Lecture Notes on Plasma Physics

Pécseli, Hans

Publication date:
1983

Document Version
Publisher's PDF, also known as Version of record

Link back to DTU Orbit

Citation (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Abstract. The following lecture notes were prepared for the course 29:195 in Plasma Physics, second semester 1979 - 1980, at the University of Iowa, Department of Physics and Astronomy, Iowa City. These notes were used together with the text book "Theory of the Unmagnetized Plasma" by D.C. Montgomery (Gordon and Breach Science Publishers, New York, 1971).

November 1983
Risø National Laboratory, DK-4000 Roskilde, Denmark
LECTURE NOTES ON PLASMA PHYSICS

H.L. Pécseli

The following lecture notes were prepared for the course 29:195 in Plasma Physics, second semester 1979-1980, at the University of Iowa, Department of Physics and Astronomy, Iowa City. These notes were used together with the text book 'Theory of the Unmagnetized Plasma' by D.C. Montgomery (Gordon and Breach Science Publishers, New York, 1971).

I would like to thank my students for their enthusiasm and colleagues in Iowa for valuable discussions. In particular, I am indebted to N. D'Angelo, K.E. Lonngren and D.R. Nicholson at the University of Iowa, and also to K.B. Dysthe and V.O. Jensen for much of the enclosed material. Parts of lecture X are based on unpublished lecture notes by A. Nielsen, The Technical University of Denmark. However, the efforts of my colleagues and myself would have been in vain if it were not for the assistance of C. Monsrud. These lecture notes became legible only through her skilled deciphering of my handwriting. I can only hope that the contents live up to the expert presentation. Thanks also to K. Thomsen for his comments on the final version of this manuscript.
CONTENTS

Electrostatic plasma oscillations as described by the linearized Vlasov equation.

Langmuir waves......................... Lectures I-IV
Ion acoustic waves...................... Lecture VI
Properties of dielectric functions........ Lectures VII-VIII
Exercises..................................... Lecture IX
The fluctuation dissipation theorem.......... Lectures X-XI
'Dressed particle' approach to plasma fluctuations.. Lecture XII
Electron waves in a strongly magnetized plasma...... Lecture XIII
Propagation of ion acoustic density perturbations
as described by the linearized ion Vlasov equation
assuming Boltzmann distributed electrons........... Lectures XIV-XV

Nonlinear waves.

Simple waves in a gas.................... Lecture XVI
Burger's equation........................ Lecture XVII
The Korteweg-deVries equation............ Lectures XVIII-XIX

Lecture V was based on material taken entirely from Theory of the Un-
magnetized Plasma, Chapter V: External Fields, 'Test' Charges, by D.C. Montgomery.
Lecture I

We consider waves in a plasma described by the Vlasov equation for electrons and ions:

\[
\frac{\partial f_{i,e}}{\partial t} + v \cdot \nabla f_{i,e} + \frac{q_{i,e}}{m_{i,e}} (E + \nabla \times B) \cdot \nabla f_{i,e} = 0 ,
\]  

(1)

where \( f_{i,e} = f_{i,e}(x,v,t) \) is the velocity distributions of the particles, \( q_{i,e} \) is their charges, and \( m_{i,e} \) is their masses. \( E \) and \( B \) are electric and magnetic fields, respectively, to be determined from Maxwell's equations

\[
\nabla \cdot E = \frac{\rho}{\varepsilon_0} ,
\]  

(2)

\[
\nabla \cdot B = 0 ,
\]  

(3)

\[
\nabla \times E = -\frac{\partial B}{\partial t} ,
\]  

(4)

\[
\frac{1}{\mu_0} \nabla \times B = J + \varepsilon_0 \frac{\partial E}{\partial t} ,
\]  

(5)

in MKS-units, \( \varepsilon_0 = 8.854 \times 10^{-12} \, \text{F/m} \), \( \mu_0 = 4\pi \times 10^{-7} \, \text{H/m} \), while \( \rho \) and \( J \) are charge and current densities, respectively. They are in general caused by charges and currents in the plasma, but may have sources \( \rho_0 \) and \( J_0 \) due to "external" sources like charged spheres, current-carrying wires, etc., so

\[
\rho = \rho_0 + q_i \int_{-\infty}^{\infty} f_i \, dv + q_e \int_{-\infty}^{\infty} f_e \, dv ,
\]  

(6)

\[
J = J_0 + q_i \int_{-\infty}^{\infty} vf_i \, dv + q_e \int_{-\infty}^{\infty} vf_e \, dv .
\]  

(7)
For the moment, we shall accept eq. (1) without further justification. Eqs. (2)-(7) need no comments.

Solving eqs. (1)-(7) is in general an impossible task, so we will make a few simplifications. Assume:

(i) No external fields $E_0, B_0$, i.e. $\rho_0 = 0, J_0 = 0$.
(ii) Consider only high frequency oscillations and ignore the ion motion, i.e. let $m_i \rightarrow \infty$ and $f_i(x,v,t) \rightarrow n_0 \delta(v)$, where $n_0$ is the (uniform) plasma density.
(iii) Linearize eq. (1) by assuming small fluctuations and a particularly simple equilibrium solution to eq. (1), namely $f_0(v)$. Then $f(x,v,t) = f_0(v) + f_1(x,v,t)$ and $\rho_1 = -e \int f_1 dv$. For $f_1$ small we have $E_1, B_1$ small too and may ignore products like $E_1 f_1, B_1 f_1$.
(iv) Consider only electrostatic fluctuations, i.e. let $B_1 = 0$. Note that in principle we can not be sure that these types of fluctuations exist at all. The consistency of our results will tell us that they do. (Note that this assumption implies $J_i = -\varepsilon_0 \partial E_i / \partial t$.)
(v) Consider one-dimensional motion only. This is not a severe restriction, but it eases the notation.

The resulting set of equations is:

\[
\frac{3f}{3t} + v \frac{3f}{3x} = \frac{e}{m} E f_0'(v) = 0 ,
\]

\[
\frac{3E}{3x} = -\frac{e}{\varepsilon_0} \int_{-\infty}^{\infty} f dv .
\]
We may solve eq. (9):

\[
E = -\frac{e}{\varepsilon_0} \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} f d\mathbf{x} d\mathbf{x}'.
\]

We omit the subscript "1" and let \(-e = q_c\). We may thus combine eqs. (8) and (9) to

\[
\hat{V}f(x,v,t) = -\frac{\partial f}{\partial t},
\]

(10)

with the operator

\[
\hat{V} = \frac{\partial}{\partial x} + \frac{\omega^2 f_0'(v)}{p_0} \int_{-\infty}^{\infty} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} dv d\mathbf{x}.
\]

(This manipulation gives an elegant formalism, but it is not really necessary.) Here \(\int f_0(v) dv = 1\). We now look for eigenvalues of \(\hat{V}\), i.e. we want to solve

\[
\hat{V}f = i\omega f,
\]

(11)

where now \(f = f(x,v)\). (In bypassing: note that the operator \(\hat{V}\) is not Hermitian.) We now note that the operator \(\partial/\partial x\) commutes with \(\hat{V}\), i.e. they have common eigenfunctions. The eigenfunction for \(\partial/\partial x\) is \(e^{ikx}\), so we end up solving the equation

\[
iki\mathbf{v} - \omega^2 \frac{f_0'(v)}{p_0} \int f dv = i\omega f,
\]

(12)

where now \(f = f(v)\).

Question: Prove eq. (12).
Note that eq. (12) could be obtained by simply inserting \( f = f(v) \exp(i(kx - \omega t)) \) in eqs. (8) and (9). However, by introducing the operator \( \hat{V} \) as in eq. (10), we have access to a well-developed formalism for the eigenvalues, eigenfunctions, etc. which may be useful for practical applications.

Equation (12) has the solution

\[
\frac{\omega^2}{k^2} f = \frac{f'(v)}{v - \omega/k} \left[ \int f dv + \lambda \delta(v - \omega/k) \right],
\]  

(13)

with \( \lambda \) arbitrary as seen by insertion into eq. (12) using that \( \xi \delta(\xi) = 0 \). \( P \) indicates that the principal value should be taken by integration. Equation (13) simply tells us that the integral \( \int f'(v)(v - \omega/k)^{-1} dv \) may take any value depending on how we deal with the singularity at \( v = \omega/k \). We specify the principal value at the expense of an arbitrary constant \( \lambda \).

We have not yet made use of the fact that \( f \) is normalizable, i.e. \( n = \int f dv \). Since \( n \) is just a constant, we may set it to unity without loss of generality, and obtain

\[
\lambda = 1 - \frac{\omega^2 P}{k^2} \int_{-\infty}^{\infty} \frac{f'(v)}{v - \omega/k} dv .
\]

(14)

Equation (13) with eq. (14) inserted gives us the eigenfunction corresponding to the eigenvalue-set \((\omega, k)\), but the result is rather surprising: first the set of eigenvalues form a continuum, (this is not so unusual), but next, for any given \( \omega \) we may come up with an arbitrary \( k \) and still be able to present an eigenfunction! In other words, there is no relation between \( \omega \)
and \( k \), or in other words: **no dispersion relation**. Looking back, we realize that there is no reason to be surprised: eq.(8) has **three** independent variables, \( x,v,t \), where we usually, in fluid equations, only encounter **two**, namely \( x,t \). (A note: usually we expect \( \delta \) functions like the one in eq.(13) locked up behind an integral sign. Note however that a velocity distribution only has physical meaning in terms of its moments, e.g. density, flux, etc., or as a convolution involving a finite analyzer resolution, and in this context \( \delta \) functions are perfectly acceptable).

Among all the real eigenvalues \( \omega \), some may take a particular position by having corresponding eigenfunctions with \( \lambda = 0 \), i.e.

\[
k^2 = \omega^2 \int_{-\infty}^{\infty} \frac{f'_0(v)}{v - \omega/k} \, dv .
\]

(15)

If such eigenvalues exist (they need not) then eq.(15) has the form of a dispersion relation connecting \( \omega \) and \( k \), but recall we still have all the other \( (\omega,k) \)'s - a double infinity of them.

Finally, complex eigenvalues \( \omega \) may exist. Still we require the corresponding eigenfunction to be normalizable, i.e.

\[
k^2 = \omega^2 \int_{-\infty}^{\infty} \frac{f'_0(v)}{v - \omega/k} \, dv ,
\]

(16)

with complex \( \omega \). Note the slight difference between eqs.(15) and (16). With complex \( \omega \), we do not need any principal value sign. From eq.(16) we find that if a solution \( \omega \) exists, then \( \omega^* \) (complex conjugate) is a solution also. Thus, if eq.(16) is fulfilled by a complex \( \omega \), we have exponentially growing fluctuations, and these will obey a dispersion relation, eq.(16).
Finally let us note that the solutions in eqs. (13) and (14) strictly speaking violate the assumption of linearization (iii). This need not worry us; in real situations we will always deal with superpositions of eigenfunctions and may then hope for well-behaved solutions. The unstable eigenvalues are, however, a worry; if they exist, the whole analysis is bound to break down sooner or later.
Lecture II

We have demonstrated that the linearized Vlasov equation, under the assumption of electrostatic waves, can be written in terms of an operator $\hat{V}$ in the form

$$\hat{V}_f = -\frac{\partial F}{\partial E},$$

and we found the eigenfunctions and eigenvalues for this equation. We found that all real $\omega$ and $k$ are eigenvalues, so for one particular $k$ we may write

$$\hat{V}_k = i\omega f$$

with

$$\hat{V}_k = ikv - \omega^2 \frac{f'_0(v)}{k} \int dv.$$  

We find the adjoint operator to be

$$\hat{V}^+_k = ikv - \omega^2 \frac{1}{k} \int dv f'_0(v),$$

now with $f'_0(v)$ under the integral sign. (Use the definition of $\hat{V}^+$, i.e. $\int g_2 \hat{V} g_1 dy = \int g_2 \hat{V}^+ g_1 dy$, insert $\hat{V}$ on the left-side and interchange the $v$ and $\gamma$ integration and identify $\hat{V}^+$. Do this for yourself!) We may solve

$$\hat{V}^+_f = i\omega f^+$$

and find, using here the more convenient normalization,

$$\frac{\omega^2}{k^2} \int_{-\infty}^{\infty} f^+ f'_0(v) dv = 1.$$
that

\[ f^+_k,\omega = \frac{P}{v-\omega/k} + \left( \frac{f'_0(v)\omega^2/k^2}{P} \right)^{-1} \lambda^+(v-\omega/k) , \tag{6} \]

with

\[ \lambda^+ = 1 - \int \frac{f'_0(v)}{v-\omega/k} \, dv . \]

Question: Check this result.

We can now prove the orthogonality of the eigenfunctions, i.e. \( \int f^+_k,\omega f_k,\omega',dv = A_\delta,\omega,\omega' \). To prove the completeness, we have to demonstrate that the evolution of any initial value of \( f(x,v,t) \) can be described as a superposition of the eigenfunctions. The proof is rather lengthy, but useful for future reference, so let us go through it.

Let us assume that the plasma is stable. Since the equations are linear, it is sufficient to prove the completeness for one given \( k \). For simplicity let us denote \( \omega/k = u \). Our problem is to demonstrate that an arbitrary initial perturbation

\[ g(v) e^{ikx} \]

evolves as

\[ g(x,v,t) = e^{ikx} \int_{-\infty}^{\infty} A_k(u)f_k(u,v)e^{-iku} \, du , \tag{7} \]

where \( A_k(u) \) is a function weighting the various eigenfunctions \( f_k(u,v) \). We have to let \( e^{-i\omega t} = e^{-iku} \). Since the eigenvalues \( \omega \) (or rather \( u \)) form a continuum, we have to integrate over them all. If we can determine \( A_k(u) \) uniquely, then we are through!
In other words: can we solve

\[ g(v, t = 0) = g(v)e^{i k x} = e^{i k x} \int_{-\infty}^{\infty} A_k(u)f_k(u, v) du \]

or

\[ g(v) = \int_{-\infty}^{\infty} A_k(u)f_k(u, v) du \]  \hspace{1cm} (8)

in terms of the eigenfunctions, eq.(13), from lecture I? We shall make use of the "well-known" (?) Titchmarsh theorem, which says that any square integrable function \( A(v) \) can be uniquely decomposed as a sum of two functions:

\[ A_+(v) = \int_0^{\infty} \phi(p)e^{i p v} dp \quad \text{and} \quad A_-(v) = \int_{-\infty}^{0} \phi(p)e^{i p v} dp \]  \hspace{1cm} (9)

where \( A_+ \) and \( A_- \) have holomorphic (i.e. no singularities) analytic continuations in the upper and lower halves of the complex \( v \)-plane, respectively. Obviously \( \phi(p) \) is the Fourier transform of \( A(v) \). I.e.

\[ A(v) = A_+(v) + A_-(v) \]  \hspace{1cm} (10)

Now introduce the Hilbert transform \( A_* \) of \( A \):

\[ A_*(v) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{A(\gamma)}{v-\gamma} d\gamma \]  \hspace{1cm} (11)

(Sometimes you see another definition of eq.(10), divided by "1".) Using
II.4

\[ \int_{-\infty}^{\infty} e^{-ipx} \, dx = \begin{cases} -ie^{ipx} & p > 0 \\ ie^{ipx} & p < 0 \end{cases}, \quad (12) \]

we find

\[ A_+ (v) = \frac{1}{2} (A_+ (v) - A_- (v)) , \quad (13) \]

or

\[ A_+ (v) = \frac{i}{2} (A + iA_\mp) , \quad (14) \]

and

\[ (A_\mp)_+ = -A. \quad (15) \]

Inserting \( f' (v, u) \) in eq.(8), we realize why these functions are so useful:

\[ g(v) = \frac{\omega^2}{k^2} f'_0 (v) \int_{-\infty}^{\infty} \frac{A_k (u)}{v-u} \, du + A_k (v) \left( 1 - \frac{\omega^2}{k^2} \int_{-\infty}^{\infty} \frac{f'_0 (\gamma)}{\gamma-v} \, d\gamma \right) , \quad (8') \]

i.e. we may write

\[ g(v) = \frac{\omega^2}{k^2} f'_0 (v) \pi A_k (v)_+ + \left( 1 - \frac{\omega^2}{k^2} \int_{-\infty}^{\infty} \frac{f'_0 (\gamma)}{\gamma-v} \, d\gamma \right) A_k (v) . \quad (16) \]

Similarly we may decompose \( f'_0 (v) \) in eq.(9) according to eq.(14) and obtain, using eqs.(13) and (14) for \( A_k (v)_+ \) and \( A_k (v) \)

\[ g(v) = \left( 1 - 12\pi \frac{\omega}{k} f'_0 (v)_+ \right) A_k (v)_+ + \left( 1 + 12\pi \frac{\omega}{k} f'_0 (v)_- \right) A_k (v)_- . \quad (17) \]

You had better verify this for yourself! Recall that

\[ \int_{-\infty}^{\infty} f'_0 (\gamma) (\gamma-v)^{-1} \, d\gamma = -\pi f'_0 (v)_+ = i\pi (f'_0 (v)_+ - f'_0 (v)_-) . \]
II.5

Eq. (17) is a given initial condition. It can also be decomposed as \( g(v) = g_+(v) + g_-(v) \), where we can determine \( g_+ \) and \( g_- \) uniquely!

But then we are through, because by using eq. (17) we can identify

\[
A_k(v) = \frac{g(v)}{1 + 2\pi \frac{P}{k^2} f'_0(v)}.
\]

For consistency, we must require that the solution of eq. (18) does have analytic continuations in the two half-planes, i.e. the denominator must not have zeroes there. In other words, the function

\[
e_k(v) = 1 + i \frac{\omega^2}{k^2} \pi \left( f'_0(v) \pm i \frac{\pi}{P} \int_{-\infty}^{\infty} \frac{f'_0(y)}{v-y} dy \right)
\]

must not have zeroes in the upper or lower half-planes, respectively. We shall later on demonstrate that eq. (19) is nothing but the dielectric function of the plasma, and its complex conjugate (+ and - sign, respectively).

Now we can use eq. (7) without hesitation and ask for instance for the time evolution of the density perturbation \( n(x,t) = \int g(v,x,t) dv \). Recalling the normalization of the eigenfunctions, we obtain the simple result:

\[
n(x,t) = e^{i k x} \int_{-\infty}^{\infty} A_k(u) e^{-i k t} du.
\]

This result is important; as \( t \to \infty \) we have \( n(x,t) \to 0 \), using Riemann's lemma, unless \( A_k(u) \) contains a \( \delta \) function. If we choose \( g(v) \) as one eigenfunction it will contain \( \delta \) functions,
but for **all** well-behaved \( g(v) \)'s we will have the well-known Landau damping. In our picture it appears as phase mixing of normal modes (or eigenfunctions). In the next lecture we shall elaborate this point a little further.

Here we conclude with some remarks concerning our eigenfunctions \( f \) and \( f^+ \). Using these, now knowing they form a complete set, we may determine \( A_k(u) \) as

\[
A_k(u) = \frac{1}{\int_{-\infty}^{\infty} f^+(v) f(v) dv} \int_{-\infty}^{\infty} f^+(u,v) g_k(v, t=0) dv.
\]

All this is now obsolete, since we have the solution already, but it is straightforward to include **unstable** plasmas if you take my word that the eigenfunctions still form a complete set. Multiple roots of I.16 will, however, present a problem, but let us not get too involved in mathematics. If the problem interests you, the enclosed reference list may be useful.
Reference list for lecture II

Lecture III

In lecture I we learned that we can solve the linearized Vlasov equation, assuming electrostatic oscillations and found solutions of the form $\mathcal{F}(v)e^{-i(\omega t-kx)}$ where

$$
\mathcal{F}(v) = \frac{\omega^2}{k^2} \frac{f''(v)}{v-\omega/k} \left( 1 - \frac{\omega^2}{k^2} \int \frac{f''(v)}{v-\omega/k} dv \right) \delta(v-\omega/k).
$$

(1)

It is important that all real $(\omega,k)$ are allowed, i.e. no dispersion relation exists. The "normal modes", eq.(1), are undamped. This is not surprising: the Vlasov equation is invariant under the transformation $t \rightarrow -t$, $v \rightarrow -v$, $x \rightarrow x$, i.e. there is no preferred direction of time. Entropy is conserved: there are no losses in the system. We demonstrated that the time evolution of any initial perturbation could be described by a superposition of normal modes, and we learned how to find the proper "weight function" $A_k(u)$. Furthermore, we recovered the well-known Landau damping by looking at the time evolution of the density $n = \int g(x,v,t)dv$. At first sight, such a result is confusing: how can an entropy conserving system give rise to damping? We realize, however, that the evolution of the distribution function show no damping. It is given as

$$
g(x,v,t) = e^{i k x} \frac{\omega^2}{k^2} f''(v) \left[ \int \frac{A_k(u)}{v-u} e^{-i u t} du \right. \\
+ A_k(v) \left( 1 - \frac{\omega^2}{k^2} \int \frac{f''(\gamma)}{\gamma-v} d\gamma \right) e^{i k(x-v t)},
$$

(2)

with the notation of lecture II. While the first term indeed damps according to Riemann's lemma, the second one oscillates
forever if we look at one particular velocity \( v \). The period of oscillation obviously depends on the chosen value for \( v \). In order to obtain the damping of density, we obviously integrate over \( v \), i.e. we eliminate one of the independent variables. Doing this, we obviously lose a lot of information and the little we are left with appears to give damping. Note the similarity to "ordinary" fluid damping; all basic laws of nature are time reversible, also those which govern the interaction of atoms, molecules in ordinary fluids. The problem is that we have no way to handle all this information; we have to make phenomenological descriptions accounting for what we believe are the most important features. The resulting picture, containing reduced information, will often exhibit damping, i.e. fading information. It can hardly be over-emphasized that the plasma oscillations we have described give us an eminent insight into a situation where we can handle the full time reversible equations and also demonstrate how we discard some information and obtain a damping. Note, however, that in order to obtain such a nice result, we had to linearize the initial equation (1) in lecture I. For stable plasmas this is not a severe restriction.

You may get a better understanding of the type of damping we are dealing with by considering the very simple equation obtained by "turning off" the charge of the electrons:

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0.
\]  
(3)
It describes particles with distribution $f(x,v,t)$ moving without any interaction whatsoever along straight orbits $x = vt$. Let us solve (3) for the initial condition

$$f(x,v,t=0) = n_0 (1 + \gamma \sin Kx) e^{-\frac{(v/v_0)^2}{\sqrt{\nu v_0}}} ,$$

(4)

with $|\gamma| < 1$. Set $f_0(v) = n e^{-\frac{(v/v_0)^2}{\sqrt{\nu v_0}}}$, this is a solution to (3). Then $n(x,t=0) = n_0 + n_0 \gamma \int_{-\infty}^{\infty} \left( v < 0 \right) e^{-\frac{(v/v_0)^2}{\sqrt{\nu v_0}}} = n_0 (1 + \gamma \sin Kx)$. Eq. (3) is a homogeneous partial differential equation with characteristic equation $v dt - dx = 0$, i.e. $vt - x = \text{const}$ is a characteristic. Any function of $vt - x$ is thus a solution, in particular

$$f_1(x,v,t) = n_0 \gamma \sin(K(x-vt)) e^{-\frac{(v/v_0)^2}{\sqrt{\nu v_0}}} .$$

(5)

This solution also satisfies the initial condition, so it is the right one. We note that it oscillates without damping for all $x,t$. Let us look at the density:

$$n_1(x,t) = \frac{n_0 \gamma}{\sqrt{\nu v_0}} \int_{-\infty}^{\infty} \sin(K(x-vt)) e^{-\frac{(v/v_0)^2}{\sqrt{\nu v_0}}} dv$$

$$= \frac{n_0 \gamma}{\sqrt{\nu v_0}} \int_{-\infty}^{\infty} \left( \sin Kx \cos Kv t - \cos Kx \sin Kv t \right) e^{-\frac{(v/v_0)^2}{\sqrt{\nu v_0}}} dv .$$

Now $\int_{-\infty}^{\infty} \sin(Kvt)e^{-\frac{(v/v_0)^2}{\sqrt{\nu v_0}}} dv = 0$ (integral of an even times an uneven function), while $(\nu v_0)^{-\frac{1}{4}} \int_{-\infty}^{\infty} \cos(Kvt)e^{-\frac{(v/v_0)^2}{\sqrt{\nu v_0}}} dv = e^{-\frac{(kv_0 t)^2}{4}}$, so

$$n_1(x,t) = n_0 \gamma e^{-\frac{(kv_0 t)^2}{4}} / \sin Kx \to 0 \text{ as } t \to \infty ,$$

(6)

i.e. analogous to Landau damping. Note that large values of $v_0$ and $K$ give larger damping rates. The following schematic $x-t$
I hope that you can see that for small t it is still possible to recognize a spatial periodicity even if you integrate over particle velocities. For large t particles originating from one point x at t = 0 are entirely messed up with particles originating from other points, so unless we keep track of the particles, i.e. retain their velocity distribution function, we will not be able to find any spatial periodicity. The full plasma case is similar, but of course more complicated.

Let us now return to eq.(18), lecture II. It demonstrates how to determine the weight function $A_x(u)$ to be used in, e.g. eq.(20)
when the initial perturbation $g(v)$ is given. However, we may adopt a different point of view: assume that we insist that we want a particular density variation, say:

$$n(x,t) = e^{ikx} e^{-\alpha|t|}.$$  

(7)

Is it possible to give an initial perturbation $g(v)$ which gives (9)? The answer is yes, we just have to use II.(18) the other way around! Equation (7), for instance, requires $A_k(u) = \frac{\alpha/k}{u^2 + (\alpha/k)^2}$ which gives

$$A_k(u)_+ = \frac{1}{2\pi} \frac{1}{\alpha/k - iv}, \quad A_k(u)_- = \frac{1}{2\pi} \frac{1}{\alpha/k + iv}.$$ 

Using II.(17) we get the corresponding $g(v)$

$$g(v) = \frac{1}{2\pi} \left( \frac{1}{\alpha/k - iv} \right) \left( 1 - i2\pi \frac{\omega^2}{k^2} f'_0(v)_+ \right) + \frac{1}{2\pi} \left( \frac{1}{\alpha/k + iv} \right) \left( 1 + i2\pi \frac{\omega^2}{k^2} f'_0(v)_- \right)$$

$$= \frac{1}{2\pi} \left[ \frac{\epsilon(k,v)}{\alpha/k - iv} + \frac{\epsilon^*(k,v)}{\alpha/k + iv} \right].$$  

(8)

Note that $g(v)$ is real as it should be. $\epsilon(k,v)$ is again the plasma dielectric function introduced in II.(19). In other words, we can get any damping, actually any temporal variation of the density provided it can be Fourier transformed, see eq.II.(20). This fact is largely "swept under the rug" in the plasma literature. Obviously we have no guarantee that the $g(v)$ we end up with is nice, but it is there. In my opinion, much of the controversy concerning the so-called "pseudo waves" (Montgomery, p.330) arises because various authors refuse to accept that initial (or boundary) perturbations $g(v)$ may have all kinds of funny shapes.
Let us summarize:

(i) A plasma is a medium which does not have any dispersion relation.

(ii) The density fluctuation associated with an initial periodic perturbation damps in time unless we choose a "pathological" velocity distribution, but we may control this damping almost as we like.

With these features in mind, we may be a bit sceptical when we see experiments showing perfect agreement with much more simplified theories. Let us not be too unfair though. In real experiments we are usually not able to produce all kinds of \( g(v) \), but usually end up with something not too far from a Maxwellian. This gives strong restrictions on the results obtained by the analysis of lectures I-III.

Finally we should pay attention to one particular unperturbed velocity distribution function, namely \( f_0(v) = \delta(v) \). (Recall the normalization \( \int f_0 dv = 1 \)). Our analysis of course also includes this case and we have the eigenfunctions

\[
 f_{k,\omega}(v) = \frac{\omega^2}{K^2} p \frac{\delta'(v)}{v-\omega/k} + \left(1 - \left(\frac{\omega}{v}\right)^2\right) \delta(v - \omega/k) .
\]  

(9)

Note, however, that we can find eigenvalues \((\omega,k)\) where \( \lambda = 0 \) in I.(14), namely

\[
 \omega = \frac{\omega}{p} \quad \text{for all } k .
\]  

(10)

Now this is a dispersion relation which picks out one particular \( \omega \) of our continuum of eigenvalues, a situation forseen in lecture
III.7

I. Now looking at (9) this does not seem to make much of a difference, but look at the normalization of $f^+_{k,\omega}(v)$ in II.(6)! It is no good, and we have to treat these cases separately (in fact all cases where $f'_{0}(v=\omega/k) = 0$). We take

$$f^+_{\omega,k}(v) = \delta(v-\omega/k) \quad \text{if } \lambda \neq 0 \quad ,$$

$$f^+_{\omega,k}(v) = \frac{1}{v-\omega/k} \quad \text{if } \lambda = 0 \quad .$$

We could also look at II.(19) to see why these cases need separate treatment: $e_k(v) = 0$ when both $f_0(v) = 0$ and

$$1 - \frac{P}{K^2} P f'_{0}(\gamma)(v-\gamma)^{-1}d\gamma = 0.$$

This gives trouble in II.(18). We call such a situation "marginally-stable" and have a solution

$$g(x,v,t) = e^{ikx} \int_{-\infty}^{\infty} A_k(u) f_k(u,v)e^{-ikt}du + \sum_{n,k} a_n,v_{n,k}(v)e^{ik(v-u_n t)} \quad ,$$

as a generalization of II.(7). To be on the safe side, we write a sum over $n$ in case there are more than one $\omega$ (i.e. $u = \omega/k$) belonging to the discrete set with the above properties. In this set we may also include complex $\omega$ (i.e. complex $u = \omega/k$) forseen in lecture II. Recall that they occur in pairs $\omega$ and $\omega^*$. Now integrating (13) to get the density, we still recover the damped contribution, but in addition we have a prevailing oscillatory component given by the sum, provided of course that we do have elements in the discrete set. Thinking back to our experience from fluid theory, we readily recognize the undamped oscillations at $\omega = \omega_p$, but in addition we get the initially damped contribution which is a purely "Vlasov phenomenon". An important
"morale" from the above example with \( f_0(v) = \delta(v) \) is the tremendous difference between the case of a zero temperature plasma \( (f_0(v) = \delta(v)) \) where we get a discrete set of undamped oscillations at \( \omega = \omega_p \), and the case where the electron velocity distribution function has a finite spread, e.g.

\[
f_0(v) = (m/2\pi T)^{1/2} e^{-mv^2/2T} \quad \text{with } T \neq 0.
\]

As \( T \neq 0 \) the discrete set immediately disappears, since in this case \( f'_0(v) \neq 0 \) for all \( v \neq 0 \), and \( k^2 - \omega_p^2 \int f'_0(v)(v-u)^{-1}dv = 0 \) no longer have any solutions with (14) inserted. The case corresponding to (13) without a sum, i.e. II.(7), is thus associated with the finite temperature of the plasma, and for this case we in general only have the damped density contribution. It is thus not possible to start with the cold plasma case and invent a kind of expansion procedure to give an approximate solution for the \( T \neq 0 \) case. You may realize this by trying to expand (14) around \( T = 0 \) for small \( T \). It does not work.

Finally, let me give you an example where \( k^2 - \omega^2 \int f'_0(v)(v-u)^{-1}dv \) have complex roots, as foreseen in lecture I. Take \( f_0(v) = \frac{1}{2}(\delta(v-v_0) + \delta(v+v_0)) \). Then we try to solve

\[
k^2 - \frac{1}{2} \omega^2 \left( \frac{1}{(v_0-u)^2} + \frac{1}{(v_0+u)^2} \right) = 0,
\]

i.e.

\[
(v_0^2 - u^2)^2 = \frac{\omega_p^2}{k^2} (v_0^2 + u^2)
\]

\[
u^2 - u^2 \left( 2v_0^2 - \frac{\omega_p^2}{k^2} \right) + v_0^2 \left( v_0^2 - \frac{\omega_p^2}{k^2} \right) = 0.
\]
This equation gives negative $u^2$, i.e. imaginary $u$ if $kv_0 < \omega_p$.

This is the well-known "two-stream" instability. Depending on our choice of $k$, we have a steadily oscillating contribution as before or an exponentially growing one. The initial damping will always be present unless we choose a pathological $g(v)$, giving $\delta$ functions in the weight function $A_k(u)$. 
References for lecture III

Lecture IV

We shall follow Montgomery, chapter III, p.28-51. It is not, strictly speaking, Landau's treatment of the problem, but very similar.

We will need another presentation of the linearized Vlasov eq.I.(8). Consider for the moment E as given. Then I.(8) is an inhomogeneous partial differential equation with characteristics $x-vt$ just as III.(3) (which is homogeneous though). A solution is therefore a function of $x-vt$. It has to satisfy an initial condition $f(x,v,t=0)$, assumed given. We find

$$f(x,v,t) = f(x-vt,v,t=0) + \int_0^t E(x-v(t-\tau),v)dt ,$$  \hspace{1cm} (1)

which obviously satisfies the initial condition.

**Question:** Verify by insertion into I.(8) that (1) is indeed a solution.

Since $x-vt$ is the unperturbed orbit of a particle, (1) is often called a solution obtained by "integrating along unperturbed orbits". Considering one Fourier component only, denoted by $k$, we may reduce (1) to

$$f(x,v,t) = g(v)e^{ik(x-vt)} + \frac{E}{m_0} \int_0^t E(t)e^{ik(x-v(t-\tau))}d\tau ,$$  \hspace{1cm} (2)

where now $f(x,v,t=0) = g(v)e^{ikx}$.

We will need some facts about Laplace transforms.

The Laplace transform $\mathcal{L}(g(t))$ of a function $g(t)$ is defined as
\[ \mathcal{L}\{g(t)\} = g(s) = \int_0^\infty e^{-st} g(t) dt . \quad (3) \]

Sometimes you see \(-i\omega\) replacing \(s\), inviting the term "half-sided" Fourier transform. It is important, however, that \(s\) is complex with some finite positive real part, \(\text{Re} s = \sigma > 0\). Obviously, (3) only works if \(g(t)\) is exponentially bounded. It can be shown that the linearized Vlasov equation only admits such solutions. The inversion of (3) reads
\[ g(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} g(s) e^{st} ds , \quad (4) \]

where \(\sigma\) must be so large that all singularities have real parts < \(\sigma\). Furthermore, we need:
\[ \mathcal{L}\left(\frac{dg(t)}{dt}\right) = s\mathcal{L}\{g(t)\} - g(t = 0) , \quad (5) \]

where the subscript + means the limit \(g(t \to 0)\) for \(t\) positive. Also
\[ \mathcal{L}\left(\int_0^t g(t) dt\right) = \frac{1}{s} \mathcal{L}\{g(t)\} , \quad (6) \]

and finally the convolution theorem
\[ \mathcal{L}\left(\int_0^t f(\tau) g(t-\tau) d\tau\right) = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\} , \quad (7) \]

The theorems (5)-(7) are not too hard to prove with (3) and (4) given.

Finally, we shall assume that some of the functions we are
dealing with are "entire functions". This means that they do not have singularities (i.e. poles) in the finite part of the complex plane. Recall that this necessarily means that they have singularities at infinity ⇒: a complex function without singularities is a constant. Note that a nice function like \( \frac{a}{\xi^2 + b^2} \) is not an entire function. A Maxwellian is, however, an entire function.

The Vlasov equation, linear and one-dimensional:

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{e n_0}{m} E f'_{0}(v) = 0.
\]

Poisson's equation:

\[
\frac{\partial E}{\partial x} = -\frac{e}{\epsilon_0} \int f dv, \quad \frac{\partial f}{\partial x} \rightarrow ikf, \quad \frac{\partial f}{\partial t} \rightarrow sf - f(0^+) ,
\]

i.e.

\[
(s + ikv)f = \frac{en_0}{m} E f'_{0}(v) + f(0^+) , \quad ikE = -\frac{e}{\epsilon_0} \int f dv
\]

combined gives

\[
E(k, s) = \frac{\frac{e}{\epsilon_0} \int f(0^+) dv}{\frac{e}{\epsilon_0} \int f_{0}'(v) dv + \frac{\omega^2}{ik} \int f_{0}'(v) dv} ,
\]

or

\[
E = \frac{1}{k^2} \frac{\frac{e}{\epsilon_0} \int g(v) \frac{d}{v - 1/s/k} dv}{\frac{\omega^2}{k^2} \int f_{0}'(v) \frac{d}{v - 1/s/k} dv} \cdot S_k(s) \cdot D_k(s) , \quad (8)
\]
with \( g(v) = f(0^+) \).

Note: no singularity in the integrals, since \( \text{Im} s > 0 \). We now want to solve

\[
E(k,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} E(k,s) e^{st} ds
\]

(9)

The integrals in \( S \) and \( D \) have a singularity at \( v = i s/k \)

above the real axis, since \( \text{Re} s = \sigma > 0 \) and \( k > 0 \). We would like to move the integration path in (9) so \( \sigma < 0 \). We therefore need to know how to deal with \( S_k(s) \) and \( D_k(r) \) in this case. See fig. 3.1, page 45 in Montgomery. We use the analytic continuation of these functions when \( \sigma \) becomes \( < 0 \), (Montgomery, page 42-43).

The resulting integration paths in \( S_k \) and \( D_k \) are denoted by the Landau contour: \( \gamma \). Landau assumes that both \( \int_{c-i\infty}^{c+i\infty} g(v)(v-is/k)^{-1} dv \) and \( \int_{c-i\infty}^{c+i\infty} f'(v)(v-is/k)^{-1} dv \) are "entire functions". Then the exponential Landau damping is derived as in Montgomery, page 47-51, since all singularities in the finite part of the complex \( s \)-plane originate from the zeroes of \( D_k(s) \), i.e. the solutions of

\[
1 - \frac{\omega^2}{k^2} \int \frac{f'(v)}{v - i s/k} dv = 0
\]
or, if you like, with \( s \rightarrow -i\omega \)

\[
1 - \frac{\omega^2}{k^2} \int_{v=\omega/k}^{\infty} f_0'(v) \, dv = 0. \tag{10}
\]

We have assumed that we **can** let \( \sigma \rightarrow 0 \) without precautions, i.e. that \( D_k(s) \) do not have zeroes (i.e. give rise to unstable oscillations) for Re \( s > 0 \). Such a case will not, however, present any problems, see fig.3.2, page 48 in Montgomery.

In particular, we emphasize that when \( \sigma = 0 \) the results of the last three pages become identical to those of lectures I-III, where we treated a stable plasma. (They had better be!!) Thus

\[
1 - \frac{\omega^2}{k^2} \int_{v=\omega/k}^{\infty} f_0'(v) \, dv = 1 - \frac{\omega^2}{k^2} \int_{v=\omega/k}^{\infty} f_0'(v) \, dv - i \frac{\omega^2}{k^2} \pi f_0'(\omega/k), \tag{11}
\]

but this is precisely what we called \( \varepsilon_k(u) \) in lecture II, page 6 with \( u = \omega/k \). Similarly, for \( \mathcal{F} g(v)(v-\omega/k)^{-1} \, dv \).

Let us now try to solve eq.(10) approximately.

First attempt: assume that \( f_0'(v = \omega/k) \approx 0 \). Not unreasonable for e.g. \( f_0'(v) \), a Maxwellian and \( \omega/k \) large.

\[
\int_{v=\omega/k}^{\infty} f_0'(v) \, dv \approx - \frac{k}{\omega} \int_{v=\omega/k}^{\infty} f_0'(v) \left[ 1 + \frac{kv}{\omega} + \frac{k^2v^2}{\omega^2} + \frac{k^3v^3}{\omega^3} \right] dv
\]

\[
\approx \frac{k^2}{\omega^2} \left[ 1 + \frac{3k^2v^2}{\omega^2} \right]
\]

1st term = 0

2nd term = 1

using \((1-x)^{-1} \approx 1 + x + x^2 + \ldots\) and \( \langle v^2 \rangle = \mathcal{V}_T^2 = \int v^2 f_0(v) \, dv \), i.e. an approximate solution to (10) is \( \omega = \omega_p \), or in the next approximation
\[ \omega^2 = \frac{\omega_p^2}{\nu_T} + 3k^2v_T^2. \] (12)

This is the dispersion relation derived by Bohm and Gross, useful for \( k^2 \ll \frac{\omega_p^2}{\nu_T^2} = \lambda_D^{-2} \), where \( \lambda_D \) is the Debye length.

Second attempt: we have not yet obtained any \( \text{Im} \omega \). Let us assume it is small, then we may use an iteration procedure.

First consider (10) with \( \omega = \Omega + i\gamma \), with \( \gamma \ll \Omega \). Make a Taylor expansion of (10) around \( \omega = \Omega \):

\[ k^2 = \frac{\omega^2}{\nu_T^2} \left\{ \int \frac{f_0'(v)}{v - \omega/k} \, dv + i\frac{\eta^2}{k} \int \frac{f_0''(v)}{v - \omega/k} \, dv + i\frac{\eta^2}{k} \left\{ f_0''(v) \right\} \right\}. \] (13)

Assume that \( f_0''(v) \) is negligible. Then solve the real part of (13), as on page 5. Likewise approximate

\[ \int \frac{f_0''(v)}{v - \omega/k} \, dv \approx -2k^3 \frac{\omega^3}{\nu_T^3}. \]

Insert in the imaginary part of (13) and obtain:

\[ \gamma \approx \pi \frac{\omega^3}{2k^2} f_0'(\frac{\omega}{k}) \] (14)

(See Montgomery, page 60-62.) Note the possibility of instability if \( f_0'(\frac{\omega}{k}) > 0 \).

When can we have unstable solutions?

Nyquist's theorem: Consider a function \( F(\omega) \). Now consider the mapping of the upper half of the complex \( \omega \)-plane on the complex \( F \)-plane.
The theorem states that the number of times the contour in the F-plane encircles the point ReF = 1 is equal to the number of poles minus the number of zeroes of \(1 - F(\omega)\) in the upper half of the complex \(\omega\)-plane. Generally there are no poles, so the theorem is easy to use. Thus

\[
F(\omega) = \frac{\omega^2}{k^2} \left\{ \int \frac{f'_0(v)}{v - \omega/k} \, dv \right\}
\]

since \(1 - F(\omega)\) now have one zero point. Note, however, that we are free to vary \(k\)! This means that if \(F(\omega)\) encircles any part of the positive ReF axis we can always find a \(k\) which makes it
encircle \( \Re F = 1 \). Now a crossing of the \( \Re F \)-axis requires that \( \Im F = 0 \), i.e. \( f'_0(v=\omega/k) = 0 \). Now any distribution function must have at least one maximum so at least once we have this condition satisfied.

Assume now that only one velocity \( v \) gives \( f'_0(v) = 0 \) and let us see whether this distribution can be unstable. For instability we require

\[
\frac{f'_0(v)}{v-\omega/k} \, dv > 0 \quad , \quad (16)
\]

satisfied simultaneously. Without loss of generality, we assume that \( \max f_0(v) \) is for \( v = 0 \); we can always change our reference system to satisfy this requirement, i.e. \( f'_0(v=0) = 0 \). Then \( f'_0(v<0) > 0 \) and \( f'_0(v>0) < 0 \). The denominator in (16) is then negative (positive) when \( f'_0(v) \) is positive (negative), respectively.

In other words, \( \Re F < 0 \), so (16) can not be satisfied for this "single humped" distribution function, so it is necessarily stable. We would be unhappy to see, for example, a Maxwellian (which is single-humped) be unstable!

Consider now a distribution \( f_0(v) \) where \( f'_0(v) \) is zero for two - and only two - velocities. This necessarily means that \( f_0(v) \) has a plateau, i.e. looks like
For convenience we again let the maximum be at \( v = 0 \). The zero crossing corresponding to \( f'_0(v=0) \) again occurs for \( \text{Re}F < 0 \), since our previous arguments remain valid. However, \( f'_1(v) \leq 0 \) in the vicinity of \( v = v_1 \), i.e. the contour can not cross the \( \text{Re}F > 0 \) axis, at most just touch it, as illustrated here:

This situation will arise if \( \int f'_0(v)(v-v_1)^{-1}dv > 0 \). Thus the plasma can not be unstable, but very well marginally stable. We have to solve (16) to see whether this is the case.

Now consider a distribution where \( f'_0(v) \) have three zeroes:
An upward crossing of the ReF-axis thus occurs at the minimum \( v = v_1 \), the downward one for \( v = v_2 \). It may very well happen, though, that one is for \( \text{ReF} > 0 \), the other for \( \text{ReF} < 0 \).

Imagine that the contour looks like

The plasma is obviously unstable (i.e. the condition (16) is fulfilled). Moreover, we note that the unstable waves necessarily have phase velocities \( \text{Re} \omega/k \) in the range \( v_1 < \text{Re} \omega/k < v_2 \), corresponding to positive slopes of the velocity distribution function, as expected. It can be shown that this is always the case for double humped distributions. For distributions with three
or more "humps", the only thing we can do is to solve (15) and (16).

Another representation of (16) may be useful. Rewrite (16) as

$$\frac{df_0}{dv} \bigg|_{v=\omega/k} = \int_{\omega/k}^{\infty} \frac{f_0(v) - f_0(\omega/k)}{v-\omega/k} dv > 0,$$

(17)

together with the condition $f_0'(\omega/k) = 0$. This relation constitutes the Penrose criterion for instability. Note that we have to be very careful when applying the usual rules for integration, differentiation, etc. when we are dealing with principal values of integrals. Maybe I should elaborate (16) in a little more detail; integrate by parts

$$-\frac{2}{\varepsilon} f_0(u) + \int_{-\infty}^{u-\varepsilon} \frac{f_0(v)}{(v-u)^2} dv + \int_{u+\varepsilon}^{\infty} \frac{f_0(v)}{(v-u)^2} dv \bigg|_{\varepsilon \to 0} = 0,$$

where, as usual, $u = \omega/k$ and $\varepsilon$ is introduced to define the principal value of the integral.

Now use that

$$\frac{1}{\varepsilon} = \int_{-\infty}^{\infty} (v-u)^{-2} dv = \int_{u+\varepsilon}^{\infty} (v-u)^{-2} dv,$$

and rewrite this expression as
\[ \begin{align*}
\left\{ \int_{-\infty}^{u-\epsilon} \frac{f_0(v) - f_0(u)}{(v-u)^2} \, dv + \int_{u+\epsilon}^{\infty} \frac{f_0(v) - f_0(u)}{(v-u)^2} \, dv \right\}_{\epsilon \to 0}.
\end{align*}\]

But in the limit \( \epsilon = 0 \), this is precisely eq. (17).

Some important results can readily be obtained from the Penrose criterion: if a "multihumped" distribution function has a minimum where \( f''_0(v=v_{\text{min}}) = 0 \), then this distribution is necessarily unstable.

If a distribution has a "cutoff" so \( f_0(v) = 0 \) for \( v \) larger than \( v_c \), then this distribution is marginally stable for \( \omega/k > v_c \). (It may or may not be unstable for other \( \omega/k \).)

Since a Maxwellian is one of the most important distribution functions we may encounter, particular attention should be paid to (16) with a Maxwellian inserted for \( f_0(v) \). Consult the tables by B. Fried and S. Conte (1961) in The Plasma Dispersion Function (Academic Press, New York).
References for lecture IV

Lecture V

Montgomery: Chapter V, External Fields, "Test" Charges,

p. 87-93.
Lecture VI

Inclusion of ion motion. We again consider electrostatic waves, but now include the linearized Vlasov equation for the ions. The full set of equations is thus (in one dimension)

\[ \frac{\partial f_e}{\partial t} + v \frac{\partial f_e}{\partial x} - \frac{e_{\text{en}}}{m} E f_{\text{en}}(v) = 0 \tag{1} \]

\[ \frac{\partial f_i}{\partial t} + v \frac{\partial f_i}{\partial x} + \frac{e_{\text{en}}}{M} E f_{\text{ei}}(v) = 0 \tag{2} \]

\[ \frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0} \int (f_i - f_e) dv \tag{3} \]

For simplicity, the ions are assumed to be singly charged. By subtracting (1) from (2), introducing \( f = f_e - f_i \), \( n = \int f dv = \int (f_e - f_i) dv \) and \( F_{\text{e}}'(v) = f_{\text{e}}'(v) + f_{\text{i}}'(v) \frac{m}{M} \), we reduce eqs. (1)-(3) to

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{e_{\text{en}}}{m} E F_{\text{e}}'(v) = 0 \tag{4} \]

\[ \frac{\partial E}{\partial x} = -\frac{e}{\varepsilon_0} \int f dv \tag{5} \]

We already know how to solve these equations! We can find the full \( x,t \) dependence of the electric field. If we then want to determine the full development of the distribution function of each species, it is advantageous to use the integral form of eqs. (1)-(2), see lecture IV.1. Thus

\[ f_{e,i}(x,v,t) = f_{e,i}(x-vt,v,0) + \frac{e_{\text{en}}}{m_{e,i}} \int_0^t E(x-v(t-\tau),\tau) d\tau \tag{6} \]
where now $E$ is known from eqs.(4)-(5). In particular, we are also able to give the full dielectric function of the plasma, using IV. (10)-(11) and the definition of $F'_0(v)$ above. It is

$$\varepsilon(k,\omega) = 1 - \frac{\omega^2}{k^2} \int \frac{f'_e(v)}{v-\omega/k} \, dv - \frac{\omega^2}{k^2} \int \frac{f'_i(v)}{v-\omega/k} \, dv,$$

(7)

where $\omega^2_{pe,i} = \frac{e^2 n_0}{\varepsilon_0 m_{e,i}}$. It is interesting to note that we may define the dielectric function for each of the plasma components, i.e.

$$\varepsilon_{i,e}(k,\omega) = 1 - \frac{\omega^2_{pi,e}}{k^2} \int \frac{f'_i,e(v)}{v-\omega/k} \, dv.$$

(8)

Then (7) indicates that the following rule for "adding" dielectric functions is valid:

$$\varepsilon = 1 + \sum \varepsilon_n \left( e^{i\omega/k} - 1 \right).$$

(9)

This relation can be proved to be generally valid! If we have a compound medium where the dielectric function of each of the components is known and they are independent of each other (as they will be in the linear approximation), then the full dielectric function of the medium is given by (9). Note that if one of the components is entirely irrelevant, it contributes with the vacuum dielectric function $\varepsilon = 1$ (and not $\varepsilon = 0$). The relation (9) can be most useful!

We may now repeat the calculations on page IV.5 and, for instance, find for IV.(12) the real part of the dispersion relation.
\[ \omega^2 = \omega^2_{pe} \left( 1 + 3k^2 \frac{v^2_{Te}}{\omega^2} \right) + \omega^2_{pi} \left( 1 + 3k^2 \frac{v^2_{Ti}}{\omega^2} \right) . \]  

(10)

Now for all practical purposes, \( \omega^2_{pe} \gg \omega^2_{pi} \) (since \( M \gg m \)) and \( v^2_{Te} \gg v^2_{Ti} \), so (10) gives only a trivial correction to IV.(12), i.e. for waves with high phase velocities \( \omega/k \).

Guided by experience from fluid theory, we now try to find solutions to \( \varepsilon (k, \omega) = 0 \) with small phase velocities - to be more explicit \( v_{Ti} \ll \omega/k \ll v_{Te} \) - again assuming \( v_{Ti} \ll v_{Te} \). Let us assume that \( f_{0e}(v) \) is a Maxwellian with \( \langle v \rangle = 0 \), or at least \( \langle v \rangle \ll v_{Te} \). Then \( f_{0e}(v = \omega/k) \approx 0 \) and we may ignore the corresponding imaginary part. For the real part we may approximate

\[ \int f'_{0e}(v) \left( \frac{\omega}{k} \right) \frac{dv}{v-\omega/k} \approx \int f'_{0e}(v) \left( 1 + \frac{\omega}{kv} \right) dv \approx \int f'(v) \frac{dv}{v} . \]

For a Maxwellian this integral is equal to: \( -\frac{1}{v^2_{Te}} \). Using \( \omega^2_{pe}/v^2_{Te} = \frac{1}{\lambda^2_D} \), the inverse Debye length, we reduce (7) to

\[ \varepsilon (k, \omega) \approx 1 + \frac{1}{(k\lambda_D)^2} - \frac{\omega^2_{pi}}{\omega^2} \int f'_{0e}(v) \frac{dv}{v-\omega/k} . \]

(11)

As in lecture IV, ignore the imaginary part of the integral in (11) in a first approximation and obtain the dispersion relation, i.e. the solution to \( \varepsilon (k, \omega) = 0 \):

\[ 1 + \frac{1}{(k\lambda_D)^2} \approx \frac{\omega^2_{pi}}{\omega^2} \left( 1 + 3 \frac{k^2v^2_{Ti}}{\omega^2} \right) . \]

(12)

In a first approximation, for small \( k \):

\[ \left( \frac{\omega}{k} \right) \approx \frac{T_e}{M} = c^2_s . \]

(13)
In the next approximation

\[ \omega^2 \approx \frac{\omega_{Di}^2}{1 + 1/(k\lambda_D)^2} \left( 1 + \frac{3v_i^2}{v_T^2} \right) = \frac{k^2}{1 + (k\lambda_D)^2} \frac{C_s^2}{C_i^2} \left( 1 + \frac{3v_i^2}{v_T^2} \right) . \]  \hspace{1cm} (14)

This is the approximate dispersion relation for ion acoustic waves. We may also find \( \text{Im} \omega = \gamma \) to be

\[ \gamma = \frac{\pi}{2} k C_s \delta f'(C_s) \]  \hspace{1cm} (15)

for small \( k \). Now for the analysis to be appropriate, we obviously require \( \gamma \) small, i.e. \( f'(C_s) \) small, but if \( T_e \sim T_i \), then \( C_s \sim v_T \), which is inconsistent. In other words, weakly damped ion acoustic waves require \( T_e \gg T_i \). In that case, we may reduce (14) to

\[ \omega^2 = \frac{k^2}{1 + (k\lambda_D)^2} \frac{C_s^2}{C_i^2} . \]  \hspace{1cm} (16)

We may now ask what kind of approximations were made in the dynamic equations for the electrons to get the approximate results (11)-(16)? We ignored electron Landau damping, so presumably a linear fluid description is sufficient, i.e.

\[ \frac{\partial v_e}{\partial t} = -\frac{T_e}{m n_0} \frac{\partial n_e}{\partial x} - \frac{e}{m} E , \]  \hspace{1cm} (17)

\[ \frac{\partial n_e}{\partial t} + n_0 \frac{\partial v_e}{\partial x} = 0 , \]  \hspace{1cm} (18)

which couple to the ion motion through Poisson's equation. Now the electron contribution to (11) was frequency independent, so we may set \( \partial/\partial t = 0 \) in (17) and (18), to obtain
\[ \frac{\partial n_e}{\partial x} = -\frac{e}{T_e} n_0 E, \]  

(19)

or by introducing \( E = -\partial \phi/\partial x \)

\[ \frac{n_e}{n_0} = \frac{e}{T_e} \phi, \]  

(20)

which simply means that the electrons assume a Boltzmann equilibrium in the potential variation associated with the ion acoustic wave. This is reasonable: ion acoustic waves have frequencies below \( \omega_{pi} \) (see (14)) and since \( \omega_{pe} \gg \omega_{pi} \), the electrons, due to their high thermal velocities, have plenty of time of adjust isothermally to any potential variation.

From (14) and (16) we note that ion acoustic waves are weakly dispersive, i.e. \( \omega/k \) deviates from \( C_s \) only at \( k \lambda_D \sim 1 \). We are usually not interested in such short wavelengths and may ignore the \( k \lambda_D \) terms in (14) and (16) - in others words, set \( \lambda_D \approx 0 \).

Now \( \lambda_D = (\varepsilon_0 T_e/e^2 n_0)^{\frac{1}{2}} \). Since \( T_e > 0 \), \( n_0 \) has to be finite and \( e \) is small, the only consistent way to let \( \lambda_D \to 0 \) is to let \( \varepsilon_0 \to 0 \) (it is small anyway: \( \varepsilon_0 = 8.85 \times 10^{-12} \, \text{F/m} \)). But now we get trouble with Poisson's equation \( \partial E/\partial x = (e/\varepsilon_0) (n_i - n_e) \). To remedy this we assume \( n_i \approx n_e = n \), the assumption of quasi neutrality. Then we may omit Poisson's equation entirely and use (20), now letting \( n_e \approx n_i = n \). It is true that \( \omega_{pe}^2 \omega_{pi} \to \infty \) for \( \varepsilon_0 \to 0 \), but this does not matter since ion acoustic waves always have frequencies well below \( \omega_{pe} \), and for \( k \lambda_D \ll 1 \) also well below \( \omega_{pi} \).

You will find that assuming quasi neutrality from the outset always simplifies the calculations considerably; one must
only make sure that the wave type in question indeed justifies the assumption. It is important to note that the assumption of quasi neutrality has nothing to do with the assumption of isothermal, Boltzmann-distributed electrons! We may easily find cases where the former assumption is justified, the latter not!
Lecture VII

Properties of dielectric functions: From the Vlasov equation for electrons/ions we obtain

\[ n_{e,i} = \frac{n_0 q_{e,i}}{m_{e,i}} \phi \left\{ \frac{f_{0e,i}^{p}(v)}{v - \omega/k} \right\} dv \]  \hspace{1cm} (1)

i.e.

\[ \varepsilon_{e,i}(k,\omega) = 1 - \frac{q_{e,i} n_{e,i}}{\varepsilon_0 k^2 \phi} \]  \hspace{1cm} (2)

This expression inserted in Poisson's equation

\[ k^2 \phi = \frac{1}{\varepsilon_0} (q_e n_e + q_i n_i) \]  \hspace{1cm} (3)

proves VI.(9). The relation (1) may sometimes prove to be useful.

Wave energy: The rate of change of electric field energy \( U \) per unit volume is given by

\[ \frac{dU}{dt} = iE \cdot \frac{dD}{dt} \]  \hspace{1cm} (4)

where \( D = \varepsilon_0 \varepsilon \cdot E \). In order to obtain a meaningful expression for the field energy, we consider a quasi-monochromatic wave, i.e. \( E = E_0(t)e^{-i\omega_0 t} \) and assume \( E_0(t \rightarrow \infty) \rightarrow 0 \), thereby getting rid of an integration constant. We assume \( E_0(t) \) to be slowly varying as compared to \( \omega_0 \). Since we made \( E \) complex, we must rewrite (4) as
where we used that products \( ED \) and \( E^* D^* \) vanish when averaged over a time \( \sim \omega_0^{-1} \). Note that \( E \) now contains Fourier components in the vicinity of \( \omega_0 \). Therefore, the properties of \( \varepsilon(\omega) \) are important not only for \( \omega_0 \), but also close to \( \omega_0 \). To include this, we make a Taylor expansion keeping the first two terms only

\[
\frac{d\zeta(t)}{dt} = \varepsilon(0) \left( -i\omega_0 \varepsilon(\omega_0) E(0) - \frac{d\omega \varepsilon}{d\omega} \bigg|_{\omega=\omega_0} \frac{dE(0)}{dt} \right) e^{-i\omega_0 t},
\]

(see e.g. Landau and Lifshitz, Electrodynamics of Continuous Media). In deriving (6) it is an advantage to use the relation between \( D \) and \( E \):

\[
D(t) = \varepsilon_0 \int_0^\infty \varepsilon(\tau) E(t-\tau) d\tau,
\]

which is quite general. (The Fourier transform of (7) gives the well-known relation \( D(\omega) = \varepsilon_0 \varepsilon(\omega) E(\omega) \).) Then assume that \( E_0(t) \) is so slowly varying that it makes sense distinguished two time scales: one for \( E_0 \) and one for \( e^{-i\omega_0 t} \).

Inserting (6) into (5) we get, with \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \),

\[
\frac{d\zeta}{dt} = \frac{1}{4} \varepsilon_0 \frac{d\omega \varepsilon_1}{d\omega} \bigg|_{\omega=\omega_0} \frac{d|E_0|^2}{dt} + 2\omega_0 \varepsilon_0 \varepsilon_2 (\omega_0) |E_2|^2 \bigg|_{\omega=\omega_0} + 2\varepsilon_0 \frac{d\omega \varepsilon_2}{d\omega} \bigg|_{\omega=\omega_0} |E_0|^2 \frac{d\phi}{dt},
\]
where \( E_0 = E_1 + iE_2 = |E|e^{i\phi} \). Integrating (8) with respect to \( t \), we obtain three terms. The first is the usual expression for the energy density (see Landau and Lifshitz), the second term accounts for the dielectric losses. The third one, however, depends on the "history" of the electric field prior to our observation time \( t \). This means that the present (at \( t \)) state of the medium does not uniquely define the energy density. This is most unfortunate, and we must require \( d\omega \varepsilon_2 / d\omega \) small in order to define the energy density of a wave! The derivation outlined here obviously allows us to include any wavenumber of interest; it was only the integration constant associated with \( d/dt \) in (5) which caused trouble.

Wave damping: Assume that the medium supports weakly damped waves \( \omega (k) = \Omega (k) + i\gamma (k), \gamma \ll \Omega \). Looking for a zero for \( \varepsilon (k, \omega) \) close to the real \( \omega \)-axis, we approximate:

\[
\varepsilon (k, \omega) \propto \varepsilon_1 (k, \Omega + i\gamma) + i\varepsilon_2 (k, \Omega + i\gamma) \\
\propto \varepsilon_1 (k, \Omega) + i\varepsilon_2 (k, \Omega) - \gamma \frac{\partial \varepsilon_2}{\partial \Omega} + i\gamma \frac{\partial \varepsilon_1}{\partial \Omega} \tag{9}
\]

Assume that the third term is small (in agreement with the assumptions on previous pages). Then \( \varepsilon (k, \omega) = 0 \) gives

\[
\varepsilon_1 (k, \Omega) \propto 0 \quad \Rightarrow \quad \text{determines} \quad \Omega = \Omega (k) \tag{10}
\]

\[
\gamma (k) \propto - \frac{\varepsilon_2 (k, \Omega (k))}{\frac{\partial \varepsilon_1}{\partial \Omega} |_{\Omega = \Omega (k)}} \tag{11}
\]

Compare with our previous results for wave damping. Note that \( \gamma \)
is a measure for the ratio between the first two terms in (8).

This result holds for the temporal damping for a wave excited by an "initial condition". Consider a boundary value problem, with \( \omega \) real and \( k = k_1 + i k_2 \). Then it can be shown that \( \gamma \) in (11) are related to \( k_2 \) by

\[
\gamma \approx -k_2 \cdot \frac{\partial \omega}{\partial k_2}.
\]  

(12)

Proof: The expansion

\[
\epsilon_1(k_1, \Omega) + i \epsilon_2(k_1, \Omega) - k_2 \cdot \frac{\partial \epsilon_2}{\partial k_1} + i k_2 \cdot \frac{\partial \epsilon_1}{\partial k_2} = 0
\]

gives as before (now \( \Omega \) is real)

\[
\epsilon_1(k_1, \Omega) \approx 0,
\]

\[
k_2 \cdot \frac{\partial \epsilon_1}{\partial k_1} \approx -\epsilon_2(k_1, \Omega).
\]

We also have

\[
\gamma \approx -\frac{\epsilon_2(k, \Omega)}{\frac{\partial \epsilon_1}{\partial \Omega}},
\]

which inserted above gives:

\[
\gamma \approx -k_2 \cdot \frac{\partial \Omega}{\partial k} Q.E.D.
\]

where we used

\[
\frac{\partial \Omega}{\partial k} = -\frac{\partial \epsilon_1}{\partial k} / \frac{\partial \epsilon_1}{\partial \Omega}.
\]

The relation (12) is often used, but it applies only when we are dealing with one dispersion relation, i.e. one specific solution to \( \epsilon_1(k, \Omega) \approx 0 \). In a plasma we may have to take into account
several roots (in particular for ion acoustic waves) and in such a case, (12) is of little value. This is why the careful analysis in Montgomery, pages 93-102, in needed. For electron waves where one solution to $\epsilon_1(k,\omega) \approx 0$ is dominant, (12) may be useful.

The Kronig-Kramers relations: Consider a function $p(\omega)$ where $p(\omega \to \infty) = p_\infty$ = real const. and assume $p(\omega)$ is analytic in the upper half of the complex $\omega$ plane. Then also

$$\frac{p(\omega) - p_\infty}{\omega - \omega_0}$$

is analytic except for the point $\omega = \omega_0$. We integrate (13) along the path shown in fig.1.

![Diagram of the complex plane](image)

We obtain, when $R \to \infty$, $r \to 0$;

$$i\pi(p(\omega_0) - p_\infty) = \int_{-\infty}^{\infty} \frac{p(\omega) - p_\infty}{\omega - \omega_0} ,$$

or, taking real and imaginary parts:
These are the Kronig-Kramers relations. Note that we again meet the Hilbert transform introduced in lecture II. In particular, (14)-(15) must also be valid for any physically acceptable dielectric functions, i.e. \( \varepsilon(k,\omega) \) where we treat \( k \) as an index. Note that we made no assumptions regarding zeroes of \( \varepsilon(\omega) \).

The relations (14)-(15) are very important since they let us determine the \( \text{Im} \) part of a function when its \( \text{Re} \) part is known and vice versa. It is good to see that they are automatically satisfied for plasma dielectric functions. Now assume that \( \varepsilon^*(\omega) = \varepsilon(-\omega^*) \), i.e. \( \varepsilon(\omega) = \varepsilon^*(-\omega^*) \). Then

\[
\text{Im} \, \varepsilon(\omega) = -\frac{2}{\pi} \int_0^\infty \frac{\text{Re} \, \varepsilon(\omega) - \varepsilon(\omega)}{\omega^2 - \omega_0^2} \, d\omega ,
\]

\[
\text{Re} \, \varepsilon(\omega) - \varepsilon(\omega) = \frac{2}{\pi} \int_0^\infty \frac{\omega \text{Im} \, \varepsilon(\omega)}{\omega^2 - \omega_0^2} \, d\omega .
\]

This implies for the plasma dielectric function that \( f_0(v) = f_0(-v) \).

Example: assume that the medium is "lossless", i.e. \( \varepsilon(k,\omega) = 0 \).

Then \( \text{Re} \, \varepsilon(\omega) = \varepsilon(\omega) \).

Note that \( \varepsilon(\omega) \) is real for physical reasons; as \( \omega \to \pm \infty \) any medium becomes lossless!
Lecture VIII

When the real part of a dielectric function is known, the Kronig-Kramers relations tell us how to determine the imaginary part (and vice versa) since the two quantities are essentially the Hilbert transforms of each other. It may be useful, however, to get an insight into the general behaviour of the Hilbert transform of a function, so we do not need to do the full integration every time, and also in cases where we are only interested in the overall picture. Consider VII.16 and 17 and introduce the transformation:

\[ u = \ln(\omega/\omega_0), \quad \omega/\omega_0 = e^u, \quad du = \omega du \]

and obtain

\[ \text{Re} p(\omega_0) - p_\infty = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^u \text{Imp}(u) - p(\omega_0)}{\sinh u} \, du \quad (1) \]

using \( d \ln \cosh |u/2|/du = -(\sinh u)^{-1} \) we get by partial integration

\[ \text{Re} p(\omega_0) - p_\infty = \frac{1}{\pi} \int_{-\infty}^{\infty} \ln \coth |u/2| \frac{du}{u} \text{Imp}(u) \, du \quad (2) \]

since \( \ln \coth u/2 \to -u \) for \( u \to \infty \) and \( \text{Imp}(\omega \to \infty) \to 0 \). By \( p(u) \) we understand \( p(\omega = \omega_0 e^u) \) throughout. By reintroducing \( \omega \) we obtain

\[ \text{Re} p(\omega_0) - p_\infty = \frac{1}{\pi \omega_0} \int_{0}^{\infty} \ln \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right| \frac{d\omega \text{Imp}(\omega)}{d\omega} \, d\omega \quad (3) \]

Thus \( \text{Re} p(\omega_0) \) depend on \( d\omega \text{Imp}(\omega)/d\omega \) at all frequencies, multiplied
by a sharply peaked (around $\omega_0$) weight function $\ln|\omega + \omega_0/\omega - \omega_0|$.

![Graph showing a sharply peaked function]

Returning to eq. (2) we write

$$\text{Re} p(\omega) - p_\infty = \pi \lim_{\omega \to \omega_0} \frac{d\text{Im} p(\omega)}{d\omega} + \int_0^\infty \ln \coth \frac{u}{2} \, du ,$$

where we used that $d/d\omega(e^{\omega \text{Im} p(u)})_{\omega = 0} = d/d\omega(\omega \text{Im} p(\omega) |_{\omega = \omega_0})$ and $\int_0^\infty \ln \coth \frac{u}{2} \, du = \pi^2/2$. The function $\ln \coth(u/2)$ is shown in fig. 2.

Note that the integrand in eq. (4) is zero for $u = 0$, i.e. at the point where $\ln \coth |u/2|$ is peaked.
The moral of eqs. (3) and (4) is that we may approximate

\[ \text{Re} \, p(\omega_0) - p_\infty = \frac{\pi}{2} \left. \frac{d}{d\omega}(\omega \text{Im} \, p(\omega)) \right|_{\omega = \omega_0} \]

provided \( d(\omega \text{Im} \, p(\omega))/d\omega \) does not vary too rapidly compared with the width of \( \ln|\omega + \omega_0/\omega - \omega_0| \) or \( \ln \coth|u/2| \). In such cases, eq. (5) may be quite helpful! Alternatively, we may want an approximation for \( \text{Im} \, p(\omega) \) with \( \text{Re} \, p(\omega) - p_\infty \) given. Using the same substitutions as before, we obtain

\[ \text{Im} \, p(\omega_0) = \frac{1}{\pi} \left[ \ln \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right| \left. \frac{d \text{Re} \, p(\omega)}{d\omega} \right|_{\omega = \omega_0} d\omega \]

and
We may argue as before that to a certain approximation

\[ \text{Imp}(\omega_0) = -\frac{\pi}{2\omega_0} \left. \frac{d \text{Re} p(\omega)}{d\omega} \right|_{\omega=\omega_0} - \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{d \text{Re} p(\omega)}{d\omega} - \omega_0 \frac{d \text{Re} p(\omega)}{d\omega} \right)_{\omega=\omega_0} \ln \coth \frac{\omega}{2} d\omega. \]  

(7)

Note that eqs. (5) and (8) are approximations, where the right-hand side is given. (It is tempting to argue that they should be identical, but this is not so.) For physical reasons we will argue that \( p_{\infty} = 1 \). We may get an expression for \( \text{Re} p(\omega = 0) = p_0 \) by using VII.17

\[ p_0 - p_{\infty} = \frac{2}{\pi} \left\{ \int_{0}^{\infty} \text{Imp}(\omega) \frac{1}{\omega} d\omega \right\}. \]  

(9)

or if we measure the frequency on a logarithmic scale

\[ p_0 - p_{\infty} = \frac{2}{\pi} \left\{ \int_{-\infty}^{\infty} \text{Imp}(\gamma = \ln \omega) d\gamma \right\}. \]

For a purely dissipative medium, \( \text{Imp}(\omega) \) corresponding to \( \text{Im} c(\omega) \) is larger than 0. This implies \( p_0 > p_{\infty} \). The value of \( p_0 \) can be determined when we know \( \text{Imp}(\omega) \).

Defining \( B_{\infty} = \lim_{\omega \to \infty} (\omega \text{Imp}(\omega)) \) we have, using VII.16

\[ B_{\infty} = \frac{2}{\pi} \left\{ \int_{0}^{\infty} (\text{Re} p(\omega) - p_0) d\omega \right\}. \]  

(10)

Now, in many cases \( B_{\infty} = 0 \), and, as mentioned, for physical reasons
\[ p_\infty = 1 \text{, so for } \text{Re} \epsilon_k(\omega) = \text{Re} \epsilon_\ast_k(\omega) \text{ we have} \]

\[ \int_0^\infty \text{Re} \epsilon_k(\omega) d\omega = \frac{\pi}{2} . \tag{11} \]

Now let us assume that \( \text{Imp}(\omega) \) is known. Using VII.15 and the relation \((\omega - \omega_0)^{-1} = -\frac{1}{\omega_0} \frac{\omega_0}{(\omega / \omega_0)^n} \) we obtain

\[ \text{Re} \rho(\omega_0) - p_\infty = - \frac{1}{\pi \omega_0} \sum_{n=0}^\infty \int \text{Imp}(\omega) \left( \frac{\omega}{\omega_0} \right)^n d\omega . \tag{12} \]

Obviously we have to assume that all the integrals are convergent. This is, for instance, the case for the common situation where \( \text{Imp}(\omega) \) varies exponentially for large \( \omega \). For sufficiently large \( \omega_0 \) we need only retain the \( n = 0, n = 1 \) terms, and get

\[ \text{Re} \rho(\omega_0) - p_\infty = - \frac{1}{\pi \omega_0} \int \text{Imp}(\omega) d\omega - \frac{1}{\pi \omega_0} \int_0^\infty \omega \text{Imp}(\omega) d\omega . \tag{13} \]

For a plasma \( \text{Imp}(\omega) \) is given by \( \text{f}_0'(\omega/k) \), so the first integral is zero even without the condition \( p^\ast(\omega) = p(-\omega^\ast) \). Obviously we may construct an asymptotic series expansion for \( \text{Re} \rho(\omega) \), using (12). (A similar expansion for \( \text{Imp}(\omega) \) may be constructed, but for this case it is much more likely to encounter cases where the integrals in the series expansion diverge.) Note that eq. (13) demonstrates that \( \text{Re} \rho(\omega_0) < p_\infty \) and \( d\text{Re} \rho(\omega)/d\omega > 0 \) for large \( \omega \) provided the medium is purely dissipative, i.e. \( \text{Imp}(\omega) > 0 \) for \( \omega > 0 \) and \( \text{Imp}(\omega) < 0 \) for \( \omega < 0 \).

Finally, let us find an expression for \( d\text{Re} \rho(\omega)/d\omega \): using e.g. VII.17
\[
\left. \frac{d \text{Re} p(\omega)}{d\omega} \right|_{\omega=\omega_0} = \frac{4}{\pi} \int_0^\infty \frac{(\omega^2-\omega_0^2)\left(\frac{d \text{Im} p(\omega)}{d\omega} + \text{Im} p(\omega)\right) + 2\omega_0 (\omega \text{Im} p(\omega) - \omega_0 \text{Im} p(\omega))}{(\omega^2-\omega_0^2)^2} \, d\omega.
\]

Let \(\text{Im} p(\omega) = 0\) and \(d\text{Im} p(\omega)/d\omega = 0\) in a certain frequency range and \(\text{Im} p(\omega) > 0\) elsewhere \((\omega > 0)\). In that frequency range we have \(d\text{Re} p(\omega)/d\omega > 0\) according to eq. (14), or more carefully:

\[
\left. \frac{d \text{Re} p(\omega)}{d\omega} \right|_{\omega=\omega_0} = \frac{8\omega_0}{\pi} \int_0^\infty \frac{\omega \text{Im} p(\omega)}{(\omega^2-\omega_0^2)^2} \, d\omega.
\]

Similarly, for \(d\text{Im} p(\omega)/d\omega\):

\[
\left. \frac{d \text{Im} p(\omega)}{d\omega} \right|_{\omega=\omega_0} = -\frac{2}{\pi} \int_0^\infty \frac{(\omega_0^2+\omega^2)(\text{Re} p(\omega) - \text{Re} p(\omega_0)) + \omega_0 \frac{d \text{Re} p(\omega)}{d\omega} \left|_{\omega=\omega_0}\right.}{(\omega^2-\omega_0^2)^2} \, d\omega.
\]

At the maximum value of \(\text{Re} p(\omega)\) we have \(d\text{Re} p(\omega)/d\omega = 0\). Here, eq. (15) gives \(d\text{Im} p(\omega)/d\omega < 0\). For the minimum value of \(\text{Re} p(\omega)\) we get similarly \(d\text{Im} p(\omega)/d\omega > 0\). In particular, for \(\omega_0 = 0\) we get

\[
\left. \frac{d \text{Im} p(\omega)}{d\omega} \right|_{\omega=0} = -\frac{2}{\pi} \int_0^\infty (\text{Re} p(\omega) - \text{Re} p(0)) \frac{1}{\omega^2} \, d\omega.
\]

Note that many of the results in this lecture were derived under the condition \(p^*(\omega) = p(-\omega^*)\). Although dielectric functions often satisfy this condition, we may have to use the relations VII.14-15 in certain cases, rather than VII.16-17.
It should be emphasized that many of the problems considered in this lecture are not specifically related to plasma media, not even necessarily to dielectric functions as such. Actually, many of the relations mentioned were first derived in connection with network analysis.

The Kronig-Kramers relations can be very time-saving, in the sense that it suffices to measure say the imaginary part of a dielectric function, i.e. the dielectric losses, for various frequencies. This can be done relatively simply. (Actually, I do not know of any easy way to measure the real part directly.) The relations mentioned above then allow the other part to be determined. The list of all the relations given in this lecture may seem lengthy and boring, but they can help you to make a quite satisfactory "free-hand drawing" of the function you would like to determine in this manner.
Reference list for lecture VIII

Exercise: Let \( f_0 = f_{0e}(v) \) and \( f_1(x,v,0) = \epsilon f_{0e}(v) \cos k_0 \cdot x \)

\[
f_{0e}(v) = \frac{a}{\pi} \frac{1}{a^2 + v_x^2} \frac{m_e}{2\pi k Te} e^{-\frac{m_e(v_y^2 + v_z^2)}{2k Te}}.
\]

Note: \( \int f_0(v) dv = 1 \).

(i) \( f_{0e}(v) \) is \textit{not} an entire function: poles for \( v_x = \pm ia \). However, for \( v_x \) fixed, it \textit{is} an entire function of \( v_y \) and \( v_z \).

(ii) \[
\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \nabla f_1 - \frac{n_0 e}{m} \mathbf{E} \cdot \mathbf{v} f_0(v) = 0,
\]

\[
(s + ik \cdot v)f_1 = \frac{n_0 e}{m} \mathbf{E} \cdot \mathbf{v} f_0(v) + n_0 \tilde{f}_1(0^+).
\]

Now \( \tilde{f}_1(0^+) = \epsilon f_{0e}(v) \cos k_0 \cdot x = \frac{1}{2} \epsilon f_{0e}(v)(e^{ik_0 \cdot x} + e^{-ik_0 \cdot x}) \), so \( \tilde{f}_1(0^+) \) is only nonzero for \( k = \pm k_0 \).

\[
n_1 = \frac{n_0 e}{m} \int_{-\infty}^{\infty} \frac{\mathbf{E} \cdot \nabla f_0(v)}{s + ik \cdot v} dv + n_0 \int_{-\infty}^{\infty} \frac{\tilde{f}_1(0^+)}{s + ik \cdot v} dv
\]

inserted into Poisson's eq. \( ik \cdot \mathbf{E} = -\frac{\epsilon}{\varepsilon_0} n_1 \) gives
Now consider explicitly \( k_0 = (k_0, 0, 0) \). Recall that \( E \) is longitudinal (electrostatic), i.e. \( \nabla \times E = 0 \): \( \mathbf{i} k \times E = 0 \rightarrow k \parallel \mathbf{E} \rightarrow \mathbf{E} = \mathbf{E}_0 / |k_0| \)

\[
\frac{i k \cdot E}{p} = \omega^2 \int_{-\infty}^{\infty} \frac{E \cdot \nabla f_0(v)}{s + i k \cdot v} \, dv = \frac{e n_s}{\varepsilon_0} \int_{-\infty}^{\infty} \tilde{f}(0^+) \, dv.
\]

Integrate over \( v, v_z \), i.e.

\[
\int_{-\infty}^{\infty} \frac{2a}{\pi} \frac{k_0 x}{(v_x^2 + a^2)^2 (s + i k_0 v_x)} \, dv_x \int_{-\infty}^{\infty} \frac{m}{2 \pi T_e} e^{-\frac{m(v_y^2 + v_z^2)}{2 \pi T_e}} \, dv_y dv_z
\]

\[
= \int_{-\infty}^{\infty} \frac{2a}{\pi} \frac{k_0 v_x}{(v_x^2 + a^2)^2 (s + i k_0 v_x)} \, dv_x.
\]

The result is then

\[
E(x, s) = -\frac{n_s e}{\varepsilon_0} k_0 \int_{-\infty}^{\infty} \frac{e}{(v_x^2 + a^2) (s + i k_0 v_x)} \, dv_x + \text{complex conjugate}.
\]

\[
i k^2 = \omega^2 \int_{-\infty}^{\infty} \frac{2a}{\pi} \frac{k_0 v_x}{(v_x^2 + a^2)^2 (s + i k_0 v_x)} \, dv_x
\]

Note again \( E = \frac{k_0}{|k_0|} \).
Problem 2: Consider the integral

\[
\frac{2a}{\pi} \int_{-\infty}^{\infty} \frac{v_x dv_x}{(v_x^2 + a^2)^2 (s + ik_0 v_x)} = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{ik_0 dv_x}{(v_x^2 + a^2) (s + ik_0 v_x)^2},
\]

use

\[
\frac{1}{v_x^2 + a^2} = \frac{1}{(v_x - ia)(v_x + ia)}.
\]

By integrating along a path shown below

noting that \( \text{Re}s > 0 \), we solve the integral and obtain

\[
- \frac{ik_0}{(s + k_0 a)^2} \quad \text{for } k_0 > 0,
\]

\[
- \frac{ik_0}{(s - k_0 a)^2} \quad \text{for } k_0 < 0 \quad \text{(use contour in upper half plane)}.
\]

Similarly, for
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{(a^2 + v^2)(s + ik_0 v)} \, dv = \begin{cases} \frac{1}{s + k_0 a} & \text{for } k_0 > 0 , \\ \frac{1}{s - k_0 a} & \text{for } k_0 < 0 . \end{cases} \]

Insert into the expression for \( E(x, s) \)

\[ E(x, s) = -\frac{e}{c_0} k_0 - \frac{\varepsilon n_0}{s + k_0 a} \frac{1}{ik_0^2 \left( 1 + \omega_p^2 \frac{1}{(s + k_0 a)^2} \right)} e^{ik_0 x} \]

\[ + \frac{e}{c_0} k_0 - \frac{\varepsilon n_0}{s + k_0 a} \frac{1}{ik_0^2 \left( 1 + \omega_p^2 \frac{1}{(s + k_0 a)^2} \right)} e^{-ik_0 x} , \]

with \( k_0 > 0 \).

\[ E(x, s) = -k_0 \frac{\varepsilon e}{c_0 k_0^2} \frac{s + k_0 a}{(s + k_0 a)^2 + \omega_p^2} \sin(k_0 x) . \]

Now

\[ \frac{s + k_0 a}{(s + k_0 a)^2 + \omega_p^2} \]

is the Laplace transform of \( \cos(\omega_p t)e^{-k_0 at} \), or rather:

\[ L(\cos \omega t) = \frac{s}{s^2 + \omega_p^2} \quad \text{and} \quad L(e^{\omega t} f(t)) = F(s - \alpha) \]

where \( L(f(t)) = F(s) \)
so

\[ E(x,t) = - \frac{\varepsilon}{\varepsilon_0 k_0^2} \cos (\omega_p t) \sin (k_0 x) e^{-ak_0 t}. \quad \text{Q.E.D.} \]

Note the oscillations have frequency \( \omega = \omega_p \) and not

\[ \omega = \sqrt{\frac{\omega^2 + 3v^2 k_0^2}{p}}. \]

However, \( v_T^2 = \int_{-\infty}^{\infty} v^2 f_0(v) \, dv. \)

With \( f_0(v) = \frac{1}{v^2 + a^2} \), \( v_T^2 \) diverges, so the usual dispersion relation becomes meaningless.

Note that the solution for \( E(x,t) \) above takes the form of a standing wave. This could be guessed from the outset: for symmetry reasons, the initial condition must give rise to a right and a left propagating wave with equal amplitudes.
Lecture X

The fluctuation-dissipation theorem

We will begin this section with a simple-minded proof of the Nyquist theorem.

Let us consider a resonance circuit as the one shown in fig.1:

![Circuit Diagram]

Fig.1.

Obviously $\frac{dq}{dt} = I$, $q = C U$ and $U = -L(\frac{dI}{dt})$. The Hamiltonian (the total energy) is $H = \frac{1}{2} C U^2 + \frac{1}{2} L I^2$. The circuit has one degree of freedom. $q$ and $p = LI$ are the generalized coordinates and generalized momentum, respectively, since they satisfy $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$, as the reader may verify. (This choice of generalized coordinate and momentum is not unique.) We can write the Hamiltonian as: $H = (1/2L)p^2 + (1/2C)q^2$. Bringing the circuit in thermal equilibrium, we find the probability for finding the system in a state $(p; p+dp, q; q+dq)$ to be
\[ P(p,q)\,dp\,dq = \frac{\exp(-H(p,q)/\kappa T)}{\int_\infty^{-\infty} \exp(-H(p,q)/\kappa T)\,dp\,dq} \, dp\,dq \quad (1) \]

where

\[ \int_\infty^{-\infty} \exp(-H(p,q)/\kappa T)\,dp\,dq = 2\pi\kappa TV\sqrt{C} \quad (2) \]

as verified by insertion of \(H(p,q)\). We use a classical (not quantum-mechanical) description. Introducing \(q = Cu\) and \(p = Li\) we calculate:

\[ \langle U^2 \rangle = \int_{-\infty}^{\infty} u^2 P(U,I)\,dU\,dI = \frac{\kappa T}{\gamma} \quad (3) \]

Similarly

\[ \langle I^2 \rangle = \int_{-\infty}^{\infty} I^2 P(U,I)\,dU\,dI = \frac{\kappa T}{L} \quad (4) \]

We will now specify how we bring the circuit into thermal equilibrium, i.e. specify the "heat reservoir". It must have many degrees of freedom compared with those of the circuit (this should not be difficult; the circuit has one). On the other hand, it must not destroy the "identity" of the resonant circuit; the damping must still be very small (without the "heat reservoir" there is no damping at all). We imagine that we bring the circuit into thermal equilibrium by filling the space between the condenser plates with some neutral gas at temperature \(T\) and some electrons with density \(n\). (We treat the problem in a rather academical way by neglecting the influence of the material
(copper plates, wires, etc. that would constitute a real circuit.) The thermal motion of the electrons then excite oscillations in the circuit. If the density $n$ is not too large, these oscillations will only be weakly damped. An equivalent circuit of the whole system is shown in fig. 2 for this particular case.

$$R = \frac{L}{\bar{A}} \frac{m}{e^2 n} \frac{(1 + \omega^2 \tau^2)}{\chi} ,$$

$$C' = C \left(1 - \frac{q^2 n}{m e_0} \frac{\chi^2}{1 + \omega^2 \tau^2} \right).$$

The oscillations in the circuit are weakly damped when $R \gg Z_0 = \frac{\sqrt{L}}{C'}$ (large $Q = RV \sqrt{C'/L}$). Evidently we can bring our circuit in fig.1 into thermal equilibrium by connecting a large resistance $R$ at temperature $T$ in parallel. $R$ may depend on $\omega$. We expand $U(t)$ in Fourier series corresponding to a very large time interval $T_0$. Then

$$U(t) = \sum_p \Re \sqrt{2} U_p \exp \{ 2\pi i p t/T_0 \} .$$
We use the "plus" sign in the exponent as is conventional in electric circuit theory. We define \( I(t) = U(t)/R \parallel C \parallel L \): the current through the parallel connection of \( R, L \) and \( C \) for given \( U(t) \). Similarly

\[
I(t) = \sum_p \sqrt{\frac{z_0}{\omega_p}} I_p \exp\left(i2\pi pt/T_0\right).
\]

\( I_p \) and \( U_p \) are connected by:

\[
U_p = \frac{i\omega_p}{\omega_p} \frac{\sqrt{z_0}}{1 - (\omega_p/\Omega)^2 + i(\omega_p/\Omega)\gamma d} \sum_{p} I_p,
\]

where

\[
z_0 = \sqrt{\frac{L}{C}}, \quad \Omega = \frac{1}{\sqrt{LC}}, \quad \gamma = \frac{1}{Q} = \frac{2\omega_p}{Q}, \quad \text{and} \quad \omega_p = \frac{2\pi p}{T_0},
\]

(\( \omega_p \) has nothing to do with a plasma frequency). Since the circuit is in thermal equilibrium, we can use eq.(3)

\[
\frac{K_T}{C} = \langle U^2 \rangle = \sum_p \langle U_p U_p^* \rangle.
\]

Equation (3) still holds since the \( Q \) of the circuit is still very large so it can be identified as a resonance circuit. Inserting eq.(5) in eq.(6) we obtain:

\[
\frac{K_T}{C} = \sum_p \frac{(\omega_p/\Omega)^2 \langle I_p I_p^* \rangle}{[1 - (\omega_p/\Omega)^2]^2 + (\omega_p/\Omega)^2 d^2 z_0^2}.
\]

When \( T_0 \) is very large, the density of \( \omega_p \)'s on the \( \omega \)-axis is very large, so eq.(7) can be written
making use of the fact that there are $\Delta \omega T_0/2\pi$ terms in the sum in eq. (7), corresponding to a frequency interval $\Delta \omega$. Assuming that $\langle I_p I_p^* \rangle$ is a slowly varying function of $\omega$, we notice that the integrand is sharply peaked around the resonance frequency $\Omega$ provided $d$ is small ($Q$ large). Therefore, we can place $\langle I_p I_p^* \rangle$ outside the integral sign, obtaining:

$$\frac{\kappa T}{C} \approx \frac{T_0}{2\pi} \int_0^\infty \left[ 1 - \frac{(\omega/\Omega)^2}{1 + (\omega/\Omega)^2} \right]^{1/2} \frac{x^2}{\sigma_0^2} \, dx$$

The integral is solved and is $\pi/2d$.

$$\frac{\kappa T}{C} \approx \frac{T_0}{2\pi} \int_0^\infty \left[ 1 - \frac{(\omega/\Omega)^2}{1 + (\omega/\Omega)^2} \right]^{1/2} \frac{x^2}{\sigma_0^2} \, dx$$

or

$$\langle I_p I_p^* \rangle = 4\kappa T \frac{1}{R} \frac{1}{T_0} \ .$$

$\langle |I_p|^2 \rangle$ corresponds to the resonance frequency $\Omega$. If $R$ is a function of $\omega$, then $R = R(\omega)$. Obviously the resonance frequency $\Omega$ is arbitrary: for a given value of $\Omega$ and a given $R(\omega)$ we can always determine $L$ and $C$, so $(LC)^{-1} = \Omega$ and $L/C \ll R(\Omega)$.

As indicated by eq. (5), we can make an equivalent circuit of the circuit shown in fig. 3, where $R$ now symbolizes an ideal noise-free resistor, while the current generator (---) describes the thermal fluctuations.
A physical resistance has therefore an equivalent circuit, as shown in fig.4, where R is a noise-free resistance (4a and 4b are equivalent: Thevenin's theorem).

\[ U = I \cdot R \]

where \( \langle |I|^2 \rangle = 4kT \frac{1}{R} \frac{1}{T_0} \) and \( \langle |U|^2 \rangle = 4kTr \frac{1}{T_0} \), or in terms of the effective value of the current and voltage, respectively

\[
I^2_{\text{eff}}(f; f + \Delta f) = 4kT \frac{1}{R} \Delta f, \\
U^2_{\text{eff}}(f; f + \Delta f) = 4kT R \Delta f. 
\] (9)

For \( R = 100 \, \text{k}\Omega \), \( \Delta f = 10 \, \text{kHz} \), and \( T = 293^\circ \text{K} \), we get \( \sqrt{U^2_{\text{eff}}} \sim 4 \, \mu\text{V} \), a rather

\[ I(t) \]

\[ R \]

\[ C \]

\[ L \]
small quantity. Experimental investigations are therefore difficult. In 1928, J.B. Johnson made some very accurate measurements and we often meet the name "Johnson noise" as a synonym for "Nyquist noise". At first sight, eq.(9) seems to indicate that it is extremely dangerous to handle a 100 MΩ resistance without a bandpass filter; the available power is, however, \(kT\Delta f\) independent of \(R\), as the reader may verify. Moreover, it is physically impossible to realize a pure resistance in the frequency interval \([0;\infty)\). A quantum mechanical description is necessary in order to avoid the ultraviolet catastrophe, but such a description is an unnecessary refinement for most practical purposes (except for masertechnics, etc.).

Finally, a word of caution: some firms supply "low noise" resistances. (Usually metal-film type and fairly expensive.) Referring to eq.(9), this designation could falsely be considered as swindle, since eq.(9) is independent of both the applied material and manufacturer. However, eq.(9) is valid only in thermal equilibrium. When we pass a current through the resistance, it is surely not in equilibrium and the noise characteristics may change drastically. Then it becomes crucial to avoid cracks, etc., in the film.
Lecture XI

We realize the resistor $R$ in fig.2, lecture X, by filling the space between two condensor plates with a conducting medium, say a plasma. In a first attempt we use a very simplified model, ignoring the pressure term in the momentum equation. The response of the medium to an electric field $E_0 e^{-i\omega t}$ is then given by

$$m \ddot{z} + \frac{m}{\tau} \dot{z} = -eE_0 e^{-i\omega t}, \quad (1)$$

where $z$ is the coordinate perpendicular to the plane-parallel condensor plates, the term $\frac{m}{\tau} \dot{z} = \frac{m}{\tau} v_z$ is a damping term. Then

$$z = \frac{-ieE_0}{mc_0(1/\tau - i\omega)} e^{-i\omega t}. \quad (2)$$

The polarization $P$ of the medium (the plasma) is then

$$P = -ez_0 = i \frac{ne^2 E_0}{mc_0(1/\tau - i\omega)} e^{-i\omega t}. \quad (3)$$

Using the definition $\varepsilon = \varepsilon_0 \varepsilon + P = \varepsilon_0 \varepsilon E$ (or $\varepsilon - 1 = \frac{P}{\varepsilon_0 E}$), we obtain

$$\varepsilon = \varepsilon_1 + i\varepsilon_2 = 1 + \frac{q^2 n}{mc_0} \frac{i\tau}{\omega(1 - i\omega)} \quad (3)$$

or

$$\varepsilon_1 = 1 - \frac{\omega^2}{p} \frac{\tau^2}{1 + (\omega \tau)^2}, \quad \varepsilon_2 = \frac{\omega^2}{p} \frac{\tau/\omega}{1 + (\omega \tau)^2}$$

where
\( \omega_p \) is the plasma frequency. The voltage between the two plates a distance \( d \) apart is \( i \cdot E_0 e^{-i\omega t} \) and the current through the condensor is

\[
A_0 \frac{d}{dt} e^{-i\omega t} = -iA(\epsilon_1 + i\epsilon_2)\epsilon_0 \omega E_0 e^{-i\omega t}, \tag{4}
\]

where \( A \) is the area of the plates. The ratio between the current and the voltage is the admittance

\[
Y(\omega) = i \frac{A \epsilon_0 \epsilon_2}{\omega \epsilon_0 \epsilon_2} - i \frac{A \epsilon_0 \epsilon_1}{\omega \epsilon_0 \epsilon_1}, \tag{5}
\]

so the plasma-loaded condensor can be represented by a resistance \( R' \) in parallel with a capacitor \( C' \) with

\[
R' = \frac{i}{A} \frac{1}{\omega \epsilon_0 \epsilon_2}, \tag{6}
\]

\[
C' = \frac{A}{\epsilon_0} \epsilon_0 \epsilon_1. \tag{7}
\]

[Recall that the "empty" capacitance is \( C'' = \epsilon_0 A/\ell \), corresponding to \( \omega_p = 0 \Rightarrow \epsilon_1 = 1 \), so \( C' \) is equal to \( C'' \) in parallel with a capacitor: \( \frac{A}{\ell} \epsilon_0 (\epsilon_1 - 1) \).] We may thus present a plasma-loaded capacitor by a circuit element with impedance \( Z \).
where now both $R'$ and $C'$ depend on $\omega$, i.e. they are not "ordinary" resistors and capacitors. This particular realization of a resistive circuit element at temperature $T$ is again connected to an $L$-$C$ circuit like the one in fig.X.1. At the resonance frequency $\Omega = (LC)^{-1}$ of this circuit we again have the relations X.(9). Having a box of LC-circuits with different $\Omega$'s, we can "map" the value of $I_{\text{eff}}^2$ and $U_{\text{eff}}^2$ as a function of $\Omega$. When using the relations X.(9) we insert for $R$ the value of $\text{Re}Z(\Omega)$, which in our case is

$$\text{Re}Z = \text{Re} \left( \frac{1}{1/R' + \frac{1}{C'}} \right) = \frac{R'}{1 + (\Omega R' C')^2}$$

(8)

or

$$\text{Re}Z = \frac{1}{\frac{1}{\varepsilon_0 \varepsilon_2} \frac{1}{\varepsilon_1 + \varepsilon_2}} = \frac{1}{\varepsilon_0 \frac{1}{\Omega} \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}$$

(9)

so, e.g.

$$U_{\text{eff}}^2 (f, f + \Delta f) = \frac{\varepsilon_0}{\varepsilon_2} \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \Delta f$$

(10)

or in terms of $\Omega$ rather than $f$:

$$U_{\text{eff}}^2 (\Omega, \Omega + \Delta \Omega) = \frac{\varepsilon_0}{\varepsilon_2} \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \Delta \Omega$$

(11)

Often we see (11) without the factor 2, if we let $\Omega$ take negative values also, i.e. $\Omega \leq -\infty$. Now obviously $l$ and $A$ refer to the particular condensor we put the plasma in. The plasma itself will be characterized by a fluctuating electric field
with a power spectrum (using $E = -V/\ell$ and $A \cdot \ell$ is the volume of the capacitor)

$$\mathcal{P}_E(\Omega) = \frac{kT}{\pi \varepsilon_0 \Omega} \frac{\varepsilon_2(\Omega)}{\varepsilon_1^2(\Omega) + \varepsilon_2^2(\Omega)} \quad .$$  \hfill (12)

I am a bit sloppy, but this is the expense of giving such a simplified proof of Nyquist's theorem (or the fluctuation-dissipation theorem) in lecture X. It can now be proved that (12) may be generalized to the case where $\varepsilon$ depend on the wave-number $k$ too:

$$\mathcal{P}_E(\Omega,k) = \frac{kT}{\pi \varepsilon_0 \Omega} \frac{\varepsilon_2(k,\Omega)}{\varepsilon_1^2(k,\Omega) + \varepsilon_2^2(k,\Omega)} \quad .$$  \hfill (13)

The power spectrum for the potential $\phi$ can be derived from (13) since $E = -ik\phi$, i.e.

$$\mathcal{P}_\phi(\Omega,k) = \frac{1}{k^2} \mathcal{P}_E(\Omega,k) \quad ,$$  \hfill (14)

and finally for the charge density $\rho$ using Poisson's equation $\rho = k^2 \phi \varepsilon_0$, so

$$\mathcal{P}_\rho(\Omega,k) = \varepsilon_0^2 k^4 \mathcal{P}_\phi(\Omega,k) = \varepsilon_0 k^2 \mathcal{P}_E(\Omega,k) \quad .$$  \hfill (15)

If we are interested in incoherent scattering of, say, microwaves from thermal fluctuations in a plasma, it is obviously $\mathcal{P}_\rho(\Omega,k)$ which have our interest since it is the fluctuations in electron density that cause the scattering.
Note that $\mathcal{F}_{E,p}$ is large around $\Omega \sim \omega_p$, corresponding to the weakly damped plasma waves. This is at first sight puzzling: we expected large fluctuations for large resistivity according to X.(9). Note, however, that an undamped resonator driven at its resonance frequency is a good absorber, so this result is consistent with X.(9). Also note that the fluctuation dissipation theorem does not distinguish between origins for the dissipation, i.e. a resistance is described by time irreversible equations, while the Landau damping giving $\text{Im}(\Omega, k) = \varepsilon_2(\Omega, k)$ originate from a time reversible Vlasov equation.

An interested reader will find the treatment by Bekefi, *Radiation Processes in Plasmas* (Wiley and Sons, 1966), stimulating.

**Problems**

(1) Consider a "drifting" Maxwellian for the electrons

\[
f_{0_e}(v) = \frac{m}{2\pi k T} e^{-m/2kT(v - v_z)^2}.
\]

(a) Write the dielectric function for this plasms assuming $m_i \sim \infty$ (immobile ions)

(b) Find the real and imaginary parts of the dispersion relation.

This is a very simple problem.

(2) Let a medium with resistivity $\sigma = \text{constant} > 0$ be given.

(a) What is the dielectric function of this medium?
Let the plasma from problem (1) be imbedded in this medium.

(b) What is the total dielectric function?

(c) Find the resulting real and imaginary parts of the new dispersion relations.

(d) Demonstrate that a sufficiently high $v_0$ leads to instability.

(e) Isn't this surprising? After all, a medium with $\sigma > 0$ is stable and dissipative!!

Solution to problems

Problem (1):

\[ \varepsilon(k, \omega) = 1 - \frac{\omega^2}{k^2} \int_{-\infty}^{\infty} \frac{f'_0(v)}{v - \omega/k} \, dv, \]

\[ f'_0(v) = -\frac{2}{\sqrt{\pi}} \left( \frac{m}{2\kappa T} \right)^{3/2} (v - v_0)e^{-m/2\kappa T(v - v_0)^2}, \]

\[ \varepsilon = \varepsilon_1 + i\varepsilon_2, \]

\[ \varepsilon_1 = 1 + \frac{\omega^2}{k^2} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left( \frac{m}{2\kappa T} \right)^{3/2} \frac{(v - v_0)e^{-m/2\kappa T(v - v_0)^2}}{v - \omega/k} \, dv. \]

Transform $\sqrt{\frac{m}{2\kappa T}(v - v_0)} = \gamma$

\[ \varepsilon_1 = 1 + \frac{\omega^2}{k^2} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\gamma e^{-\gamma^2}}{\gamma - (\omega/k - v_0)\sqrt{m/2\kappa T}} \, d\gamma. \]

Now proceed as in lecture IV, page 7:
\[ \varepsilon_1 \approx 1 - \frac{\omega_p^2}{k^2} \frac{2}{\sqrt{\pi}} \frac{m/2kT}{(\omega/k - v_0)^2} \int_{-\infty}^{\infty} \gamma e^{-\gamma^2} \left( 1 + \frac{\gamma}{(\omega/k - v_0)^2} \right) \left( \frac{\gamma}{(\omega/k - v_0)^2} \right)^2 + \text{etc.} d\gamma \]

(1) term in the integral: \[ \int \gamma e^{-\gamma^2} d\gamma = 0 \]

(2) \[ \int \gamma^2 e^{-\gamma^2} d\gamma = \sqrt{\pi} \]

(3) \[ \int \gamma^3 e^{-\gamma^2} d\gamma = 0 \]

(4) \[ \int \gamma^4 e^{-\gamma^2} d\gamma = \frac{3\sqrt{\pi}}{2} \]

\[ \varepsilon_1 \approx 1 - \frac{\omega_p^2}{k^2} \left( \frac{1}{(\omega/k - v_0)^2} + \frac{3}{2} \frac{\omega_p^2}{m/2kT} \right) \]

The real part of the dispersion relation is obtained from \[ \varepsilon_1(k, \omega) = 0 \], i.e.

\[ 1 - \frac{\omega_p^2}{(\omega - kv_0)^2} - 3 \frac{kT}{m} \frac{\omega_p^2}{k^2(\omega/k - v_0)^2} \approx 0 \]

in the first approximation, \[ 1 - \frac{\omega_p^2}{(\omega - kv_0)} \approx 0 \quad \Rightarrow \quad (\omega - kv_0)^2 = \frac{\omega_p^2}{p} \]

in the next approximation, \[ 1 - \frac{\omega_p^2}{(\omega - kv_0)^2} \left( 1 + 3 \frac{kT}{m} \frac{k^2}{\omega_p^2} \right) = 0 \]
or

\[(\omega - kv_0)^2 = \omega^2 + 3 \frac{k^2}{m} \Rightarrow \omega \approx \pm \omega_p + kv_0 \pm \frac{3}{2} \frac{kT}{m} \frac{k^2}{\omega_p} ,\]

i.e. \(v_0\) simply introduces the Doppler shift \(kv_0\) in the result from lecture IV, as expected.

We find the damping, using VII.(11). First:

\[\frac{\partial \epsilon_1}{\partial \omega} \approx \frac{2\omega^2}{(\omega - kv_0)^2} \approx \frac{2}{\omega} , \text{ since } \omega - kv_0 \approx \pm \omega_p ,\]

inserted in

\[\gamma = \frac{\epsilon_2(k,\omega)}{\partial \epsilon_1(k,\omega)/\partial \omega} ,\]

i.e.

\[\gamma \approx - \frac{\omega_p^2}{2\pi} \frac{2\pi}{(k^2)} \left( \frac{m}{2kT} \right)^{3/2} (\omega/k - v_0) e^{-m/2kT(\omega/k - v_0)^2} \frac{\pm 2/\omega_p}{\omega} .\]

Now for "+" in \(\partial \epsilon_1/\partial \omega \Rightarrow \omega - kv_0 = + \omega_p\)

\[\gamma_+ \approx - \sqrt{\pi} \left( \frac{m}{2kT} \right)^{3/2} \frac{\omega_p^4}{k^2} e^{-m/2kT(\omega_p/k)^2} .\]

For "-" in \(\partial \epsilon_1/\partial \omega \Rightarrow \omega - kv_0 = - \omega_p\)

\[\gamma_- = - \sqrt{\pi} \left( \frac{m}{2kT} \right)^{3/2} \frac{\omega_p^4}{k^2} e^{-m/2kT(\omega_p/k)^2} = \gamma_+ .\]

The wave is damped in both cases.
The real part of the dispersion relation looks like:

Problem (2):

Given a medium with resistivity $\sigma = \text{const} > 0$. The dielectric function of this medium is

$$
\varepsilon = 1 + i \frac{1}{\varepsilon_0 \omega}.
$$

The dielectric function for plasma + resistive medium is obtained using $\varepsilon_{\text{total}} = 1 + \frac{2}{i} (\varepsilon - 1)$, i.e.

$$
\varepsilon_{\text{total}} = \varepsilon_{\text{plasma}} + \frac{1}{\varepsilon_0 \omega} = 1 - \left(\frac{\varepsilon}{\varepsilon_0 \omega}\right)^2 \int_{-\infty}^{\infty} \frac{f_0'(v)}{v} dv - i \left[ \pi \left(\frac{\omega}{k}\right)^2 f_0'\left(\frac{\omega}{k}\right) - \frac{1}{\varepsilon_0 \omega} \right].
$$

Obviously, the real part of $\varepsilon$ is unchanged, so the real part of
the dispersion relation remains the same as in problem (1).

However, Im$\varepsilon$ is now different, so $\gamma_z$ in problem (1) are modified. The change is simply:

$$\gamma_{+\text{new}} = \gamma_{+} + \frac{1}{2\varepsilon_0} \frac{\omega_p}{k} = \gamma_{+} - \frac{1}{2\varepsilon_0} \frac{\omega_p}{k}$$

$$\gamma_{-\text{new}} = \gamma_{-} + \frac{1}{2\varepsilon_0} \frac{\omega_p}{k}$$

Obviously, $\gamma_{+\text{new}}$ remains negative, while $\gamma_{-\text{new}}$ may change sign, becoming positive $\Rightarrow$ instability for sufficiently small $\sigma$, and $k > \omega_p/v_0$.

Now I claim that this is strange, at least at first sight, since we start with a stable plasma, introduce dissipation and get instability. The answer is that the slow wave $\omega = -\omega_p + kv_0$ has negative energy for $k > \omega_p/v_0$. Recall that the wave energy is given as $(\partial \omega_1/\partial \omega)|E|^2$.

The concept of a negative energy wave can be understood as follows: Assume a wave with ($\omega, k$) given. Let the DC-drift velocity be $v_0$. From the linearized continuity e.g. we get

$$\frac{\partial n}{\partial t} + v_0 \frac{\partial n}{\partial x} + n \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{n(\omega - v_0)}{n_0(k - v_0)} = v$$

(i) if $v_0 < \omega/k$, we have schematically:
In the latter case, $n$ is larger where $v$ is small or $n$ is small where $v$ is larger, i.e. the energy density with the wave is smaller than without a wave, therefore the term "negative energy". If we try to damp a negative energy wave, i.e. extract energy from it (e.g. by the dissipative medium), its amplitude will grow. However, the positive slope of the displaced Maxwellian is putting energy into the wave, thereby decreasing its amplitude, i.e. the two effects are competing.
It is important to get a "feeling" for the concept of negative energy waves, so try to get used to the idea.
Lecture XII

Fluctuations in plasmas: a different point of view.

Introduce a "test charge" in the plasma, as, for example, Montgomery, page 88. The potential response is

$$\phi(k,\omega) = \frac{1}{\varepsilon_k k^2} \frac{\rho(k,\omega)}{\varepsilon_1(k,\omega) + i\varepsilon_2(k,\omega)}.$$  \hspace{1cm} (1)

We consider $\varepsilon = \varepsilon_1 + i\varepsilon_2$ as a given characteristic of the plasma. Assume now that $\rho(k,\omega)$ is fluctuating and that we know $\langle \rho^2(k,\omega) \rangle$. The power spectrum of the associated potential fluctuations is obviously:

$$\langle \phi^2(k,\omega) \rangle = \frac{1}{\varepsilon_k k^2} \frac{\langle \rho^2(k,\omega) \rangle}{\varepsilon_1(k,\omega) + i\varepsilon_2(k,\omega)}.$$  \hspace{1cm} (2)

Assume now that the test charges move independently without interaction (i.e. collisions). Then the fluctuations in the density will be that of an ideal (Knudsen) gas, i.e.

$$\langle n^2 \rangle = n_0 \int F(v) \delta(\omega-kv) dv,$$  \hspace{1cm} (3)

where $n_0$ is the density of the test charges and $F(v)$ is their velocity distribution (see, e.g. Montgomery, page 269). Solving eq. (3) we get

$$\langle \rho^2 \rangle = \varepsilon^2 \langle n^2 \rangle = \frac{n_0 \varepsilon^2}{k} F\left(\frac{\omega}{k}\right),$$  \hspace{1cm} (4)

(recall that $\delta(\alpha x) = \frac{1}{|\alpha|} \delta(x)$). Using eq. (2) we get:
Now consider each test charge as one of the electrons constituting the plasma. If we ignore any interaction, then the potential associated with their fluctuations in density is clearly given by eq. (2) or (5) with $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ inserted. In the next approximation we consider each electron as a "dressed particle", taking into account its polarizing effect on the remaining electrons in the plasma, by introducing $\varepsilon(k, \omega)$ given by

$$
\varepsilon(k, \omega) = 1 - \frac{\omega^2}{k^2} \int_{-\infty}^{\infty} \frac{F'(v)}{\sqrt{\omega/k}} \, dv - \frac{i \pi P}{k^2} F'(\omega/k) ,
$$

since obviously $F(v)$ is just the velocity distribution function we denoted $f_0(v)$ in previous lectures. At first sight, this procedure seems inconsistent: we treat each electron as an independent "dressed particle", but the polarization of the plasma obviously involves all the other electrons! Note, however, that it takes a very little displacement in the electron trajectory to build up the polarization, so the description is likely to be correct - at least in the linear approximation - and this is all we are concerned with here, since eq. (6) is derived on the basis of the linearized Vlasov equation. In Montgomery, chapter 10, you will find this approach elaborated in considerable detail. Here we shall be content with demonstrating that eq. (5) reproduces XI.(14) as a special case. The result XI.(14) was derived using very basic thermodynamic arguments, so it must be correct (again within a linear description). In other words,
in order to be able to trust eq.(5) we want it to reproduce XI.(14) if we insert a Maxwellian for $F(v)$. Now in that case, $F'(v) = -v(m/kT)F(v)$, so $\langle \rho^2 \rangle = -\frac{kT}{m} k/\omega F'(\omega/k)$; note that $k/\omega F'(\omega/k) \leq 0$: we need not worry about the minus sign. However, we may write $\varepsilon_2(k,\omega)$ from eq.(6): $\varepsilon_2(k,\omega) = -\pi(\omega_p^2/k^2)F'(\omega/k)$, so

$$\varepsilon_2(k,\omega) = \frac{\pi}{k^2} \frac{\omega}{\varepsilon_0 kT} \langle \rho^2(k,\omega) \rangle$$

when $F(v)$ is a Maxwellian. Inserting eq.(7) into XI.(14) gives eq.(2), Q.E.D.

Now recall that in deriving eq.(2) we only made one (implicit) assumption, namely that the plasma is stable; otherwise $\langle \phi^2 \rangle$ would not stay finite for long times. Apart from this we may insert any $F(v)$ and have thus obtained a considerable generalization of the fluctuation-dissipation theorem. Note, however, that eq.(2) is only valid for a plasma; XI.(14) could be generalized to any medium.

In this lecture we have only considered the electrons and assumed that the ions constitute an immobile neutralizing background of positive charge. If we only consider $\omega > \omega_{pi}$, their effect on the spectrum is negligible. A generalization is straightforward, see for example Bekefi, *Radiation Processes in Plasmas*, chapter 4.

Finally we make a remark on the averaging: $\langle \rangle$. For the fluctuation-dissipation theorem we are free to consider time or ensemble averages. When the system is not in thermal equilibrium, we must consider ensemble averages. It is a general misunder-
standing that for stationary systems the two averages are equal: it need not be so. Another thing is that we often have to be content with time averages and then hope for the best!
Lecture XIII

Electron waves in a strongly magnetized plasma

Consider frequencies $\omega \sim \omega_p \ll \omega_{ce}$. The electron velocity $\parallel$ to $B$ is $v_\parallel \sim E_n / \omega$, while $\perp$ to $B$ it is $v_\perp \sim E_\perp / B$. i.e. we use for $v_\perp$ the guiding center velocity $E \times B / B^2$. Then $v_\parallel / v_\perp \sim (\omega_{ce} / \omega) E_n / E_\perp \gg 1$.

From the outset we can only argue that this inequality is satisfied when $\omega \sim \omega_p$ and $E_n \sim E_\perp$. We will see that $\omega \ll \omega_p$ is a sufficient condition. It is then a good approximation to ignore the $\perp$ motion of the electrons altogether and write the linearized electron Vlasov equation in the one-dimensional form, with $z$ denoting the coordinate along $B$ (which we take to be homogeneous and uniform)

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} - \frac{e}{m} E_n f'_{\parallel}(v) = 0 \quad .$$ (1)

In eq. (1) $f = f(v, z, t)$ where $v$ is the velocity along $B$. Since $v_\perp$ does not enter the equation according to the arguments above, we may as well "integrate it out" and $f$ in eq. (1) thus represents this reduced distribution function. In other words: we assume that the electrons move as pearls on a string.

Poisson's equation of course still contains the full three-dimensionality of the problem, i.e.

$$v \cdot E = \frac{\partial E_n}{\partial z} + \nabla_\perp E_\perp = -\frac{e}{\varepsilon_0} n(r) \quad ,$$ (2)

where in general $E_n = E_n(r)$ and $E_\perp = E_\perp(r)$. As usual, we only con-
sider electrostatic (or longitudinal) oscillations, so $E = -\nabla \phi = -\frac{\partial \phi}{\partial z} - \nabla \phi$. Then

$$\frac{\partial^2 \phi}{\partial z^2} + \nabla^2 \phi = \frac{e}{\varepsilon_0} n,$$  \hspace{1cm} (2)

or

$$\frac{\partial^2 \phi}{\partial z^2} + \nabla^2 \phi = \frac{e}{\varepsilon_2} \frac{dn}{dz}.$$  

Since $E_n = -\frac{\partial \phi}{\partial z}$ we may thus express eq. (2) in terms of $E_n$ as

$$\frac{\partial^2}{\partial z^2} E_n + \nabla^2 E_n = -\frac{e}{\varepsilon_2} \frac{dn}{dz}.$$  \hspace{1cm} (4)

We now look for plane wave solutions of the form $E_n = A e^{-i (\omega t - k_n z - k_\perp \cdot \mathbf{L})}$ we obtain from eq. (1)

$$n = \frac{\varepsilon}{m} A \int \frac{f_0'(v)}{I (v k_n - \omega)} \, dv$$  \hspace{1cm} (5)

inserted into eq. (4):

$$k_n^2 + k_\perp^2 = k^2 = \frac{e^2}{c \varepsilon m} \int \frac{f_0'(v)}{\sqrt{v^2 - \omega^2/k_n^2}} \, dv,$$  \hspace{1cm} (6)

or by normalizing $\int f_0 \, dv = 1$

$$1 = \frac{\omega^2}{k^2} \int \frac{f_0'(v)}{\sqrt{v^2 - \omega^2/k_n^2}} \, dv.$$  \hspace{1cm} (7)

Although the result looks familiar, it is not the same as in lecture IV since only $k_n$ enters the integrand! Assume now that
the plasma is located between two conducting, infinite, plane parallel metal plates a distance \( \ell \) apart where \( \mathbf{B} \) is parallel to the plates. Then \( k_\parallel \) can only take the values \( n2\pi/\ell \cdot k_{\perp n} \) since \( E_\parallel \) have to be zero at the plates. Assume \( k = 0 \) for the third direction. Then

\[
k^2_{\parallel} + k^2_{\perp} = \omega^2 \frac{\int_{\infty}^{\omega} f_0^i(v) \, dv}{\omega - \omega/k_\parallel}.
\]  

or approximately, for the real part of the dispersion relation

\[
\omega^2 = \omega^2 \frac{k_{\parallel}^2}{p k_{\parallel}^2 + k_{\perp}^2} + 3v^2 k_{\parallel}^2.
\]

Note that for \( k_\parallel \to 0 \) we have \( \omega^2 \sim \omega^2 k_{\parallel}^2/k_{\perp}^2 \). The dispersion relation looks like

![Graph of the dispersion relation](image)

The dispersion relation is similar for a strongly magnetized plasma in a cylindrical waveguide. Now recall that \( v_\parallel/v_\perp \sim (\omega_{ce}/\omega)E_\parallel/E_\perp \). Since \( E = -i\mathbf{k} \phi \), \( |E_\parallel/E_\perp| = k_\parallel/k_\perp \), i.e. \( v_\parallel/v_\perp \sim \omega_{ce} k_\parallel/\omega k_\perp \). Note again that \( \omega \sim \omega_p k_\parallel/k_\perp \) as \( k_\parallel \to 0 \), i.e. \( v_\parallel/v_\perp \sim \omega_{ce}/\omega_p \). Our calculations are consistent under the assumption \( \omega_{ce}/\omega_p \gg 1 \).
Problem

Consider an electron plasma with the one-dimensional velocity distribution

\[ f_0(v) = \frac{0.75}{\pi} \frac{a}{v^2 + a^2} + \frac{0.25}{\pi} \frac{a}{(v-v_0)^2 - a^2}. \]

(i) Make a sketch of \( f_0(v) \).

(ii) What is \( \int f_0(v) dv \)?

(iii) Determine the boundary curve of growing and damped plasma oscillations on the plane \( v_0/a \) versus \( k^2a^2/\omega_p^2 \).

Solution to problem

\[ v_0 = 3a \]

(i)

(ii) \( \int_{-\infty}^{\infty} f_0(v) dv = 1. \)

(iii) Take \( \text{Re} \omega > 0 \). Then instability will only occur if \( k > 0 \), i.e. the phase velocity is in the positive \( v \)-direction. Consider only the dielectric function with \( \epsilon \) sign.
Solve

\[ \int \frac{f'_0(v)}{v - \omega/k} \, dv = \frac{k^2}{\omega^2} \]

with

\[ f'_0(v) = - \frac{1.5}{\pi} \frac{va}{(v^2 + a^2)^2} - \frac{0.5}{\pi} \frac{(v - v_0)a}{[(v - v_0)^2 + a^2]^2} \]

The denominators have zeroes for \( v = \pm ia \) and \( v = v_0 \pm ia \), respectively.

Let \( \omega \) have a small positive imaginary part, and solve the integrals by contour integration along:

\[
\text{pole for } v = \omega/k \quad \text{above the axis}
\]

See the solution to problem 1, page 85 in Montgomery. It is most convenient to integrate by parts, i.e.

\[ \int \frac{f'_0(v)}{v - \omega/k} \, dv = \int \frac{f'_0(v)}{(v - \omega/k)^2} \, dv. \]

The result is, e.g.
\[
0.25 \pi \int \frac{dv}{[(v-v_0)^2+a^2](v-\omega/k)^2} = -a \int \frac{dv}{(v-\omega/k)^2[v-v_0+ia][v-v_0-ia]}
\]

\[
= \frac{0.25}{\pi} \neq \frac{2\pi I}{(v_0-ia-\omega/k)^2[1+2\pi I]} = \frac{0.25}{(v_0-ia-\omega/k)^2}.
\]

The full result is thus:

\[
\frac{0.25}{(v_0-ia-\omega/k)^2} + \frac{0.75}{(ia+\omega/k)^2} = \frac{k^2}{\omega^2 p},
\]
or

\[
\frac{0.25}{(v_0/a-1-\omega/ak)^2} + \frac{0.75}{(1+\omega/ak)^2} = \frac{a^2k^2}{\omega^2 p}.
\]

Introduce \(v_0/a = x, \quad \omega^2/ak^2 = y^2, \quad \omega/\omega_p = d\). Then the dispersion relation reads:

\[
\frac{0.25}{(i+dy-x)^2} + \frac{0.75}{(i+dy)^2} = \frac{1}{y^2}.
\]

Special case \(x = 0\):

\[
y^2 = (dy+i)^2, \quad y = \pm(dy+i),
\]
or

\[
\frac{\omega}{\omega_p} = \begin{cases} 
\omega/ak + 1 & \text{use only } \omega > 0 : \quad \omega = \omega_p - i ak \\
-\omega/ak - 1 & \text{always damped.}
\end{cases}
\]

Recall we assumed \(k > 0\).

Necessary condition for instability: a minimum for \(f_0(v)\), i.e.
\[
\frac{3v}{(v^2 + a^2)^2} + \frac{(v-v_0) a}{((v-v_0)^2 + a^2)^2} = 0 \quad \text{for some } v = \omega/k \neq \infty ,
\]
such that:

\[
\frac{dy}{3[dy^2 + 1]^2} + \frac{(dy-x)}{[(dy-x)^2 + 1]^2} = 0 .
\]

The criterion for the existence of the minimum is approximately:

\[v^2 > 4a^2\] obtained graphically.

Rewrite the dispersion relation as

\[
\frac{1}{4} \frac{(dy-x-i)^2}{[(dy-x)^2 + 1]^2} + \frac{3}{4} \frac{(dy-i)^2}{[(dy)^2 + 1]^2} = \frac{1}{y^2} .
\]

For marginal stability \(\text{Im} \omega = 0\), i.e. \(d = \omega/\omega_p\) is real. Take the real and imaginary parts of the expression above:

\[
\frac{1}{4} \frac{dy-x}{[(dy-x)^2 + 1]^2} + \frac{3}{4} \frac{dy}{[(dy)^2 + 1]^2} = 0 ,
\]

\[
\frac{1}{4} \frac{(dy-x)^2-1}{[(dy-x)^2 + 1]^2} + \frac{3}{4} \frac{(dy)^2-1}{[(dy)^2 + 1]^2} = \frac{1}{y^2} .
\]

In principle we may find \(dy\) from the first expression, insert in the second one and obtain the desired relation between \(x\) and \(y^2\). However, a graphical solution is preferable. Rewrite the dispersion relation as

\[-\frac{1}{4} \frac{W(dy-x)}{4} \frac{3}{4} \frac{W(dy)}{y^2} \].
where

\[ W(\xi) = \frac{(\xi - 1)^2}{(\xi^2 + 1)^2} = \frac{\xi^2 - 1}{(\xi^2 + 1)^2} - \frac{2\xi}{(\xi^2 + 1)^2} \]

(see enclosed fig.). For fixed \( y^2 \) draw \(-\frac{1}{4}W(dy-x)\) and \(\frac{3}{4}W(dy)-1/y^2\):

![Diagram](image)

Determine A and B and the corresponding values for \( dy - x \) and \( dy \), say:

For A: \( dy - x = a_1 \), \( dy = a_2 \).

For B: \( dy - x = b_1 \), \( dy = b_2 \).

We thereby obtain two points on our stability curve, namely \( x = a_2 - a_1 \) and \( x = b_2 - b_1 \), for the particular value of \( y^2 \) we have chosen. Now take a new value for \( y^2 \), find A and B, etc. The resulting stability curve looks something like this:
Let us consider a case which we can solve analytically, namely:

\[ f_0(v) = \frac{0.5}{\pi} \frac{a}{v^2 + a^2} + \frac{0.5}{\pi} \frac{a}{(v-v_0)^2 + a^2}. \]

With the previous notation, the real and imaginary part of the dispersion relation become

\[
\frac{1}{2} \frac{\text{d}y - x}{[(\text{d}y - x)^2 + 1]^2} + \frac{1}{2} \frac{\text{d}y}{[(\text{d}y)^2 + 1]^2} = 0,
\]

\[
\frac{1}{2} \frac{(\text{d}y - x)^2 - 1}{[(\text{d}y - x)^2 + 1]^2} + \frac{1}{2} \frac{(\text{d}y)^2 - 1}{[(\text{d}y)^2 + 1]^2} = \frac{1}{y^2}.
\]

Obviously, the first equation gives \( \text{d}y = x/2 \); inserted into the second one gives
The special case where \( a = 0 \):

\[
\frac{1}{2} \left( \frac{1}{v_0 - \omega/k} \right)^2 + \frac{1}{2} \left( \frac{1}{\omega/k} \right)^2 = \frac{k^2}{\omega_p^2}
\]

or

\[
\frac{1}{(kv_0 - \omega)^2} + \frac{1}{\omega^2} = \frac{2}{\omega_p^2}
\]

For simplicity, introduce \( \Omega = \omega - \frac{1}{4}kv_0 \),

\[
\frac{1}{(\Omega - \frac{1}{4}kv_0)^2} + \frac{1}{(\Omega + \frac{1}{4}kv_0)^2} = \frac{2}{\omega_p^2}
\]

or

\[
\left( \Omega^2 - \frac{k^2v_0^2}{4} \right) = \omega_p^2 \left( \Omega^2 + \frac{k^2v_0^2}{4} \right)
\]

\[
\Omega^2 - \Omega^2 (\omega^2 + \frac{1}{4}k^2v_0^2) + \frac{1}{2}k^2v_0^2 (\frac{1}{4}k^2v_0^2 - \omega^2) = 0
\]
\( \Omega^2 \) has a negative root of \( k^2 v_j^2 < 4 \omega_p^2 \) leading to instability. Note that \( \Re \Omega = 0 \) for the instability. Also note that for sufficiently small \( k \), the plasma will always be unstable if \( v_0 > 0 \). This we could tell right away from the Penrose criterion: the distribution function has a minimum where \( f_0(v) = 0 \).

In the "two-stream" instability, where \( f_0(v) = \delta(v-v_0) + \delta(v-v_0) \), all the particles participate in the instability. In the small "bump-on-tail" case, where \( f_0(v) \) looks like

![Diagram](image-url)

only the particles at the phase velocity drive the instability.

Intermediate cases, e.g. a double humped distribution like the ones considered in the problem, can probably not be understood in simple terms.
Nyquist's criterion for stability revisited

Consider for instance

\[
f_0 = \frac{3}{4\pi} \frac{a}{a^2 + v^2} + \frac{1}{4\pi} \frac{a}{a^2 + (v - v_0)^2}.
\]  

(1)

Note: \[ \int f_0(v) dv = 1. \] For definiteness \( v_0 > 0. \)

For marginally unstable \( \omega/k = u \), we have

\[
\int \frac{f_0'(v)}{v - u} dv = \frac{k^2}{\omega_p^2}.
\]  

(2)

No principal value, since marginal instability implies \( f_0'(u) = 0. \)

Solve eq. (2) with eq. (1) inserted (see previous problem) assuming \( k > 0 \). NB!!!

\[
\frac{0.25}{(v_0/a - i - \omega/ak)^2} + \frac{0.75}{(i + \omega/ak)^2} = \frac{a^2k^2}{\omega_p^2}.
\]

Define \( v_0/a = V, \omega/ak = u, a^2k^2/\omega_p^2 = K^2 \), and a function

\[
W(\xi) = \frac{1}{(1 + \xi)^2} = \frac{(\xi - i)^2}{(\xi^2 + 1)^2} = \frac{\xi^2 - 1}{(\xi^2 + 1)^2} - i \frac{2\xi}{(\xi^2 + 1)^2},
\]

i.e.

\[
\frac{1}{4} W(U - V) + \frac{3}{4} W(u) = K^2.
\]

Clearly \( W \) have no poles in the upper half of the complex \( u \)-plane (but it has one in the lower half, namely \( \xi = -i \)).
If $\frac{1}{2}W(u-V) + \frac{3}{2}W(u)$ assume any real positive value, say $q > 0$, in the upper half of the complex $u$-plane (for some obviously complex $u$-value $= \mu$) we have instability, since we can always take $K^2 = q$ with a corresponding $\omega = \mu \omega_k$.

Nyquist's criterion for the occurrence of such a situation is:

map the curve:

on the complex $\frac{1}{2}W(u-V) + \frac{3}{2}W(u)$-plane, i.e.

If the curve (as shown) encircles the real axis for some positive numbers, the instability criterion is satisfied. Obviously it will always encircle the negative axis, but this is uninteresting since $K^2 > 0$. The case $U = \pm \infty$ gives marginal stability for $K = 0$, i.e. these oscillations are neither damped nor amplified. This is nothing but the singular case of $\omega = \omega_p$. It is not really relevant since $K = 0$ can not be realized in any physical system of finite dimensions.
In particular, for $V = 0$:

$$W(u) = K^2.$$  

The mapping of this function I have shown you already.

The enclosed figure shows the cases $V = 3$ (see graph for $f_0(v)$ from previous problem; note that the minimum for $f_0(v)$ corresponds to $v \sim 2a$) and $V = 5$. 


Lecture XIV

In this lecture we shall consider ion acoustic perturbations in some detail. As in lecture VI, we assume that the electrons are Boltzmann distributed at all times, i.e.

$$\frac{n_e}{n_0} = \frac{e}{T_e} \phi$$.

We also assume that wavelengths of interest are much larger than the Debye length $\lambda_D$, and thus make the assumption of quasi-neutrality, i.e. $n_e = n_i = n$.

Consider first a (linear) fluid treatment, in one dimension. Our set of equations is thus

$$\frac{n}{n_0} = \frac{e}{T_e} \phi$$, \hspace{1cm} (1)

$$\frac{\partial n}{\partial t} + \frac{\partial \psi}{\partial x} = 0$$, \hspace{1cm} (2)

$$\frac{\partial \psi}{\partial t} = -\gamma \frac{T_i}{M} \frac{\partial n}{\partial x} - \frac{e}{M} \frac{\partial \phi}{\partial x}$$, \hspace{1cm} (3)

where $\gamma$ takes the value 1 or $\frac{3}{2}$ for isothermal or adiabatic ion motion, respectively. We assume that the ions have no zero order velocity, such as $v_0 = 0$. Equations (1)-(3) are easily reduced to

$$\frac{\partial^2 n}{\partial t^2} - C_s^2 \frac{\partial^2 n}{\partial x^2} = 0$$, \hspace{1cm} (4)
where \( c_s^2 = (T_e + \gamma T_i)/M \). The linear dispersion relation corresponding to eq. (4) is \( \omega = \pm c_s k \), i.e. the waves are non-dispersive within the present description, i.e. for \( 2\pi/k \gg \lambda_D \).

Consider a particular initial value problem where \( n/n_0 = a\delta(x) \) for \( t = 0 \). Then, using Laplace transform in time, Fourier transform in space, we obtain:

\[
 s^2/n_0 + c_s^2k^2n/n_0 = sa \quad \Rightarrow \quad \frac{n(s,k)}{n_0} = \frac{sa}{s^2 + k^2c_s^2} ,
\]

so

\[
 n(t,k)/n_0 = a \cos k_c t
\]

and

\[
 n(t,x)/n_0 = \frac{a}{2\pi} \int_{-\infty}^{\infty} \cos k_c t \ e^{ikx}dk .
\]

Using that the Fourier transform of \( \delta(x-x_0) \) is \( e^{ikx_0} \), we readily find

\[
 n(t,x)/n_0 = \frac{a}{2\pi} \int_{-\infty}^{\infty} \cos k_c t \ e^{ikx}dk .
\]

i.e. the pulse breaks up into two, each with amplitude \( a/2 \), propagating in opposite directions with velocity \( c_s \).

Let us now consider the same problem from a kinetic point of view, i.e. apply the linearized Vlasov equation but still retain eq. (1) and the assumption of quasi-neutrality. The Vlasov ion equation, inserting \( E = -\partial\phi/\partial x \),
\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{n_0 e}{M} \frac{\partial f_0}{\partial x} = 0 \]

reduces to, using eq. (1)

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{T_e}{M} \frac{\partial n}{\partial x} f_0'(v) = 0 \] (7)

with \( n = \int f dv \). We solve this equation with the initial condition \( f(x,v,t=0) = g(v) \delta(x) \), i.e. \( n(x,t=0) = \delta(x) \int g(v)dv \). Like before we get, with \( C^2 = T_e/M \)

\[ (s + ikv) f - ikC^2 n f_0'(v) = g(v) \] (8)

or

\[ n(k,s) = \frac{\int_{-\infty}^{\infty} g(v)dv}{s + ikv} \cdot \frac{1}{1 - iC^2 \int_{-\infty}^{\infty} \frac{f_0'(v)}{s + ikv}dv} \] (9)

Now \( n(x,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} n(k,s)e^{ik\cdot x}dk \). For \( k > 0 \) we have the singularities in the integrands of eq. (9) above the real \( v \)-axis, for \( k < 0 \) below (recall \( \Re s > 0 \)). It is an advantage to split up the contribution to \( n(x,s) \) accordingly:

\[ n(x,s) = \frac{1}{2\pi} \int_{-\infty}^{0} n_1(k,s)e^{ik\cdot x}dk + \frac{1}{2\pi} \int_{0}^{\infty} n_2(k,s)e^{ik\cdot x}dk \] (10)

Assuming that the plasma is stable (e.g. \( f_0(v) \) single-humped) we may deform the integration contour in the complex \( k \)-plane as:
The contribution from the half circles vanishes as $R \to \infty$, so the integrals along 1 and 1' (and 2 and 2') are identical. Introducing the variable transformation $k = is/\gamma$, $dk = isd\gamma/\gamma^2$ we reduce eq. (10) to

$$n(x,s) = \frac{1}{2\pi i} \int_0^\infty \left[ n_1(\gamma,s) - n_2(\gamma,s) \right] \frac{e^{-sx/\gamma}}{\gamma} d\gamma$$

Now

$$n_2(k,s) = \frac{1}{ik} \left\{ \frac{q(v)}{v-\frac{i\gamma}{k}} \frac{dv}{1-C^2} \right\}$$

and similarly for $n_1(k,s)$ with $\int$ inserted. Now with $k = is/\gamma$ inserted, we have

$$n_2(\gamma,s) = -\frac{\gamma}{s} \left\{ \frac{q(v)}{v-\gamma} \frac{dv + i\pi q(v)}{1-C^2} \right\} = -\frac{\gamma}{s} H_2(\gamma)$$

and similarly for $n_1(\gamma,s)$. We note that $H_1(\gamma) = H_2^*(\gamma)$, where $H_2$ is
defined above. Thus, eq. (11) is reduced to

\[ n(x,s) = \frac{1}{2\pi i} \int_0^\infty (H_2(\gamma) - H_1(\gamma)) \frac{1}{\gamma} e^{-sx/\gamma} d\gamma. \] (13)

Now the \( s \)-dependence of the integrand is very simple and the inversion of the Laplace transform is easy. We recall that the Laplace transform of \( \delta(t-a) \) is \( e^{-sa} \), i.e.

\[ n(x,t) = \frac{1}{2\pi i} \int_0^\infty (H_2(\gamma) - H_1(\gamma)) \frac{1}{\gamma} \delta(t-x/\gamma) d\gamma, \]

or using \( \delta(t-x/\gamma) = \frac{\gamma}{t} \delta(\gamma-x/t) \) since both \( t \) and \( \gamma \) are \( \geq 0 \). Then

\[ n(x,t) = \frac{1}{2\pi i} \left[ H_2 \left( \frac{x}{t} \right) - H_1 \left( \frac{x}{t} \right) \right] \frac{1}{t}. \]

However, \( (H_2 - H_1)/2\pi i = \text{Im} \ H_2/\pi = h \left( \frac{x}{t} \right) \), so our final result is

\[ n(x,t) = \frac{1}{t} h \left( \frac{x}{t} \right), \] (14)

where

\[ h(\gamma) = \frac{1}{\pi} \text{Im} H_2 = \frac{C^2 f'_0(\gamma) \left( \frac{g(\gamma)}{\gamma} \right) d\gamma + g(\gamma) \left[ 1 - C^2 \int \frac{f'_0(\gamma)}{\gamma} d\gamma \right]}{\left[ 1 - C^2 \int \frac{f'_0(\gamma)}{\gamma} d\gamma \right]^2 + \left[ \pi C^2 f'_0(\gamma) \right]^2}. \] (15)

Although this function looks rather complicated, it is not too difficult to evaluate for the cases where \( g(\gamma) \) and \( f'_0(\gamma) \) are Maxwellians. In this case, with \( g(\gamma) = f_0(\gamma) \), we have schematically...
Note the $x/t$ dependence of $n(x,t)$; this is called "self-similarity".

If $T_e = 0 = C = 0$, $h(\gamma) = g(\gamma)$, i.e. the evolution is determined by particle free streaming, as shown in lecture III.
Reference list for lecture XIV

Lecture XV

At first sight there seems to be an inconsistency in lecture XIV: we assume quasi-neutrality, which clearly only applies to long wavelength perturbations (as compared to the Debye length), but we apply the equations to a perturbation $n(x,t=0)\sim \delta(x)$ where the Fourier transform of $\delta(x)$ contains all wavenumbers with equal weight. Note, however, that given the response to a $\delta$-function, we can construct the response to any initial condition $n(x,t=0) = F(x)$, since we can write

$$F(x) = \int F(\xi) \delta(x-\xi) d\xi,$$

i.e.

$$n_{\xi}(x,t) = \int F(\xi) n_{\delta}(x-\xi,t) d\xi , \quad (1)$$

where $n_{\delta}$ is the density response to the $\delta$-function. Consider e.g. a perturbation to the average density $n_0$ in the form of $\Delta(1-\epsilon(x))$ where $\epsilon(x)$ is Heaviside's "step function", i.e. an initial situation as

```
\begin{figure}
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Initial condition for a perturbation $\Delta(1-\epsilon(x))$.}
\end{figure}
```

\begin{itemize}
\item $t = 0$
\item $n_0$
\item $\Delta$
\end{itemize}
We recall that the Fourier transform of a step-function is proportional to 1/k, i.e. becomes small for large k (i.e. small wavelength). We need no longer worry about the assumption of quasi-neutrality. Using eq. (1) with \( F(\xi) = \Delta(1 - \theta(\xi)) \), we obtain

\[
\begin{align*}
  n(x,t) &= \Delta \int n_0(x-\xi,t) d\xi = \Delta \int \frac{1}{\xi} h\left(\frac{x-\xi}{t}\right) d\xi \\
  &= \Delta \int \frac{1}{\xi} h\left(\frac{x}{t} + \frac{x}{t}\right) d\xi - \int h(\gamma) d\gamma.
\end{align*}
\]

(2)

Since the maximum available information is in the ion velocity distribution function, we want an expression for this. We could, for instance, integrate along unperturbed orbits, i.e. solve

\[
f(x,v,t) = g(v) e(vt-x) + C^2 f_0'(v) \int_0^t \frac{\partial n(x';t')}{\partial x'} dt',
\]

where \( x' = x - v(t - t') \). This procedure is somewhat cumbersome. Instead we proceed as follows: guided by the \( x/t \)-dependence of \( n \) in eq. (2), i.e. the self-similarity, we assume that \( f \) is self-similar also, i.e. \( f(v,x/t) \). Then the Vlasov equation becomes:

\[
(v - \xi) \frac{\partial f}{\partial \xi} = C^2 f_0'(v) \frac{\partial n}{\partial \xi},
\]

(3)

where \( \xi = x/t \). Using eq. (2) we get

\[
(v - \xi) \frac{\partial f}{\partial \xi} = -C^2 f_0'(v) \Delta h(\xi)
\]

or, with the usual trick of introducing a function \( \Delta h(v - \xi) \)
Integration with respect to $\xi$ gives

$$f(\xi, v) = \Delta C^2 \xi' \left( v \right) \int_{\xi'}^v \frac{h(\xi')}{v-\xi'} \, d\xi' + \lambda \epsilon(v-\xi)$$  \hspace{1cm} (5)

since $\int \delta(v-\xi') \, d\xi' = \epsilon(v-\xi)$. Now for fixed $v, t$ and $x \to \infty$ we expect $f(\xi, v) = 0$ since the perturbations have not reached that far. This is consistent with eq. (5) for $x/t \to \infty$. Similarly, for $x \to -\infty$ we expect $f(\xi, v) = g(v)$, i.e.

$$\lambda = g(v) - \Delta C^2 \xi'(v) \int_{-\infty}^v \frac{h(\xi')}{v-\xi'} \, d\xi'$$  \hspace{1cm} (6)

inserted into eq. (5) gives the complete expression for $f(\xi, v)$. Note the singularity at $v = x/t$. This arises because the E-field is $\to$ at $t = 0$. This result is of course unphysical and can be remedied by considering an initial condition

$$f(x, v, t = 0) = g(v) \frac{1}{1 + e^{x/d}}.$$  

Note that in eq. (5) we do not make use of the condition $\int f \, dv = n$; we have already inserted the correct expression for $n$ given by eq. (2).

The singularity at $v = x/t$ need not worry us; any finite resolution of an energy analyzer will smear it out. Note that the singularity is logarithmic, i.e. integrable.
Question: Determine the expression for the ion flux: \( J_{Vf}(v) \, dv \).

Hint: For heaven's sake, do not start to integrate eqs. (5) and (6) - use eq. (3) instead!!

The following figures show theoretical and experimental results for a drifting Maxwellian; a problem relevant for a single-ended Q-machine. In the figure presented together with the experimental results, the finite resolution of the energy analyzer is taken into account.

Question: How would you check the self-similarity of a, say, density variation, experimentally in the simplest way?

The following figures refer to the experimental conditions of Ref XV.1.
The perturbed ion distribution function, \( f(I/I, v) \), as a function of \( t/x \) and with \( v \) as a parameter.
The ion velocity distribution function \( f(v_x, v_y, v_z) \) in the perturbation; \( z_0 = 6 \text{ cm} \); 10 \( \text{msec} \)/large div. (a) \( u_y = 1030 \text{ m/sec} \); (b) \( u_y = 1780 \text{ m/sec} \); (c) \( u_y = 1900 \text{ m/sec} \); (d) \( u_y = 1420 \text{ m/sec} \); (e) \( u_y = 1220 \text{ m/sec} \); (f) calculated results. The letters written on the curves correspond to experimental curves (a) through (e).
Reference list for lecture XV

Lecture XVI

Nonlinear waves

Simple waves in a gas.

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0 ,
\]

(1)

\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2)}{\partial x} = 0 ,
\]

(2)

\[p = p(\rho) \quad \text{that is: some equation of state.} \]

(3)

Define \( dp/d\rho = C^2 \) (\( \rho \) is the gas density).

Solution to linearized equations

\[
\frac{\partial \rho}{\partial t} + \frac{C^2}{\rho} \frac{\partial \rho}{\partial x} = 0 ,
\]

\[
\frac{\partial (\rho v)}{\partial t} + \rho \frac{\partial (\rho v)}{\partial x} = 0 ,
\]

i.e.

\[
\frac{\partial^2 \rho}{\partial t^2} - C^2 \frac{\partial^2 \rho}{\partial x^2} = 0 \quad \Rightarrow \quad \omega^2 = C^2 k^2
\]

\[\omega = \pm C k \quad ,
\]

or

\[
\frac{\partial^2 \rho}{\partial t^2} - C^2 \frac{\partial^2 \rho}{\partial x^2} = 0 .
\]

Note that in the linear case \( \rho = \rho_0 v/C. \) Assume, in the nonlinear case, a somewhat more general relation, namely \( \rho = \rho(v) \), where \( v = v(x,t) \). This is the assumption of simple waves.
(1) $\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{C^2}{\rho} \frac{\partial \rho}{\partial v} \frac{\partial v}{\partial x} = 0$ ,

(2) $\frac{d\rho}{dv} \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + \rho \frac{\partial v}{\partial x} = 0$ ,

using $v \frac{\partial \rho}{\partial x} = v \frac{d\rho}{dv} \frac{\partial v}{\partial x}$.

From (4) and (5): $C^2 \left( \frac{d\rho}{dv} \right)^2 = \rho^2$ ,

or

$$\frac{d\rho}{dv} = \pm \frac{\rho}{C}$$

inserted into eq.(5) gives

$$\frac{\partial v}{\partial t} + (v \pm C) \frac{\partial v}{\partial x} = 0 \ .$$

Linearize (6) $\Rightarrow \omega = \pm Ck$ as before.

Change the frame of reference using, say, $+C$ in (6) so

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0 \quad \text{or} \quad \frac{Dv}{Dt} = 0, \text{ with } \frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \ .$$

This is probably the simplest nonlinear equation we can imagine.

It is exact solution for a given initial condition, say $v(x,t=0) = F(x)$ is formally

$$v(x,t) = F(x-vt) \ ,$$

or referring to (6) considering only one of the signs ±, e.g.
outgoing waves only (+)

\[ v(x,t) = F(x - (v \pm C)t) \]  \( (9) \)

Note that for nonlinear problems, "superposition" is not possible. Equations (8) and (9) are most simply understood by a graphical construction, since, for example, eq.(7) implies that a point in a \( v-x \) diagram which at, say \( t=0 \), is located at \( (v,x_1) \) is to be found at \( (v,x_1+vt) \) at a later time \( t \).

We may make (8) or (9) look more convenient by introducing \( F^{-1}(x) \) defined so that \( F^{-1}(F(x)) = x \), i.e. (for example)

\[ x + Ct = vt + F^{-1}(v) \]  \( (10) \)

where by inserting \( v \) and \( t \) we determine the corresponding \( y \).

We may also solve eqs.(6) and (7) for a boundary value problem, i.e. let \( v(x=0,t) = G(t) \) be given, again considering in (-) or out (+)-going waves only.
\[ v(x,t) = G(t - \frac{x}{v + C}) , \quad (11) \]

or by introducing \( G^{-1} \) as \( G^{-1}(G(v)) = v \)

\[ t - \frac{x}{C} = G^{-1}(v) - \frac{v}{C} \frac{x}{C} . \quad (12) \]

**Question:** Prove eq. (12).

Figure 1 clearly demonstrated that any initial wave form will break for sufficiently large \( t \), i.e.

![Figure 2](image)

This is physically unacceptable, since \( v \) is a fluid velocity which must be single valued for all \( x \), in particular \( \partial v/\partial x \rightarrow \infty \) is unacceptable. Clearly the concept of simple waves breaks down and must be remedied by phenomena left out in the derivation, e.g. dissipation or dispersion. However, the initial evolution is quite well described.

**NOTE:** We have nowhere claimed that all linearly non-dispersive waves will ultimately break!
Lecture XVII

Burgers equation

In the previous lecture, we considered "simple waves" and derived a nonlinear equation (in one dimension)

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0 \quad .$$

(1)

This equation will give an adequate description of the initial evolution of any well-behaved initial condition, but we learned that ultimately, as $t \to \infty$, $v$ becomes multivalued, which is physically unacceptable. This "breaking" of the wave will be inhibited by terms left out in lecture XVI. One such term accounts for the viscosity of the fluid. We therefore modify eq.(1) as

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \mu \frac{\partial^2 v}{\partial x^2} \; ,$$

(2)

where $\mu$ is the coefficient of viscosity.

Linearize (2), such that: $\frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2}$ : the "heat" equation ,

Dispersion relation: $\omega = -i \mu k^2$ .

All the waves are damped; short wavelengths (large $k$) damps most rapidly. Note that $\mu < 0$ leads to instability, but we know that $\mu > 0$ in order to represent a true viscosity.

Consider a particular initial condition
The solution is

\[ v = v_2 + \frac{v_1 - v_2}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4\mu t}} e^{-\zeta^2} d\zeta. \]  

(3)

**Question:** Prove eq. (3) by insertion into \( \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2} \).

This result represents a smoothed-out step approaching values \( v_1; v_2 \) as \( x \to \pm \infty \) and with slope decreasing like \( (\mu t)^{-1} \).

Return now to the nonlinear equation (2).

The Cole (1951), Hopf (1950) transformation.

1st step: introduce \( v = \frac{\partial \psi}{\partial x} \) in eq. (2) and integrate once

\[ \frac{\partial \psi}{\partial t} + \frac{1}{2} \left( \frac{\partial \psi}{\partial x} \right)^2 = \mu \frac{\partial^2 \psi}{\partial x^2}, \]  

(4)

using

\[ \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{2} \frac{\partial}{\partial \phi} \left( \frac{\partial \phi}{\partial x} \right)^2 \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{1}{2} \frac{\partial v^2}{\partial x}. \]

2nd step: introduce \( \psi = -2\mu \ln \phi \), to obtain

\[ \frac{\partial \phi}{\partial t} = \mu \frac{\partial^2 \phi}{\partial x^2}, \]  

(5)
again the linear "heat" equation. Note that the transformation $v = \partial \psi / \partial x = -2\mu \frac{\partial^3}{\partial x^3} \ln \phi$ is nonlinear, and simply eliminates the nonlinear term in eq. (2). Equation (5) can be solved as before. Now the solution to (5) is single valued (we know). The transformation from $v$ to $\phi$ gives a one to one correspondence, so we conclude that viscosity inhibits the wave breaking. This we could expect just by inspection of eq. (2): the importance of the $\partial^2 v / \partial x^2$ term increases rapidly as the wave steepens.

Example 1. Initial condition where $v \to 0$ as $x \to \pm \infty$, e.g. $v(x, t=0) = F(x)$

\[
\int_{-\infty}^{\infty} v(x, t) \, dx = 2\pi \int F(x') \, dx'
\]

or

\[
\phi(x, t=0) = \phi(x) = e^{\frac{1}{2\mu} \int F(x') \, dx'}
\]

Integral of motion

\[
A = \int v(x, t) \, dx = 2\mu \ln \frac{\phi_{-\infty}}{\phi_{\infty}}
\]

or

\[
\phi_{\infty} = \phi_{-\infty} e^{-A/2\mu}
\]
The solution to the heat equation is

\[ \phi = \frac{1}{\sqrt{4\pi \mu t}} \int_{-\infty}^{\infty} e^{\frac{1}{2\mu} \int F(x') dx'} - \frac{(x-n)^2}{4\mu t} \, dn, \quad (6) \]

or for \( v \):

\[ v = \frac{\int \frac{x-n}{t} e^{-G/2\mu} \, dn}{\int e^{-G/2\mu} \, dn}, \quad (7) \]

with

\[ G(n; x, t) = \int_{-\infty}^{\infty} F(x') dx' + \frac{(x-n)^2}{2t}. \]

For any fixed \( x \) we have \( v \to 0 \) as \( t \to \infty \), as long as \( \mu \neq 0 \), i.e. the initial perturbation "spreads out" in order to conserve \( \int_{-\infty}^{\infty} v(x, t) dx \).

Example 2. Consider the initial condition

Assume \( v = h(x-ut) \) with \( u = \) some constant velocity. Now \( \int_{-\infty}^{\infty} v(x) dx \) is infinite.
Equation (8) has a one-parameter family of solutions

\[ h = h(x - ut) = \frac{2u}{1 + e^{\frac{u}{2}(x - ut)}} \]  

This is the shock solution

where the shock thickness is \( u/u \), i.e. with the given initial condition \( 2u = v(-\infty) \)

\[ v(x,t) \rightarrow h(x-v(-\infty)t/2) \] 

The fine details of the initial condition are smeared out by viscosity and asymptotically we obtain eq. (10). The energy dissipated by viscosity is continuously replaced by the source for \( v \) at \( x = -\infty \). Such an agency was absent in example 1.

The shock itself is a balance between the wave steepening described in lecture XVI, and the "smearing out" effect of viscosity demonstrated by the linear example, see eq. (3).
Since viscous dissipation is of minor interest in plasma physics, we shall not elaborate shock formation in detail (although it is certainly an interesting problem). The interested reader is referred to G.B. Whitham, Linear and Nonlinear Waves (Wiley, 1974), see for example chapter 4.

Problems:

1. Consider ion acoustic oscillations. Assume that the electrons are Boltzmann distributed at all times, but do not assume quasi-neutrality.
   (a) Write down the full nonlinear set of equations for the problem, in one dimension.
   (b) Linearize these equations.
   (c) Demonstrate that "normal mode" solutions exist for arbitrary $(\nu,k)$.
   (d) What is the condition for unstable solutions?

2. Consider the equation $\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0$ with the initial condition:

   $v(x,t=0)$

   (a) At what time will the wave break?
Solution:

1(a)

\[
\frac{2f}{c} + v \frac{2f}{\partial x} - e \partial \phi \frac{2f}{\partial v} = 0
\]

(1)

\[
- \frac{n_e}{n_e} \frac{\partial n_e}{\partial x} + e \frac{\partial \phi}{\partial x} = 0
\]

or

\[
n_e = n_0 e^{\phi/T_e}
\]

(2)

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} (n_e - n_i)
\]

(3)

\[
n_i = \int f(v) dv
\]

(4)

(b)

Eqs. (1)-(2) linearized, such that: \( f \rightarrow f_c + f \), \( \phi \rightarrow 0 + \phi \),

\[
n_{e,i} = n + n_{e,i}
\]

\[
\frac{2f}{c} + v \frac{2f}{\partial x} - e \partial \phi \frac{2f}{\partial v} f'(v) = 0
\]

(5)

use \( e^{\phi/T_e} = 1 + \frac{\phi}{T_e} \), such that:

\[
n_e = n_0 \frac{\phi}{T_e}
\]

(6)

Eqs. (3) and (4) are linear from the outset.

(c)

Look for solutions

\[
f \rightarrow f(v) e^{-i(\omega t - kx)}
\]

\[
n_e \rightarrow n_e e^{-i(\omega t - kx)}
\]

etc.

inserted into (3)-(6)

\[
(v - \omega/k)f - \frac{e}{\mu} \phi f'(v) = 0
\]

\[
-k^2 \phi = \frac{e}{\varepsilon_0} (n_e - n_i) = \frac{e}{\varepsilon_0} \left( n_0 \frac{\phi}{T_e} \int f dv \right)
\]
or

\[ \phi = \frac{\int f dv}{e^2 n_e / \epsilon_0 T_e} , \]

i.e.

\[ (v - \omega/k) = \frac{e}{H} \frac{\int f dv}{k^2 + e^2 n_e / \epsilon_0 T_e} f'_0(v) , \]

or with

\[ \lambda_D = \sqrt{e^2 n_e / \epsilon_0} \]

\[ (v - \omega/k) = \frac{T_e}{M} \left( \int f dv \right) \frac{f'_0(v)}{1 + (k \lambda_D)^2} - \]

Solve this equation as usual, i.e.

\[ f = \frac{\frac{T_e}{M}}{1 + (k \lambda_D)^2} \left( \int f dv \right) \frac{\int f'_0(v)}{v - \omega/k} + \lambda \delta(v - \omega/k) . \]

Normalize \( \int f dv = 1 \) and find

\[ \lambda = 1 - \frac{\frac{T_e}{M}}{1 + (k \lambda_D)^2} \int f'_0(v) , \]

valid for all \( k, \omega \), Q.E.D.

(a)

The criterion for unstable oscillations is that

\[ \int f'_0(v) \frac{1 + (k \lambda_D)^2}{v - \omega/k dv} = \frac{T_e}{M} \]

has solutions for complex \( \omega \). NOTE the difference from
electron oscillations, where the condition is

\[ \int_{v - \omega/k}^{v} \frac{f'(v)}{\omega^2/k} dv = k^2/\omega_p^2. \]

2. The wave breaks as \( t = 5 \) sec.
Lecture XVIII

In the previous lecture we demonstrated that wave breaking can be inhibited by dissipation. Alternatively, we shall demonstrate that dispersion has similar effects. We therefore add the simplest dispersive term we can imagine to XVI.(7), namely $\partial^3 v / \partial x^3$, i.e.

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \alpha \frac{\partial^3 v}{\partial x^3} = 0 ,
\]  

(1)

where $\alpha$ is a constant included for dimensional reasons. This is the Korteweg-deVries equation (or KdV equation). Linearize (1):

\[
\frac{\partial v}{\partial t} + \alpha \frac{\partial^3 v}{\partial x^3} = 0 .
\]  

(2)

Dispersion relation: $\omega = -\alpha k^3$.

Solve eq.(2) for the initial condition $v(x,t=0) = \delta(x)$. The general solution to (2) is

\[
\int_{-\infty}^{\infty} F(k) e^{-i\omega(k)t + ikx} \, dk ,
\]  

(3)

where $F(k)$ is the Fourier transform of the initial condition, in particular $F(k) = a/2\pi$ in our case.

Question: Prove eq.(3) by insertion into eq.(2).

Inserting the linear dispersion relation $\omega = -\alpha k^3$ in (3) we get
\[ v(x,t) = \frac{a}{2\pi} \int_{-\infty}^{\infty} e^{i(ak^3t - kx)} \, dk = \frac{a}{\pi} \int_{-\infty}^{\infty} \cos(kx - ak^3t) \, dk . \]  

Introducing \( s = (3at)^{1/3} \) we may express (4) as

\[ v(x,t) = \frac{a}{\pi} \frac{1}{(3at)^{1/3}} \int_{0}^{\infty} \cos\left(s(3at)^{1/3} - s^3\right) \, ds \]

\[ = \frac{a}{\pi} \frac{1}{(3at)^{1/3}} \text{Ai}\left(x/(3at)^{1/3}\right) , \]  

where we have introduced the Airy function, having the form:

\[ \text{Since the phase velocity is now different for different } k, \]  

the initial pulse will spread out (i.e. "disperse") and consequently damp since eq. (2) conserves \( \int v(x,t) \, dx \).  

Now return to the nonlinear equation (1): Just like for the Burgers equation we expect that the situation may arise where the steepening effect of the nonlinearity is balanced by the dispersion, i.e. (1) may have steady state solutions in some reference system moving with velocity \( u \). Consequently we look for solutions to (1) of the form \( v = v(x-ut) \):

\[ (v-u)\frac{dv}{dx} + \frac{d^2v}{dx^2} = 0 , \]
or using \( \frac{dv}{dx} = \frac{1}{2} \frac{d^2v}{dx^2} \) and integrating once.

\[
\frac{1}{2} v^2 - uv + \alpha \frac{d^2v}{dx^2} = 0 .
\] (7)

Multiplying with \( \frac{dv}{dx} \), using \( \frac{d}{dx} \frac{d^2v}{dx^2} = \frac{1}{2} \left( \frac{d}{dx} \frac{dv}{dx} \right)^2 \), and integrating once, we get

\[
\frac{1}{2} \left( \frac{dv}{dx} \right)^2 = \frac{u}{2a} v^2 - \frac{1}{a^6} v^3 + A ,
\] (8)

where \( A \) is an integration constant. A corresponding integration constant in eq.(7) is set to zero for simplicity; it may however be retained without difficulty.

We may interpret eq.(8) as follows: consider \( x \) as a "temporal" variable and \( v \) as a "coordinate". Then (8) simply describes a "particle" moving in a potential well given by

\[
v = \frac{1}{6a} v^3 - \frac{u}{2a} v^2 ,
\] (9)

where we take \( a > 0 \). Or
where we have a "family" of curves for varying $A$. Bounded motion of our "particle" clearly requires $v > 0$. We may alternatively make a figure like

\[ A > 0 \]

\[ A = 0 \]

\[ A < 0 \]

$A > 0$ gives periodic solutions (waves) oscillating between two values for $v$. Obviously we will disregard cases where $A < 0$. We will pay particular attention to the case where $A = 0$: This value corresponds to the KdV soliton. Note that the solitons have $v(x,t) > 0$! For $v < 0$ we have no well-behaved steady state solutions! It is intuitively clear that $v$ increases from $v = 0$ at $x = \infty$, rises to a maximum at $v = 3u$, and then returns symmetrically to $v = 0$ at $x = -\infty$. Analytically one finds, by solving eq. (8) that the soliton is given by

\[ v(x,t) = 3u \sech^2\left[ \frac{x - ut}{\Delta} \right] , \]
where $\Delta = 2\sqrt{3a/u}$. An interesting feature is that the peak amplitude "a" of the soliton is related to the velocity by $a = 3u$, and to the width of the soliton $\Delta = 2\sqrt{3a/a}$: large solitons are narrower and move faster than small ones! At first sight one may wonder whether such a peculiar object, corresponding to one particular integration constant, deserves that much attention. The KdV equation may, however, be solved exactly by the so-called "inverse scattering method" (which we shall not elaborate) and the result is that any initial perturbation where $v \to 0$ as $x \to \pm \infty$ always ends up as one or more solitons, e.g.

where the dotted line reminds you of the amplitude-velocity relation. The small "wiggles" at $x = 0$ are well described by linear theory. (N.B. Note that the KdV equation conserves $\int_\infty v(x,t)dx$).

It is important that only initial conditions containing positive $v$ values produce solitons (in general at least one),
thus a rarefactive perturbation does not give rise to solitons. In this latter case, dispersion will always dominate the non-linearity. This apparent lack of symmetry is due to the fact that the KdV equation only describes waves propagating in one direction (with the corresponding linear dispersion relation \( \omega = -a k^3 \)).

We may rewrite all the present results in the laboratory frame of reference simply by letting \( x \rightarrow x - ct \), i.e. a soliton is always "supersonic", i.e. having Mach number > 1. Do not confuse it with a shock. Recall that the initial condition considered here did not give rise to shock solutions in lecture XVII.

The previous results were derived under the assumption \( a > 0 \). It may be instructive for the reader to consider the case \( a < 0 \).

The literature concerning the KdV equation is overwhelming, and seems to increase steadily. We have no time to go into detail, but only mention phenomena like recurrence, invariants, soliton interaction, etc.

Experimental results demonstrating the latter phenomenon are shown in Montgomery, page 338. The corresponding chapter provides some valuable discussions of the phenomenon. Personally I should like to emphasize that the term "collisionless shocks" used in that section is, in my opinion, a misnomer. We have seen that classical shocks are inherently connected to viscosity, i.e. collisions, so the term appears to be contradictory. Unfortunately the nomenclature is now established in the literature.
Lecture XIX

We will apply the results of lectures XVI - XVIII to waves in plasmas. These lectures dealt with non-dispersive or weakly dispersive waves. An obvious candidate among plasma waves is therefore ion-acoustic waves (although we could name others).

Consider a fluid model for these waves:

\[
\frac{\partial n}{\partial t} + \frac{\partial n v}{\partial x} = 0 ,
\]

(1)

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{e}{M} \frac{\partial \phi}{\partial x} ,
\]

(2)

\[
n = n_0 e^{\phi/T_e} ,
\]

(3)

where we have assumed cold ions, i.e. \( T_i = 0 \) and quasi-neutrality, i.e. \( n_e \sim n_i = n \).

For linear waves \( n = n_0 v/C_s \), \( C_s = \sqrt{eT_e/M} \). In the spirit of lecture XVI, we now assume \( n = n(v) \), i.e.

(2) \(\rightarrow\)

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{T_e}{M} \frac{1}{n} \frac{\partial n}{\partial x} = 0 ,
\]

(4)

(1) \(\rightarrow\) (5)

\[
\frac{1}{n_0} \frac{\partial n}{\partial v} \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + \frac{n}{n_0} \frac{\partial v}{\partial x} = 0 ,
\]

(4) \(\rightarrow\) (5)

\[
\left( \frac{\partial n}{\partial v} \right)^2 = \frac{n^2}{C_s^2} = \frac{\partial n}{\partial v} = \frac{n}{C_s} .
\]

Equation (6) inserted into (4) gives:
\[ \frac{\partial v}{\partial t} + (v + C_s) \frac{\partial v}{\partial x} = 0 . \] \hspace{1cm} \text{Q.E.D.} \tag{7}

**Example:** Change the reference system to one moving with \( C_s \)

\[ (7) \rightarrow \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0 . \]

Consider an initial perturbation like

![Graph showing wave frequency and period](image)

Let \( n/n_0 \sim 10\% \) associated with the wave. As an estimate use the linear relation \( n/n_0 = v/C_s \sim 10\% \). Take \( C_s \sim 3 \cdot 10^5 \text{ cm/s} \) (i.e. argon and \( T_e \sim 2 \text{ eV} \)). The wave will break at \( t \sim 2 \text{ cm}/3 \cdot 10^6 \text{ cm/s} \approx 6.6 \cdot 10^{-5} \text{ sec} \).

The wave frequency is \( f = C_s/\lambda \approx 37 \cdot 10^3 \text{ Hz} \), i.e. its period is \( 2.6 \cdot 10^{-5} \text{ sec} \), so the wave breaks within 3 periods. Although it is not a fully realistic example since, for example, damping is ignored (which obviously delays the breaking), it demonstrates that in order to have linear waves we must require \( n/n_0 \ll 1 \!

We have seen that introducing dispersion in eq.(7) will inhibit the breaking of the wave. In lecture XVIII we just postulated such a term, here we shall attempt to derive the resulting KdV equation analytically. Note that the linear dispersion re-
The dispersion is there! The approximation in eq. (8) for small $k$ readily implies that a KdV equation is only an approximation valid for long wavelengths! We are (of course?) particularly interested in soliton formation. Recall that this phenomenon exhibits a balance between nonlinear steepening and dispersion. We want to model a situation where these two phenomena appear on the same level. This is the philosophy of the so-called "reductive perturbation method" outlined in the following. Consider the equations

\[
\frac{\partial n}{\partial t} + \frac{\partial n \nu}{\partial x} = 0 ,
\]

\[
\frac{\partial \nu}{\partial t} + \nu \frac{\partial \nu}{\partial x} = -\frac{e}{M} \frac{\partial \phi}{\partial x} ,
\]

\[
n_e = n_0 e^{e\theta/T_e} ,
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} (n_e - n) .
\]

Inclusion of Poisson's equation (12) is mandatory, since without it we do not have dispersion. Let us now normalize $n$ with $n_0$, $\phi$ with $e/T_e$, $\nu$ with $C_s$, while $x$ (the spatial displacement) is normalized with $L$, where $L$ is a macroscopic length characterizing the perturbation. Time is measured in units of $L/C_s$. We
thus obtain

\[
\frac{\partial n}{\partial t} + \frac{\partial n v}{\partial x} = 0 ,
\]  

(13)

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{\partial \phi}{\partial x} ,
\]  

(14)

\[
n_e = e^\phi \approx 1 + \phi + \frac{1}{2} \phi^2 + \ldots.
\]  

(15)

\[
\left( \frac{\lambda_p}{L} \right)^2 \frac{\partial^2 \phi}{\partial x^2} = (n_e - n) ,
\]  

(16)

where all the quantities are now normalized. We now expand the dependent variables as

\[
n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \ldots , \quad v = \epsilon v_1 + \epsilon^2 v_2 + \ldots , \quad \phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \ldots , \quad etc.
\]

The quantity \( \epsilon \) is a small expansion parameter, serving to identify the order of magnitude of the corresponding perturbations. Ignoring all terms with \( \epsilon^n, n \geq 2 \), we obviously recover linear results. In order to ensure that dispersion enters on the same level as nonlinearity, we assign \( (\lambda_p/L)^2 = \epsilon \). We now proceed by considering terms containing the same powers of \( \epsilon \):

Zero order : \( \epsilon^0 \) is trivially satisfied.

First order : \( \epsilon^1 \).

\[
(13) \rightarrow \frac{\partial n_1}{\partial t} + \frac{\partial v_1}{\partial x} = 0 ,
\]  

(17)

\[
(14) \rightarrow \frac{\partial v_1}{\partial t} + \frac{\partial \phi_1}{\partial x} ,
\]  

(18)
(15) \[ n_{e1} = \phi \quad , \] (19)

(16) \[ n_{e1} = n_1 \quad , \text{i.e. quasi-neutrality!} \] (20)

Solving (17)-(20), we obtain \( \partial^2 n_1 / \partial \xi^2 + \partial^2 n_1 / \partial \eta^2 = 0 \), i.e. \( \omega / k = \pm 1 \) (recall: velocities are normalized with \( C_s \!) \) and

\[ n_{e1} = n_1 = \phi_1 = v_1 \] (21)

This result, however, implies that to order \( \varepsilon^1 \) all quantities propagate without distortion (as expected). In particular, if we change the frame of reference to one moving with the velocity \( v \) (i.e. \( C_s \)) they do not change at all. Thus in this frame of reference any time variation is of higher order. We therefore introduce the new variables

\[ \xi = x - t \quad \text{and} \quad t = \varepsilon t \] (22)

in the original set of equations (13)-(16). Usually (22) is assumed from the outset with the justification given here, to be understood.

Second order : \( \varepsilon^2 \).

Using \( \partial n_1 / \partial \xi = \varepsilon \partial n_1 / \partial t - \partial n_1 / \partial \xi \), etc., we obtain from eqs. (9)-(12) the following \( \varepsilon^2 \)-order equations, in the new reference frame:
Adding (23) and (24), using (21) in order to eliminate \( n_1 \) and \( v_1 \), in favor of \( \phi_1 \) we get

\[
\frac{3\phi_1}{\partial t} + \frac{3\phi^2}{\partial \xi} = \frac{\partial n_2}{\partial \xi} - \frac{\partial \phi_2}{\partial \xi}, \tag{27}
\]

Using (25) and (26) we get

\[
\frac{\partial^3 \phi_1}{\partial \xi^3} = \frac{\partial n_2}{\partial \xi} = \frac{1}{3} \frac{\partial \phi_2}{\partial \xi} - \frac{\partial^3 \phi_1}{\partial \xi^3},
\]

or

\[
\frac{\partial n_2}{\partial \xi} = \frac{1}{3} \frac{\partial \phi_2}{\partial \xi} - \frac{\partial^3 \phi_1}{\partial \xi^3}, \tag{28}
\]

i.e. the \( n_2, \phi_2 \) quantities "drop out" most fortunately. This inserted on the right-hand side of eq. (27) gives:

\[
\frac{3\phi_1}{\partial t} + \frac{3\phi^2}{\partial \xi} + \frac{\partial^3 \phi_1}{\partial \xi^3} = \frac{\partial \phi_1}{\partial t} + \phi_1 \frac{\partial \phi_1}{\partial \xi} + \frac{\partial^3 \phi_1}{\partial \xi^3} = 0, \tag{28}
\]

which is the desired KdV equation.
By integrating eq. (28) we readily see that $\int_{-\infty}^{\infty} d\xi$ is conserved. There is, however, an infinite number of conserved quantities: see, for example, Whitham. Linear and nonlinear waves, e.g. $\int_{-\infty}^{\infty} d\xi$, etc.

As mentioned: we were fortunate that the $n_2, \phi_2$ terms could be eliminated. What if this did not happen? Well, then the problem could not be treated by a simple KdV equation!

A derivation of eq. (28), based on the ion Vlasov equation, would be more in the spirit of these lectures. Such a derivation is straightforward, provided we ignore the effect of Landau damping and the effect of ions (and electrons) reflected by the potential. The procedure is outlined below:

Order $\epsilon^1$:

$$\frac{\partial f_1}{\partial \xi} - (v-u) \frac{\partial f_1}{\partial v} - \frac{\partial f_1}{\partial \xi} f'_0(v) = 0 . \quad (29)$$

Using eqs. (19) and (20), which are of course still valid, we get

$$f_1 = \frac{n_1 f'_0(v)}{v-u}$$

or

$$\int_{v-u}^{f'_0(v)} dv - 1 = 0 , \text{ no principal value } : f'_0(v-u) = 0 \text{ is assumed,}$$

which determine the linear propagation velocity $u$ (i.e. $C_s$) in the following normalized to unity.
Second order $\epsilon^2$:

$$\frac{\partial f_1}{\partial \tau} - \frac{\partial f_2}{\partial \xi} + v \frac{\partial f_2}{\partial \xi} - \frac{\partial f_0}{\partial \xi} f'_0(v) - \frac{\partial f_1}{\partial \xi} \frac{\partial f_1}{\partial v} = 0 \quad , \quad (31)$$

completed with eqs. (25) and (26). Using (21) we reduce (31):

$$(v-1) \frac{\partial f_2}{\partial \xi} - \frac{\partial f_2}{\partial \xi} f'_0(v) = \phi_1 \frac{\partial f_0}{\partial \xi} \frac{\partial f_0}{\partial v} - \frac{\partial f_0}{\partial \xi} \frac{\partial f_0}{\partial v} - \frac{\partial f_0}{\partial \xi} \frac{\partial f_0}{\partial \tau} (v-1)^2 \quad .$$

Integrating, using eqs. (30) and (27), we get

$$\frac{1}{2} \frac{\partial \phi_1^2}{\partial \xi} - \frac{\partial \phi_1}{\partial \xi} = \phi_1 \frac{\partial \phi_1}{\partial \xi} \left[ \frac{1}{v-1} \frac{\partial f'_0(v)}{\partial v} - \frac{\partial f'_0(v)}{\partial \tau} \right] \quad ,$$

or

$$\frac{\partial \phi_1}{\partial \tau} \int_{-\infty}^{\infty} \frac{f'_0(v)}{(v-1)^2} dv + \phi_1 \frac{\partial \phi_1}{\partial \xi} \left( 1 - \int_{-\infty}^{\infty} \frac{f'_0(v)}{(v-1)^2} dv \right) = \frac{\partial \phi_1}{\partial \xi} \quad ,$$

which again have the form of a KdV equation. In particular, for $f_0(v) = \delta(v)$ it reduces to eq. (27).

Problem

Consider a strongly magnetized plasma $\omega_{pe} \ll \omega_{ce}$ (as in lecture XIII) confined between two plane parallel, conducting plates ($\parallel B$). Ignore ion motion and assume that the electrons are cold. Your basic set of equations are:
Take $k^2$ to be one given constant in the following:

(a) What is the physical meaning of $k^2$? (see lecture XIII).
(b) Derive the linear dispersion relation.
(c) Prove that the set of equations 1)-3) have stationary ("soliton-like") solutions moving with a certain velocity $u$. (Do not try to give the analytic expression for the "soliton").
(d) What is the relation between $u$ and the amplitude of the pulse?

Solution

(a)

$k^2 = \frac{\pi}{L}$ describes the standing wave component between the two plates. The wave is propagating parallel to
the plates. In the problem we assumed it to propagate in the x-direction, i.e. along B.

(b) 
\[
\frac{\partial n}{\partial t} + n \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} = \frac{e}{\mu} \frac{\partial \phi}{\partial x}, \quad \frac{\partial^2 \phi}{\partial x^2} - k_{1}^2 \phi = \frac{e}{\varepsilon_0} (n - n_0)
\]

\[
\frac{\partial}{\partial t} \rightarrow -i\omega, \quad \frac{\partial}{\partial x} \rightarrow ik \quad : \quad \omega^2 = \frac{k_s^2}{\mu} \frac{k_s^2 + k_{1}^2}{k_{1}^2 + k_{1}^2}
\]

(c) Assume \( n = n(\xi), \ v = v(\xi), \ \phi = \phi(\xi) \) with \( \xi = x - ut \):

\[
\frac{\partial n}{\partial t} \rightarrow -u \frac{\partial n}{\partial \xi}, \ etc. \quad \frac{\partial n}{\partial x} \rightarrow \frac{\partial n}{\partial \xi}, \ etc.
\]

1) \(-u \frac{\partial n}{\partial \xi} + \frac{\partial n}{\partial \xi} = 0\), \quad 2) \(-u \frac{\partial v}{\partial \xi} + \frac{\partial v^2}{\partial \xi} = \frac{e}{\mu} \frac{\partial \phi}{\partial \xi}\),

3) \(\frac{\partial^2 \phi}{\partial \xi^2} - k_{1}^2 \phi = \frac{e}{\varepsilon_0} (n - n_0)\).

Integrate eqs. 1) and 2) with respect to \( \xi \); it may help you to identify the integration constants if you define \( n = n_0 + n \), i.e. rewrite eq. 1) as

\[-u \frac{\partial n}{\partial \xi} + n \frac{\partial v}{\partial \xi} + \frac{\partial n}{\partial \xi} = 0.\]

Upon integration \( n(v - u) + n_0 v = C \), now \( v = 0 \Rightarrow n = 0 \), i.e. \( C = 0 \). From 2)

\[v^2 - 2uv - 2 \frac{e}{m} \phi = 0.\]
where again the integration constant is 0. Now

\[ v = u - \sqrt{u^2 + 2 \frac{e}{m} \phi} \]  
(we take the \(-\) sign, corresponding to the linear result)

inserted above, i.e.

\[ n = \frac{-n_0 u + n_0 \sqrt{u^2 + 2 \frac{e}{m} \phi}}{-\sqrt{u^2 + 2 \frac{e}{m} \phi}} \]

inserted in 3)

\[ \frac{d^2 \phi}{d \xi^2} = k^2 \phi + \frac{e \phi_0}{\epsilon_0} \frac{u - \sqrt{u^2 + 2 \frac{e}{m} \phi}}{\sqrt{u^2 + 2 \frac{e}{m} \phi}} , \]

or

\[ \frac{d^2 \phi}{d \xi^2} = -\frac{dV(\phi)}{d\phi} \]

with \( V(\phi) = -\frac{k_\perp^2 \phi^2}{2} - \frac{m \phi_0}{\epsilon_0} \sqrt{u^2 + 2 \frac{e}{m} \phi} + \frac{e \phi_0}{\epsilon_0} \phi \).

Now the usual trick: multiply with \( d\phi/d\xi \), use \( \frac{\partial \phi}{\partial \xi} \frac{\partial^2 \phi}{\partial \xi^2} = \frac{1}{2} \frac{\partial}{\partial \xi} (\frac{\partial \phi}{\partial \xi})^2 \)
and integrate with respect to \( \xi \):

\[ \frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 + V(\phi) = C . \]

Normalize \( \xi \) with \( k_\perp^{-1} \)

\( \phi \) with \( \frac{m \phi_0^2}{e(k_\perp)} \)

\( u \) with \( \frac{\omega_p}{k_\perp} \).
\[ \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2 - \frac{1}{2} \phi^2 - u \sqrt{u^2 + 2 \phi} + \phi = C' \]

V(\phi) has a minimum (or maximum) for

\[ 1 - \phi - u \frac{1}{\sqrt{u^2 + 2 \phi}} = 0 \]

or

\[ \phi^2 - \phi (2 - \frac{1}{2} u^2) + 1 - u^2 = 0, \quad \text{or} \quad \phi = 0 \]

Depending on C', we may have periodic solutions with wavelength \( \lambda \) given through:

\[ \frac{b}{2} \int_{a}^{b} \frac{d\phi}{\sqrt{C' + \phi^2 + u \sqrt{u^2 + 2 \phi} - \phi}} = \lambda \]
where $a$ and $b$ are defined in the figure. For $C' = -u^2$ we have "solitary solutions" where $\lambda \to \infty$.

(d) Consider the "solitary solution". For $|\phi| = \phi_{\max}$ we have $d\phi/d\xi = 0$, i.e.

$$-\frac{1}{2} \phi_{\max}^2 - u \sqrt{u^2 - 2\phi_{\max}} - \phi_{\max} = -u^2$$

or

$$u^2 - \frac{1}{2} \phi_{\max}^2 - \phi_{\max} = u \sqrt{u^2 - 2\phi_{\max}}$$

$$\left(u^2 - \frac{1}{2} \phi_{\max}^2 - \phi_{\max}\right)^2 = \left(u^2 - 2\phi_{\max}\right)u^2$$

or

$$u = 1 + \frac{1}{2} \phi_{\max}.$$

It is interesting to note that the velocity is proportional to $\phi_{\max}$ just as for KdV solitons although the object is clearly not a KdV soliton!
Problem:

Find the errors, mistakes, misprints, etc. in these lecture notes and communicate them to the author!
Problem

Communication to the author of any errors, mistakes, misprints, etc. found in these lecture notes and which could conceivably be due to the secretary (unlikely) will produce dire results/retaliation.
Title and author(s)

Lecture Notes on Plasma Physics
by
H.L. Pécsei
Physics Department, Risø National Laboratory

Date November 1983
Department or group
Physics

Group's own registration number(s)

Abstract

These lecture notes were prepared for the course 29:195 in Plasma Physics, second semester 1979-1980, at the University of Iowa, Dept. of Physics and Astronomy, Iowa City. These notes were used together with the text book 'Theory of the Unmagnetized Plasma' by D.C. Montgomery (Gordon and Breach Science Publishers, New York, 1971).

Available on request from Risø Library, Risø National Laboratory (Risø Bibliotek), Forsøgsanlæg Risø), DK-4000 Roskilde, Denmark
Telephone: (02) 37 12 12, ext. 2262. Telex: 43116