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The Lateral Dynamics of a Nonsmooth Railway Wheelset Model

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In this paper, we investigate the lateral dynamics of a railway wheelset suspended under a moving car with linear springs and dry friction dampers. Both theoretical and numerical methods are used to complement each other. The car runs on an ideal, straight and perfect track with a constant speed. A nonlinear relation between the creepages and the creep forces is used in this paper. The nonsmoothness of this model is due to the dry friction dampers. The speed is selected as the bifurcation parameter. The one-dimensional bifurcation diagram, which gives a general view of the dynamics of the system, is presented. Both symmetric and asymmetric periodic motions, quasi-periodic motions and chaotic motions are found. In addition to bifurcations that can exist in both smooth and nonsmooth systems, a kind of sliding bifurcations that are unique to nonsmooth systems is found. Bifurcation diagrams, phase portraits, Poincaré sections and Lyapunov exponents are presented to ensure that no contradictory results are given. The influence of the conicity of the wheel tread on the Hopf bifurcation type is examined.

Keywords: Railway wheelset; nonlinearity; nonsmoothness; bifurcations; chaos.

1. Introduction

Railway vehicle dynamics as an interesting topic in railway engineering has been investigated by researchers for more than a century. The fundamental railway guidance system consists of a flanged rigid wheelset and two rigid rails. When the wheelset runs above a certain critical speed it may oscillate laterally combined with a yaw motion that is known as the hunting motion of the wheelset. Based on the kinematic instability the basic analysis of the hunting motion was done by [Klingel 1883]. Since no dynamical forces were considered in his paper, he could not explain the difference between the theory and the experimental results. Carter [1914] first included the dynamical forces in the wheelset stability problem. Later, for the first time, Huilgol [1978] introduced the bifurcation analysis from nonlinear dynamics into the railway vehicle dynamical problems, which initiated a new trend of investigations among the railway engineers around the world. Possel et al. [1990] presented a best stability analysis of a two-axle railway vehicle in 1990.

\(^{\ast}\)Author for correspondence
and won a prize provided by the Office for Research and Experiments of the Union of International Railways.

With the spread and development of bifurcation theory, many dynamical railway applications are possible. True and Kaas-Petersen [1984] found that the bifurcation to periodic motion is subcritical and the critical speed must be found by an investigation of the existence of multiple attractors. They used Kaas-Petersen’s program PATH [Kaas-Petersen, 1986], for the investigations of both the stable and unstable stationary and periodic solutions. Kaas-Petersen [1986] used this program for the investigation of the dynamical motion of a railway bogie model and discovered chaos in this model. Knudsen et al. [1992] discovered both symmetric and asymmetric oscillations and chaos in a model of a rolling railway wheelset that were used to explain a lopsided wear of the wheelset. Knudsen et al. [1994] extended the investigations in another paper and discussed the different transitions to chaos. Gao et al. [2012] put forward the “resultant bifurcation diagram” method to investigate the symmetric/asymmetric periodic and chaotic motions in the symmetric railway vehicle models. They also found that symmetry breaking bifurcations and symmetry restoring bifurcations happened repeatedly in the symmetric railway vehicle models, and finally chaotic attractors appeared through a series of period doubling bifurcations or quasi-periodic motions.

The nonsmoothness of the former mentioned researches was mainly from the flanged wheels. In real life, however, the nonsmoothness often exists in the form of impact, sliding, switching and other discrete state transitions. Xia [2002] investigated the dynamics of a nonsmooth three-piece-freight truck in his PhD thesis. Because of the dry friction, sticking and sliding motions exist in this model, which leads to a discontinuity in the behavior of the dynamical system, and leads to a collapse of the state space. Hoffmann and Petersen [2003] studied the dynamics of a Hbiilss 311 freight wagon with dry friction and impact. With the development of the nonsmooth dynamical theory in recent years, a lot of research has appeared. Andronov et al. [1986] investigated nonsmooth equilibrium bifurcations, Filippov [1988] introduced C-bifurcations into the piecewise-smooth dynamical systems.
The Lateral Dynamics of a Nonsmooth Railway Wheelset Model

\[ F_R = \mu_N \left\{ \begin{array}{ll}
\frac{1}{3}u^2 + \frac{1}{27}u^3, & u < 3, \\
1, & u \geq 3,
\end{array} \right. \] (2)

with the longitudinal component \( F_x \) and the lateral component \( F_y \) as:

\[ \begin{align*}
F_x &= \xi_x F_R \\
F_y &= \xi_y F_R
\end{align*} \] (3)

where \( \xi_R = \sqrt{(\xi_x/\Phi)^2 + (\xi_y/\Phi)^2} \) is the resultant creepage, and \( u = (G + \alpha \beta_v / \mu N) = C \xi_R / \mu N \).

As for the dry friction dampers, the dry friction model from the paper [True & Asmund 2002], that is convenient for numerical realization without losing the essential characteristics, is selected. The expression for the dry friction dampers in this model is:

\[ F_a = F_d \text{sech}(\alpha \psi_a) + F_s (1 - \text{sech}(\alpha \psi_s)). \] (4)

Therefore, the mathematical model of the wheelset can be formulated as:

\[ \begin{align*}
\dot{m} \psi_w + 2F_y - 2K \psi_w - \text{sign}(\dot{\psi}_w) F_a &= 0, \\
I \ddot{\psi} + 2a \dot{F}_x &= 0.
\end{align*} \] (5)

Table 1. Wheelset parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Comment</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>Mass of the wheelset</td>
<td>1022 kg</td>
</tr>
<tr>
<td>( I )</td>
<td>Yaw moment of wheelset</td>
<td>678 kg \cdot m^2</td>
</tr>
<tr>
<td>( a )</td>
<td>Half the distance between the contact points</td>
<td>0.75 m</td>
</tr>
<tr>
<td>( C )</td>
<td>Constant related to the resultant creep force</td>
<td>6.5630 MN</td>
</tr>
<tr>
<td>( K_a )</td>
<td>Lateral spring stiffness</td>
<td>13 MN \cdot m^{-1}</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>Lateral wheel-rail contact parameter</td>
<td>0.54219</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>Longitudinal wheel-rail contact parameter</td>
<td>0.60252</td>
</tr>
<tr>
<td>( r_0 )</td>
<td>Centered wheel rolling radius</td>
<td>0.4752 m</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Coarness</td>
<td>0.15</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Coefficient of adhesion</td>
<td>0.15</td>
</tr>
<tr>
<td>( N )</td>
<td>N is the vertical force between wheel and rail</td>
<td>10 kN</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>A scaling factor</td>
<td>50 m/s</td>
</tr>
<tr>
<td>( F_L )</td>
<td>Kinetic friction force of dry friction dampers</td>
<td>1000 N</td>
</tr>
<tr>
<td>( F_a )</td>
<td>Static friction force of dry friction dampers</td>
<td>1200 N</td>
</tr>
<tr>
<td>( V )</td>
<td>Speed of the wheelset</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 1. Schematic diagram of a nonsmooth wheelset model.
We make a change of variables, $[x_1, x_2, x_3, x_4] = [y_0, y_4, v, w]$, to obtain the following four first-order autonomous piecewise-smooth differential equations:

$$
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\frac{2F_y}{m} + \frac{2K}{m}x_1 + \text{sign}(x_2)\frac{F_y}{m}, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -\frac{2aF_y}{I}.
\end{align*}
$$

(6)

### 3. The Method of Investigation

The dynamical system of the wheelset model can be viewed as an initial value problem of a set of first-order autonomous piecewise-smooth differential equations. Here we select the speed of the wheelset as the control parameter with other parameters constant. Due to the nonlinearities and the non-smoothnesses in this model, it is impossible to investigate the bifurcations and chaos in a purely analytic way. Since the system is of low dimension, we can combine the analytic method with the numerical analysis.

The pseudo-arclength continuation method and Newton’s iteration are used to follow the stationary solutions. The bifurcation of periodic orbits from the stationary solutions, that is known as a Hopf bifurcation, is located according to the method proposed by [Zon et al. 2008], and the Hopf bifurcation type is determined from the first Lyapunov coefficient [Kuznetsov 2004]. Because of the dry friction dampers in the system, a special sliding bifurcation is determined based on the theory of sliding bifurcations in Filippov systems [Di Bernardo et al. 2008].

For the numerical integrations we apply the standard ode45 function in MATLAB with a variable step size. According to a trial-and-error approach both the absolute and relative error are set to $10^{-7}$ to make a compromise between the accuracy of the solutions and the time consumption. The bifurcation diagram is constructed through an increase and decrease process of the speed based on a Poincaré map. Here the Poincaré map is defined by $\prod = \{(x, V) \in \mathbb{R}^4 \times \mathbb{R}^* \mid x_1 \geq 0, x_2 = 0\}$.

In order to investigate the quasi-periodic and chaotic behaviors of the wheelset model we continue the integration after the transients have died out. Phase portraits, Poincaré sections and Lyapunov exponents are computed to distinguish the quasi-periodic motions from the chaotic motions. We emphasize here that different methods are used so that no contradictory results are given. The Lyapunov exponents, which measure the mean convergence or divergence of the nearby trajectories in the phase space, are the most efficient indicators to determine if the motion of the system is quasi-periodic or chaotic. In this paper, we use Wolf’s method [Wolf et al. 1985] to calculate the Lyapunov exponents of the wheelset system. We choose a time step of 0.5 s for the Gram–Schmidt renormalization procedure. Refer to [Wolf et al. 1985] for the details of the numerical calculation of the Lyapunov exponents of dynamic systems from time series.

### 4. Some Results

The full bifurcation diagram in the speed range between 35 m/s and 65 m/s was computed and shown in Fig. 2 according to the numerical methods mentioned in the last section. Compared with the wheelset without dry friction dampers [Knudsen et al. 1992], a set of stationary points takes the place of the globally stable trivial solutions. The qualitative behavior of the system at $V = 35$ m/s is shown in Fig. 2 from which we can see that in the stationary state the lateral displacement is zero, but the yaw angle is different from zero. From the point of view of physics, because of the existence of the friction damper in the model, the wheelset can run without hunting motions under a low running speed.
even if its yaw angle with respect to the track center line is different from zero.

Assume that the linearized system of both sides has a pair of complex conjugate eigenvalues \( \alpha \pm (V) \pm i\omega \pm (V) \), where + and − indicate the right and the left sides respectively, and \( i = -1 \). Let

\[
B(V) = \frac{\alpha^+(V)}{\omega^+(V)} + \frac{\alpha^-(V)}{\omega^-(V)}
\]  
(7)

As the speed is increased, a subcritical Hopf bifurcation [Eva, 1992; True, 1993] will be reached at \( V = 44.4571 \text{ m/s} \) which is confirmed by Fig. 4 with \( B(V) = 0 \) and a positive first Lyapunov co-efficient \( \lambda_1(0) = 1.3738 \times 10^{-10} \). An unstable limit cycle bifurcates to the left side of the Hopf bifurcation point, and it continues until \( V = 44.4450 \text{ m/s} \) where the unstable limit cycle regains its stability through a saddle-node bifurcation. From this point and increasing the speed with a small step, a crossing-sliding bifurcation, where the two sliding segments vanish, will be reached at \( V = 44.5080 \text{ m/s} \). At this point

\[
[x_1, x_2, x_3, x_4] = [0.0026, 0.0000, 0.0509 \text{ rad/s}],
\]

which satisfies all the conditions from [di Bernardo et al., 2008] for a crossing-sliding bifurcation to happen. Orbits of the system before and after the crossing-sliding bifurcation are shown in Fig. 5. To give a better view of the transition at the crossing-sliding bifurcation point, an enlargement of the 2D phase portraits in Fig. 5 is shown in Fig. 6. From Fig. 5 we can see that the system undergoes a periodic motion in a large speed range. A variety of bifurcation phenomena happen in the speed range 56–65 m/s. In the following, the bifurcation analysis in four important speed ranges is elaborated, and transitions between different bifurcations are discussed.

4.1. Speed range 56.2–57 m/s

A blow-up of the bifurcation diagram in the speed range 56.2–57 m/s is shown in Fig. 7. The transitions in this speed range are of interest because of the sequences of bifurcations in this narrow speed interval, that finally lead to asymmetrical periodic orbits through a crisis. The wheelset undergoes quasi-periodic and periodic motions alternatively during this speed range. The first bifurcation happens around \( V = 56.2380 \text{ m/s} \) where the stable limit cycle loses its stability, and an invariant two-dimensional torus appears through a Neimark–Sacker bifurcation. This is illustrated in Fig. 8 where the Poincaré maps were plotted under two different speeds before and after the Neimark–Sacker bifurcation respectively. The Poincaré map is constructed after the transients have died out. It is seen that a point in the Poincaré map converts into a closed curve, which indicates the existence of a biperiodic oscillation.

From Fig. 7 we can see there are six periodic windows during this speed range. Here we only choose the first periodic window to give a detailed analysis of the transitions near this periodic window. A refined bifurcation diagram near this periodic window is shown in Fig. 8. The most noticeable phenomenon is the period-20 window around \( V = 56.2843 \text{ m/s} \). The Poincaré map under
Fig. 5. Orbits before and after the crossing-sliding bifurcation.

Fig. 6. Enlargement of 2D phase portraits in Fig. 5.
Fig. 7. A blow-up bifurcation diagram in the range 56.2–57 m/s.

Fig. 8. Poincaré maps near the Neimark–Sacker bifurcation point.

As the speed increases further, the system enters into chaos through a torus breakdown in the speed range 56.80–56.90 m/s. Comparing the Poincaré maps in Fig. 12, it can be seen that foldings are displayed at the speed of \( V = 56.90 \) m/s, which are typical features of chaos. The largest two Lyapunov exponents of the system shown in Fig. 12 indicate that chaos occurs at the speed of...
Fig. 11. Poincaré maps on both sides of the periodic window.

$a) v = 56.2841 \text{ m/s}$
$b) v = 56.2845 \text{ m/s}$

Fig. 12. (a) and (b) Poincaré maps and (c) and (d) the two largest Lyapunov exponents.

$c) v = 56.80 \text{ m/s}$
$d) v = 56.90 \text{ m/s}$
V = 56.90 m/s. At around V = 56.9665 m/s two asymmetric periodic solutions bifurcate from the chaos through a crisis which is confirmed by the phase portraits of the model near the crisis shown in Fig. 13 where the initial conditions for Figs. (b) and (c) are

\[ [x_1, x_2, x_3, x_4] = [0.0179 \text{ m}, -0.5681 \text{ m/s}, -0.0059 \text{ rad}, -0.5144 \text{ rad/s}] \]

and

\[ [x_1, x_2, x_3, x_4] = [-0.0179 \text{ m}, 0.5681 \text{ m/s}, 0.0059 \text{ rad}, 0.5144 \text{ rad/s}] \]

respectively. It can be seen that Fig. (c) is a reflection of Fig. (b) around the axis ψ (rad) followed by a reflection around the axis \( \dot{\psi} \) (m/s).

4.2. Speed range 59.5–61.5 m/s

From Fig. we can see that the asymmetrical periodic solutions exist in a broad speed range. In this section, we give a detailed bifurcation analysis in the speed range 59.5–61.5 m/s. A blow-up bifurcation diagram in this speed range is shown in Fig. The transitions in this speed range are simple, where the asymmetrical periodic solutions go through period doublings. Therefore, four asymmetrical periodic solutions are created. The asymmetrical periodic solutions undergo a complete period doubling cascade to chaos, which will be explained in the next section. A variety of the characteristic periodic windows are shown in Fig. It can be seen that the two period doubling cascades cross with the others four times (here we only show the first one at around V = 60.77 m/s. Except for the left-top dot the other three dots consist of two points each.

4.3. Speed range 61.6–62.2 m/s

With the increase of the speed the system enters into chaos through a period doubling cascade. A refined bifurcation diagram in speed range 61.6–62.2 m/s is shown in Fig. constructed with an
increase of the speed. To give a clear description of the transitions we divide it into four regions as shown in Fig. 14. As the transitions in these four regions are similar, we only give a detailed description of region 1. Four-band chaos develops after the period doubling cascade. Two pairs of asymmetric chaotic attractors merge into two asymmetric chaotic attractors with the increase of the speed followed by another asymmetry breaking of the chaotic attractors, where only one chaotic attractor exists. When the speed reaches $V = 61.9684 \text{ m/s}$ a period-3 solution bifurcates from the chaotic attractor through a crisis followed by a period doubling cascade again, which leads the system into a broad-band chaos. Another crisis happens around $V = 62.1473 \text{ m/s}$ where the system jumps from the chaotic attractor to a limit cycle.

Fig. 14. A blow-up bifurcation diagram in the range 59.5–61.5 m/s.

Fig. 15. Poincaré maps.
From Fig. 16 it is seen that this system possesses hysteresis phenomenon in this speed range. We construct the bifurcation diagram again by decreasing the speed, which is shown in Fig. 18. It can be seen that another jump happens at around $V = 61.878$ m/s with the decrease of the speed, where a period-4 attractor jumps to the four-band chaotic attractor that can be recognized in Fig. 18.

### 4.4. Speed range 62.5–65 m/s

A blow-up bifurcation diagram in this speed range is shown in Fig. 19, which shows that the system alternates between periodic windows and chaotic attractors. At around $V = 62.7125$ m/s, the stable period-4 limit cycle loses stability through a crisis to two asymmetric chaotic attractors, which is
Fig. 18. Bifurcation diagram in the range 61.6–62.2 m/s by decreasing the speed.

Fig. 19. A blow-up bifurcation diagram in the range 62.5–65 m/s.

Fig. 20. (a) and (b) Poincaré maps and (c) and (d) the two largest Lyapunov exponents of the system near the first crisis.
confirmed by the Poincaré maps and the largest two Lyapunov exponents displayed in Fig. 20. The four points in Fig. 20(a) indicate a period-4 solution, and the stripes in Fig. 20(b) indicate chaos. These are confirmed by the two largest Lyapunov exponents of the system shown in Figs. 20(c) and 20(d), where the largest Lyapunov exponent changes from zero to a positive value. When the speed increases to \( V = 62.8212 \text{ m/s} \) the two asymmetric chaotic attractors lose stability through another crisis to a period-8 attractor, which is illustrated in Fig. 21 by the Poincaré maps similar to Fig. 20 but in a reserve transition. After a small range of periodic windows the asymmetric chaotic attractors regain stability. At still higher speed around \( V = 63.0100 \text{ m/s} \) the asymmetric chaotic attractors merge into a chaotic attractor. With the increase of the speed the chaotic attractor loses stability into a period 13 attractor. After several times of alternating between periodic windows and symmetric chaos, the system enters into asymmetric chaotic motions followed by a reverse period doubling cascade into two asymmetric limit cycles, which can be seen from the phase portrait of the system at \( V = 65 \text{ m/s} \) shown in Fig. 22 under different initial values \([x_1, x_2, x_3, x_4] = [0.0167 \text{ m}, 0.6299 \text{ m/s}, 0.0112 \text{ rad}, -0.2821 \text{ rad/s}]\) and \([x_1, x_2, x_3, x_4] = [-0.0167 \text{ m}, -0.6299 \text{ m/s}, -0.0112 \text{ rad}, 0.2821 \text{ rad/s}]\) respectively.

5. Investigation of the Influence of the Conicity

In this section, we investigate the influence of the conicity of the wheels on both the Hopf bifurcation point and the first Lyapunov coefficient, which is used to determine the bifurcation form of the system. From Fig. 23 we can see that the speed...
corresponding to the Hopf bifurcation point, which is the linear critical speed of the wheelset system, decreases with the increase of the conicity. The value of the conicity has a significant influence on the Hopf bifurcation point when it is smaller than 0.1. Because of the wear of the wheels in the running process, the conicity of the wheels will change. Therefore, it is of interest to study the influence of the conicity on the bifurcation forms of the system. To have a clear knowledge of the influence of the conicity on the Hopf bifurcation forms of the system, the first Lyapunov coefficient of the system under different values of the conicity is calculated.

From Fig. 24 we can see that when the conicity increases to 0.3420 the system changes from a subcritical Hopf bifurcation into a supercritical Hopf bifurcation. When the conicity reaches 0.4180, the first Lyapunov coefficient of the system changes sign from negative to positive, which means the system changes from a supercritical Hopf bifurcation into a subcritical Hopf bifurcation. It can be concluded that a subcritical Hopf bifurcation is common in the railway vehicle system in a wide range of the conicity.

6. Conclusion
In this paper, we investigate the lateral dynamics of a nonsmooth railway wheelset model, that consists of two degrees of freedom with linear characteristic springs and dry friction dampers. The nonlinear forces between the wheel and the rail are calculated by combining the linear kinematic relation between the wheel and the rail with nonlinear creepage-creep force relation [Vermeulen & Johnson [1964]]. This simple model has rich dynamical features. A variety of possible motions such as stable stationary motions, periodic motions, quasi-periodic motions and chaotic attractors are illustrated in this paper.

Unlike wheelset models with smooth dampers, the wheelset model with dry friction dampers has a set of stationary points under a low running speed. With the increase of the speed the stationary points lose stability through a subcritical Hopf bifurcation, where an unstable limit cycle bifurcates to the left side. The unstable limit cycle regains its stability through a fold bifurcation where a stable limit cycle bifurcates to the right side. The crossing-sliding bifurcation, which is special for nonsmooth dynamical systems, happens at \( V = 44.5080 \) m/s, where two sliding segments disappear.

At higher speed many complicated dynamical motions can happen. A Neimark-Sacker bifurcation occurs at \( V = 56.2380 \) m/s, where a two-dimensional torus develops. After several transitions between periodic and aperiodic motions, the two-dimensional torus loses its stability through a torus breakdown into a chaos followed by a crisis into two asymmetric periodic oscillations. Through a period doubling cascade the system enters into chaos again followed by another crisis around \( V = 62.1474 \) m/s, where the system enters into a period-4 oscillation. With a further increase of the speed, the motion of the system transits between chaos and periodic windows alternatively until a reverse period doubling cascade leads the system into two asymmetric limit cycles.

Since the conicity of the wheel, which will change with the wear of the wheel, is an important parameter for the lateral dynamics of the wheelset system, it is desirable to study the influence of this parameter value on the Hopf bifurcation point and
the bifurcation forms of the system. From the calculated results it can be seen that the conicity has a significant influence on the Hopf bifurcation point when it is lower than 0.1.

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References


