast analysis using Geometric Singular Perturbation Theory.

2 Singular analysis ($\varepsilon = 0$)

Layer problem - fast flow:

$$z_2' = -e^{-z_2}\left(y_2 + \frac{x_2+z_2}{\xi}\right), \qquad (x_2, y_2)(t) = (x_2^0, y_2^0).$$

Critical manifold:

$$C_0: \; y_2 + \frac{x_2+z_2}{\xi} = 0, \qquad \left.\frac{\partial z_2'}{\partial z_2}\right|_{C_0} = -\xi^{-1}e^{-z_2} < 0.$$

C_0 everywhere normally hyperbolic, not for $z_2 \to \infty$.
Reduced problem - slow flow:

$$\mathbf{F}(y_2, z_2) := \begin{cases} \dot{y}_2 &= e^{z_2} - 1, \\ \dot{z}_2 &= \xi + e^{z_2}(\alpha z_2 - \xi y_2 - \xi). \end{cases}$$

Fixed point: $(y_2, z_2) = (0, 0)$, (faults steady sliding).
Degenerate Hopf bifurcation for $\alpha = \xi$.
⇒ Hamiltonian system with closed periodic orbits for $H = h$, $h \in]0, 1[$.

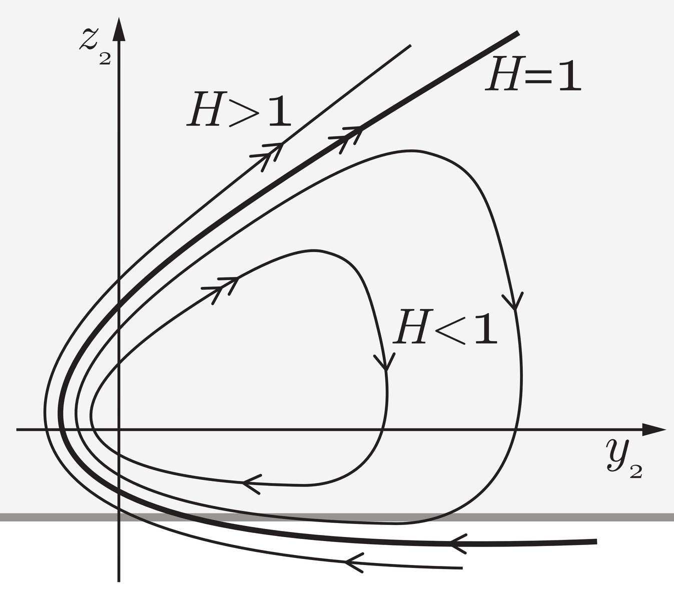

Figure 2: Phase space on C_0 for $\varepsilon = 0$, $\alpha = \xi$.

3 Perturbation from singular case ($0 < \varepsilon \ll 1$)

By Fenichel's theory find supercritical Hopf bifurcation for

$$\alpha_H = \xi - \varepsilon\xi^2 + \mathcal{O}(\varepsilon^2).$$

1st Lyapunov coefficient: $K = -\frac{\varepsilon}{8}\xi^3(\xi+1) + \mathcal{O}(\varepsilon^2)$, proportional to $\varepsilon \Rightarrow$ valid for $|\alpha - \alpha_H| \leq c\varepsilon$.
For $\alpha - \alpha_H > c\varepsilon$ continue with a Melnikov-type computation from the Hamiltonian system.

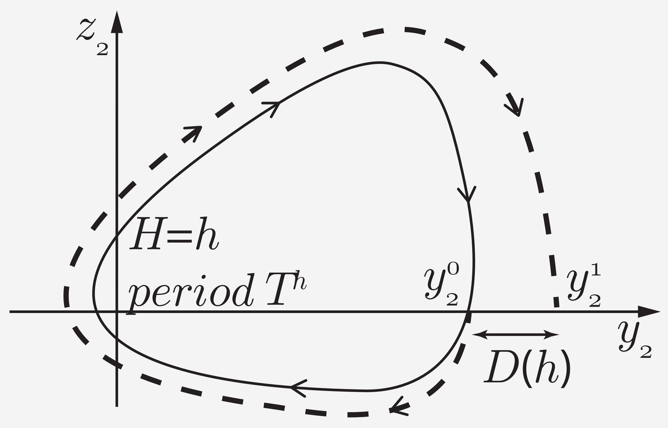

Figure 3: Perturbation from the Hamiltonian system.

Taylor expand $D(h) = H(y_2^1, 0) - H(y_2^0, 0)$:

$$D(h) = (\alpha - \xi)D_\alpha(h) + \varepsilon D_\varepsilon(h) + \mathcal{O}((\varepsilon + (\alpha - \xi))^2). \qquad (2)$$

$$D_\varepsilon(h) = \int_0^{T_h} \nabla H(h) \cdot \mathbf{F}_\varepsilon(y_2, z_2)|_{\varepsilon=0,\alpha=\xi}\, dt,$$

$$D_\alpha(h) = \int_0^{T_h} \nabla H(h) \cdot \mathbf{F}_\alpha(y_2, z_2)|_{\varepsilon=0,\alpha=\xi}\, dt.$$

For $\varepsilon > 0$ closed periodic orbits (Fig.4) for:

$$\alpha(h) = \xi - \varepsilon\frac{D_\varepsilon(h)}{D_\alpha(h)} + \mathcal{O}(\varepsilon^2), \quad \text{if } D_\alpha(h) \neq 0. \qquad (3)$$

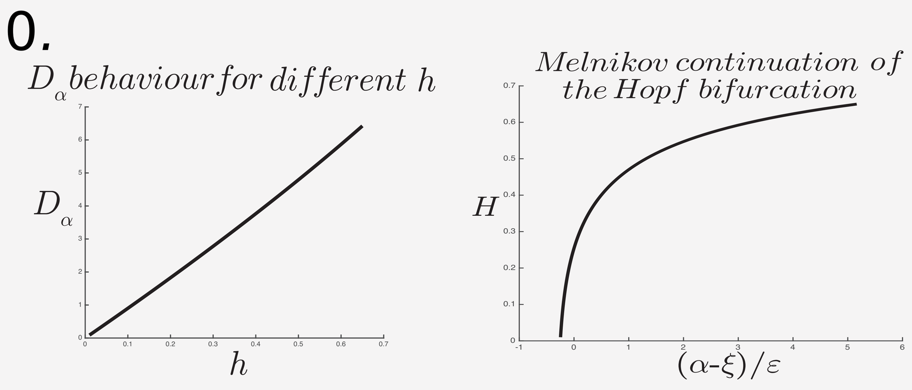

Figure 4: Left: $D_\alpha(h)$ for $h \in]0, 1[$, ξ fixed. Right: Plot of eq. (3).

Contradiction:
- In eq. (2) for $\varepsilon = 0$, $\alpha > \xi$ unbounded orbits on C_0 because $D_\alpha(h) > 0$, see Fig. 4.
- In Fig. 1 we have limit cycles.

⇒ Study the behaviour at infinity.

4 Behaviour at infinity

Introduce the Poincaré sphere

$$\mathcal{S}_3^+ := \{X^2 + Y^2 + Z^2 + W^2 = 1, \quad W \geq 0\}.$$

Chart K_2 for eq. (1), study chart K_1 for $y_2 > 0$:

$$K_2 := \mathcal{S}_3^+ \cap \{W = 1\}, \quad x_2 = \frac{X}{W}, y_2 = \frac{Y}{W}, z_2 = \frac{Z}{W}.$$

$$K_1 := \mathcal{S}_3^+ \cap \{Y = 1\}, \quad x_1 = \frac{X}{Y}, z_1 = \frac{Z}{Y}, w_1 = \frac{W}{Y}.$$

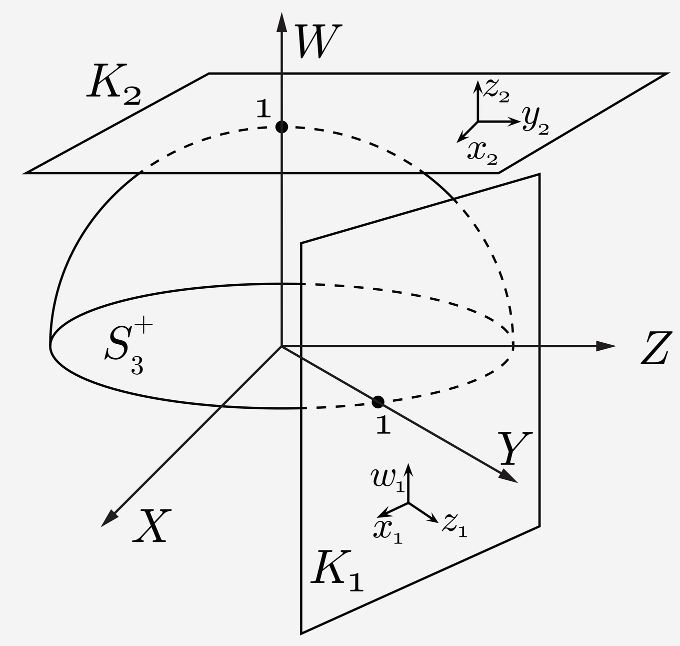

Figure 5: Poincaré sphere and directional charts K_2, K_1.

Singular analysis on K_1. Two invariant planes:
- $\varepsilon = 0$: critical manifold $K_{21}(C_0)$, continue the Hamiltonian orbits up to "infinity", i.e. $w_1 = 0$.
- $w_1 = 0$: line of fixed points, a center manifold can be continued for $w_1 > 0$.

For $\varepsilon = 0$, $\alpha > \xi$ creation of limit cycles at infinity.

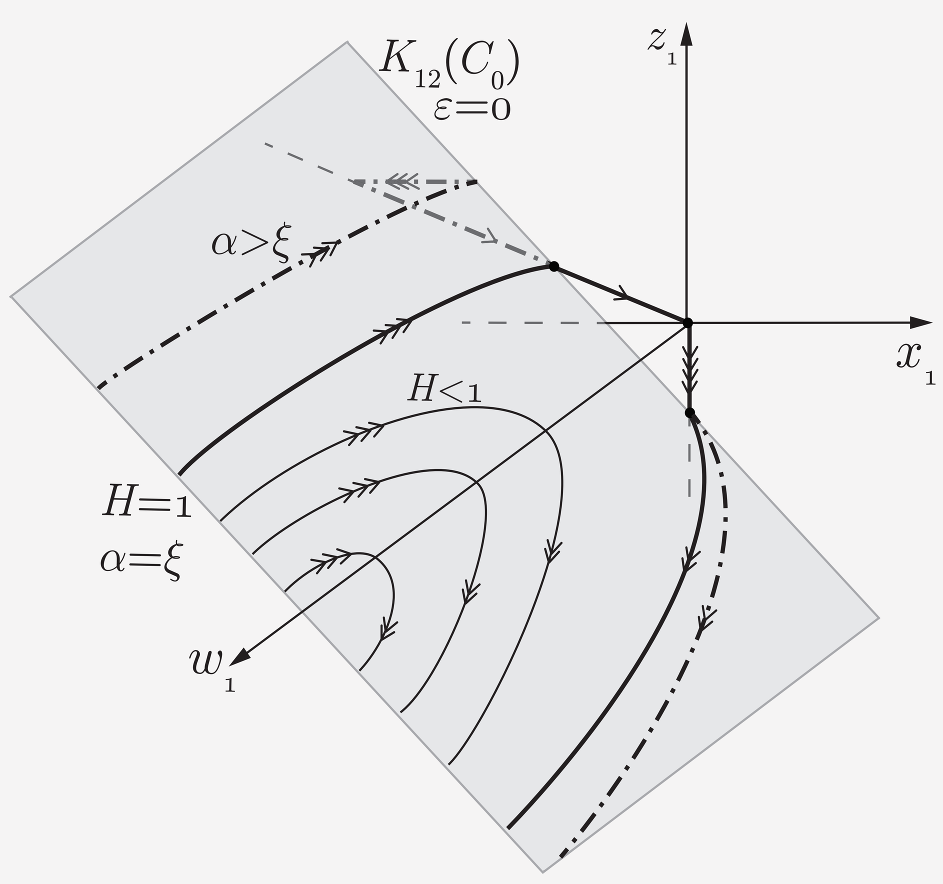

Figure 6: Phase space in chart K_1, in grey: $K_{12}(C_0)$.

5 Conclusion and Future work

Conclusion:
Mechanism for generation of limit cycles at infinity:
- $\varepsilon = 0$, $\alpha > \xi$: unbounded singular cycles, Fig. 7.
- $\varepsilon \neq 0$: we conjecture the creation of limit cycles from the singular ones.

Future work:
Blow-up the origin in chart K_1 to study the case $\varepsilon > 0$.

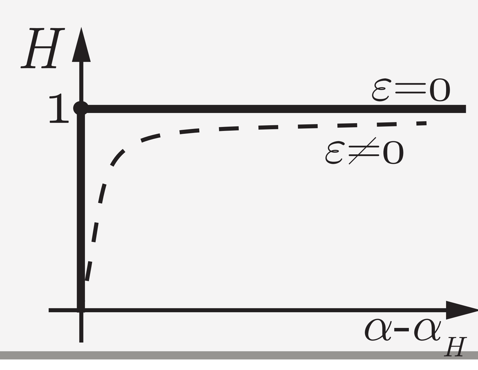

Figure 7: Conjecture on the orbits' amplitude for $\varepsilon \geq 0$. $H = 1$ corresponds to infinite amplitude.