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Abstract

We prove the conjecture made by G.Wegner in 1977 that the square of every planar, cubic graph is 7-colorable. Here, 7 cannot be replaced by 6.

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1   Introduction

We prove the conjecture made by G.Wegner [11] in 1977, mentioned by Gionfriddo [5] and listed in the monograph by Jensen and Toft [6], that the square of every planar, cubic graph is 7-colorable. To see that this bound is best possible, consider first the cubic prism graph with six vertices. Then subdivide an edge which is not contained in a triangle. The square of this graph is a complete graph with seven vertices. Now we take two copies of this graph and add an edge between them so that we obtain a cubic graph. This cubic graph is planar and its square has chromatic number 7.

The proof is based on a decomposition method: We color the vertices of the planar, cubic graph by two colors, red and blue, such that the blue

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square-graph is 3-colorable, and the red square-graph is planar and hence 4-colorable, by the 4-color theorem.

Wegner's 7-color conjecture proved in the present paper is part of a more general problem on the chromatic number of squares of planar graphs. After submission of the present paper a number of papers have been written on this subject, see e.g. [2], [3], [4], [7], [10] and the references in these papers. A computer aided proof of the 7-color theorem has recently been obtained in [1].

2   Terminology and notation

The terminology is the same as in [6] and [9].

A *k-path* is a path with k vertices. A *k-cycle* is defined analogously.

In a plane embedding of a connected graph every face boundary is a walk called a *facial walk*. A *facial path* is a path which is a subgraph of a facial walk. If C is a cycle in a plane graph, then *the interior of C*, denoted $int(C)$, consists of the edges and vertices inside C. Thus, an edge joining a vertex in C with a vertex inside C is also in $int(C)$. Sometimes $int(C)$ also refers to a graph, namely the subgraph of G induced by the vertices inside C. The precise meaning will always be clear from the context.

If G is a graph, then the *square G^2* of G is obtained from G by adding all edges joining vertices of distance 2 in G. If we color the vertices of G red or blue, then the *red subgraph* (or just the *red graph*) is the subgraph of G induced by the red vertices. The *red square-subgraph* (or just the *red square-graph*) is the subgraph of G^2 induced by the red vertices. Similar notation is used for the blue vertices.

If some vertices of G are colored $1, 2, 3$ such that the coloring is proper in G^2, then we say that *vertex v can see color i* if there is a vertex u of color i such that u is a neighbor of v in G^2. A *Kempe chain with colors i, j* is a connected component in the subgraph of G^2 induced by the vertices of colors i, j.

We shall also use the following notation: If we have already named a sequence $v_1, v_2, \ldots$ of vertices in the cubic graph and that sequence includes say two neighbors of v_1, then the neighbor of v_1 which is not in the list is called *the third neighbor of v_1*. If precisely one neighbor of v_2 is in the list, then the two neighbors of v_2 not in the list are called *the two other neighbors of v_2*.

3 Decomposing the vertex set of a cubic graph

We shall now indicate the idea in the proof of Wegner's conjecture. We begin with a conjecture.

Conjecture 1 *If G is a 3-connected, cubic graph, then the vertices of G can be colored blue and red such that the blue subgraph has maximum degree 1 (that is, it consists of a matching and some isolated vertices) and the red subgraph has minimum degree at least 1 and contains no 4-path.*

Suppose Conjecture 1 is true. Assume now that G is planar cubic and 3-connected, and consider the blue vertices. Using the fact that the red subgraph has no isolated vertex, it follows easily that the blue square-graph has maximum degree at most 3. As G is 3-connected it is easy to see that the blue square-graph contains no complete graph with 4 vertices (unless G has a very special structure that makes it easy to complete the proof by induction). Hence the blue square-graph is 3-colorable, by Brooks' theorem. Consider next the red subgraph. Add all edges in the red square-subgraph. Every new edge can be added such that it does not cross any edge of G. Two new edges may cross, though. But, as there is no red facial 4-path, no two additional edges cross. So the red square-subgraph is planar and hence 4-colorable, by the 4-Color Theorem. This implies that G^2 is 7-colorable.

The method of this paper is to prove a technical and less elegant version of Conjecture 1 which is strong enough, though, to prove Wegner's conjecture.

4 Decomposing the vertex set of a planar cubic graph

We shall consider a graph where some vertices are colored red or blue, and some vertices are uncolored. A *forbidden cycle* is a cycle C' such that either the length of C' is not a multiple of 3 and all the vertices of C' are all blue, or the length of C' is congruent to 2 modulo 3 and all but precisely one of the vertices are blue. Note that the blue vertices cannot be properly colored in three colors in the square of a forbidden cycle. We say that a cycle is a *dangerous cycle* if and only if it has at least one non-blue vertex, and if we change the color of any vertex from non-blue to blue, then we obtain a forbidden cycle.

It is easy to see that a cycle C''' is dangerous if and only if C''' has length congruent to 1 modulo 3 such that each vertex of C''', except precisely one, is blue, or C''' has length congruent to 2 modulo 3 such that each vertex of C''', except precisely two, is blue.

Theorem 1 *Let G be a connected planar graph with chordless outer cycle C. Assume that the following hold:*

(c_1): If v is a vertex in $int(C)$ and E is a set of at most two edges in $int(C)$, then $G - E$ has a path from v to C.

(c_2): All vertices of G have degree at most 3, and all vertices in $int(C)$ have degree precisely 3.

(c_3): Each vertex of C is precolored by red or blue such that at most one vertex b_0 of C is blue.

(c_4): If b_0 exists, then at least one of its neighbors on C has degree 2 in G.

(c_5): If b_0 does not exist, that is, all vertices of C are red, then some vertex r_0 on C is called either left-forbidden or right-forbidden or 4-forbidden.

(c_6): $G - V(C)$ is connected and contains a vertex joined to b_0 (if b_0 exists) or r_0 (otherwise).

(c_7): Every vertex v in $int(C)$ which has a red neighbor on C distinct from r_0 is colored blue. If r_0 exists and is either right-forbidden or left-forbidden and its neighbor r_0' in $int(C)$ has two neighbors in $int(C)$, then one of these neighbors is precolored blue as follows: If r_0 is right-forbidden (respectively left-forbidden) and the path $r_0 r_0' a$ turns sharp right (respectively left) at r_0', then a is precolored blue. No other vertex in $int(C)$ is precolored. (To clarify, we emphasize some special consequences of this: If the neighbor r_0' of r_0 in $int(C)$ has two neighbors on C, then r_0' is precolored blue, and the third neighbor of r_0' is not precolored unless that neighbor is joined to $C - r_0$. If r_0 is 4-forbidden and r_0' has two neighbors in $int(C)$, then r_0' is not precolored. Also, a neighbor of r_0' in $int(C)$ is not precolored unless that neighbor of r_0' has a neighbor on $C - r_0$.)

(c_8): There is no forbidden cycle, and there is no dangerous cycle in G, except possibly a dangerous cycle that contains b_0 and precisely one other vertex of C.

(c_9): If r_0 exists and is right-forbidden or left-forbidden and r_0' is the neighbor of r_0 in $int(C)$ and r_0' has no other neighbor in C, then G has a facial cycle which contains r_0' and is disjoint from C.

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Then the red-blue coloring (of $V(C)$ and the blue vertices inside C) can be extended to a red-blue coloring of $V(G)$ such that the following conditions hold:

(i): G has no red facial 4-path whose edges are in $int(C)$. (Note that, if r_0 exists and is left-forbidden, then G has no red facial 3-path starting at r_0 and with edges in $int(C)$ and turning sharp left at the neighbor of r_0. In other words, we allow a red facial 3-path starting at r_0 and with edges in $int(C)$ provided it turns sharp right at the neighbor of r_0. Similarly, if r_0 exists and is right-forbidden, then G has no red facial 3-path starting at r_0 and with edges in $int(C)$ and turning sharp right at the neighbor of r_0. If r_0 is neither right-forbidden nor left-forbidden, then the only condition on r_0 is that there is no red facial 4-path starting at r_0, and therefore we call it 4-forbidden.)

(ii): the blue vertices of G can be 3-colored with colors $1, 2, 3$ such that this coloring is proper in G^2.

(iii): If r_0 is right-forbidden or left-forbidden, and the neighbor r_0' of r_0 in $int(C)$ has no other neighbor on C, then the red-blue coloring can be chosen such that r_0' is red.

Note that $(c_1), (c_2), (c_6)$ imply that G is obtained from a planar, cubic, 3-connected graph by subdividing edges on the outer cycle. The condition (c_1) is included only to reduce the amount of case analysis. It is a triviality to check (c_1) in the induction steps. We may assume that each edge in the outer cycle, except possibly one, is subdivided many times, since subdividing edges of C (except possibly one) does not affect the conditions nor conclusion of Theorem 1. This is useful when we wish to add an edge from $int(C)$ to C. If b_0 exists, then, by (c_4), b_0 may be joined to precisely one vertex d on C which has degree 3 in G. We are not allowed to subdivide the edge b_0d. The reason for the condition (c_9) is that we keep $int(C)$ connected if we delete r_0'. The reason for (iii) is that it is convenient when we color the blue vertices by colors $1, 2, 3$ in the induction step.

In the induction step we sometimes introduce a new blue vertex. This will never create a forbidden cycle, but it may create a new dangerous cycle which we then have to dispose of. In case we make more than one vertex blue we make sure that the other new blue vertices are not part of a dangerous cycle. This is done by letting the other new blue vertices have two neighbors on the outer cycle.

Proof of Theorem 1

The proof is by induction on the number of edges inside C. Suppose (reductio ad absurdum) that Theorem 1 is false. Select a counterexample such that the number of edges in $int(C)$ is minimum.

We may assume the following:

Claim 1: $int(C)$ does not contain a path $v_1 v_2 v_3 v_4 v_5$ such that each of v_1, v_2, v_3, v_4, v_5 is joined to $C - r_0$ (if r_0 exists) or $C - b_0$ (if b_0 exists).

For, if such a path exists, then we delete the vertices v_2, v_3, v_4 add the edge $v_1 v_5$ and use induction. If v_1, v_5 get blue colors $1, 2$, respectively, then v_1, v_2, v_3, v_4, v_5 can be colored $1, 2, 3, 1, 2$, respectively. This contradiction proves Claim 1.

Case 1: r_0 exists and is either right-forbidden or left-forbidden.
Assume without loss of generality that r_0 is right-forbidden. Let r_0' be the neighbor of r_0 in $int(C)$.

Subcase 1.1: r_0' has at least two neighbors on C.

If r_0' has three neighbors on C, then (c_6) implies that $int(C)$ consists of r_0' only and there is nothing to prove. So assume that r_0' has a neighbor a in $int(C)$. We define a new graph G' by deleting r_0' and adding the edge $a r_0$. We now apply the induction hypothesis to G' where r_0 plays the role of b_0. We subdivide some edges of C incident with r_0 so that there is no dangerous cycle in G' and such that (c_4) is satisfied. After the induction we transfer the blue color of r_0 to r_0', and we give r_0 its red color back.

Subcase 1.2: r_0' has only one neighbor r_0 on C.

Let a, b be the neighbors of r_0' in $int(C)$ (and thus distinct from r_0) such that b is blue, that is, the path $r_0 r_0' b$ turns sharp right at r_0'. (In other words, the facial path $r_0 r_0' b$ is part of a facial cycle traversed clockwise.) Let c, d be the neighbors of r_0 on C such that the path $c r_0 d$ is anticlockwise around C. Hence the paths $c r_0 r_0' a$ and $d r_0 r_0' b$ are facial. Now we define a new graph G' with a new outer cycle C' as follows: We delete r_0, r_0', we add two new vertices x, y, we add a path $cxyd$, and we add the edges xa, yb. Let u, v be the neighbors of a distinct from x such that $cxau$ is a facial path.

Subcase 1.2.1: u is in C.

Let G'' be obtained from G' by deleting a and adding the edge xv. We

apply induction to G'' where x plays the role of b_0. (We explain below why induction is possible.) After the induction we transfer the blue color of x (which is distinct from the blue color of b) to a, and we let r'_0 be red. Now we argue why we can apply induction to G''. The only problem is that there may be a forbidden or dangerous cycle C''' in G''. As G has no forbidden or dangerous cycle, C''' must contain x or y or both. As we may subdivide the edges xc, yd, we may assume that C''' contains the path $vxyb$. The corresponding cycle in G contains the path $var'_0 b$ which has the same length and same number of blue vertices as C'''. Hence that cycle in G is forbidden or dangerous, a contradiction to (c_8).

So, we may assume that

Subcase 1.2.2: u **is not in** C.

Note that also v is not in C because of (c_9). Possibly, v, b are neighbors. Possibly $v = b$. By (c_1), $u \neq b$, and u, b are nonneighbors.

We now try to apply induction to G' where x plays the role of r_0 and is left-forbidden. If G' satisfies $(c_1) - (c_9)$, then we apply induction. In that case a will be red by (iii), and then we have completed Case 1 because we can use the coloring of G' (which satisfies (i),(ii),(iii)) for G.

It is a simple matter to show that G' satisfies $(c_1) - (c_7)$. We now consider the cases where (c_9) or (c_8) fail for G'. These are the cases 1.2.2.1, 1.2.2.2, respectively, below.

Subcase 1.2.2.1: **The facial cycle of** G' **which contains** a **but not** x **intersects the outer cycle** C'.

In this case a is a cutvertex in $int(C') = G' - V(C')$. Let H_1, H_2 be the two components of $int(C') - a$ such that H_1 contains u and hence H_2 contains b, v. We now apply induction to H_1, H_2 separately. First we draw H_i inside a cycle C_i for $i = 1, 2$. The vertices in H_i which have a neighbor on C' each has a neighbor on C_i. Moreover u is joined to a vertex a_1 in C_1 and v is joined to a vertex a_2 in C_2. We may draw H_2, C_2 such that the neighbor of b in C_2 is a neighbor of a_2. Now we use induction such that a_i plays the role of b_0 for $i = 1, 2$. (It is easy to see that induction is possible. The only problem is a possible forbidden or dangerous cycle. There is no forbidden cycle as that would be dangerous in G. There may indeed be a dangerous cycle containing a path of length 2 from b to a_2. Such a dangerous cycle is allowed.) We may assume that a_1, a_2 have the same blue color, and we give

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that color to a in G. All other colors are transferred to G in the obvious way. Note that a, b get distinct blue colors. It is possible that u, v have the same blue color. In that case we interchange the two blue colors (distinct from the color of a_1) in H_1 so that u, v get distinct blue colors.

As noted above, G' satisfies $(c_1)-(c_7)$. As have disposed of Subcase 1.2.2.1 we may assume that G' also satisfies (c_9). Now we consider the subcase where G' does not satisfy (c_8).

Subcase 1.2.2.2**: The facial cycle of G' which contains a but not x does not intersect the outer cycle C'.**

As G' does not satisfy (c_8), G' contains a dangerous cycle C_d. Clearly, C_d contains u, since G has no dangerous cycle.

Subcase 1.2.2.2.1**: C_d contains a.**

As C_d becomes dangerous when we make u blue it follows that u is not joined to C' (since otherwise, C_d would be dangerous in G.) Also, v is not joined to C' (unless $v = b$) because each of G and G' satisfies (c_9). It follows that all vertices of C_d except v, u are joined to C' (unless $v = b$). We may assume that u, v are not neighbors since otherwise, C_d has length 2 modulo 3 and hence, by Claim 1, C_d is of the form $uavbz_1u$ or $uavbz_1z_2z_3z_4u$, where z_1, z_2, z_3, z_4 are joined to C'. In the former case the edge from z_1 to C and the edge r_0r_0' violate (c_1). Therefore $C_d = uavbz_1z_2z_3z_4u$. Hence $int(C')$ consists of C_d and the edge uv. In this case r_0' is the only vertex in $int(C)$ not in C_d. It is easy to complete the proof with r_0', a, v being the only red vertices in $int(C)$. Finally, the neighbors of u, v not in C_d are outside C_d because of (c_1).

Subcase 1.2.2.2.1.1**: $v \neq b$.**

Because u, v are the only vertices of C_d without neighbors in C', C_d is a facial cycle. Also, u, v are nonconsecutive on C_d. As $v \neq b$, $int(C') - V(C_d)$ has precisely two components H_1, H_2 where H_2, say, contains b and a neighbor of v, and H_1 contains a neighbor of u. The components H_1, H_2 are both outside C_d because C_d is facial. We now try to apply induction to H_1, H_2 separately. First we draw H_i inside a cycle C_i for $i = 1, 2$. Each vertex of H_i has as many neighbors on C_i as it has on C'. Moreover the neighbor of u in H_1 is joined to a vertex (which we also call u) in C_1, and the neighbor of v

in H_2 is joined to a vertex (which we also call v) in C_2. We let each of u, v play the role of r_0, and we call each of u, v 4-forbidden. After the induction we make a blue and r_0' red. We then color the blue vertices of C_d by the colors $1, 2, 3$ starting with a (which gets a color distinct from those of b and the third neighbor of v) and then the other neighbor of v finishing with the other neighbor u' of u in C_d. When we make this blue coloring we ignore the color of the third neighbor u'' of u. If u'' is blue we permute the blue colors in H_1 so that u'' gets a blue color distinct from those of a, u'.

Subcase 1.2.2.2.1.2: $v = b$.

If $int(C') - V(C_d)$ has precisely two components H_1, H_2 where H_1 contains a neighbor of u, then we repeat the argument in Subcase 1.2.2.2.1.1 except that now u plays the role of b_0. The vertex of C_d joined to H_2 again plays the role of r_0 and is 4-forbidden. After we have applied the induction hypothesis we color the blue vertices of C_d with colors $1, 2, 3$ so that there is no blue color conflict with H_1. The vertex in H_2 joined to C_d may be blue and it may have a color conflict with a blue vertex in C_d. In that case we permute the blue colors in H_2 so that this conflict disappears.

We now consider the case where $int(C') - V(C_d)$ has precisely one component H_1 containing a neighbor of u. Consider the case where $C_d : uavw_1w_2u$ where u has a neighbor u' in H_1, and w_2 has a neighbor w_2' in H_1, and w_1 is joined to C. (It is also possible that C_d has length 4 in which case w_2 does not exist and u is the only vertex of C_d joined to H_1 because a dangerous cycle with 4 vertices has only one vertex that is not precolored blue. That case is similar and easier. It is also possible that C_d has length 7 in which case u is the only vertex of C_d joined to H_1. Again, that case is similar and easier. Note that C_d cannot have length 6 by the definition of a dangerous cycle, and C_d cannot have length at least 8 because of Claim 1.) Now we apply induction to H_1, u, and an outer cycle C'''. Each vertex of H_1 has as many neighbors on C''' as it has on C. The vertex u is joined to two vertices of C''' which implies that u is precolored blue, and C''' can be suitably subdivided so that u is not contained in any dangerous cycle. w_2' is joined to a vertex which is also called w_2. This vertex w_2 plays the role of r_0 and is 4-forbidden. After the induction r_0', a are made red, and we can give w_1, v blue colors. This coloring satisfies the conclusion of the theorem except that w_2', u may have the same blue color. In that case we first try to interchange colors of u, a so that the color conflict between w_2', u no longer exists. The

problem with this color change is that we may create a red facial 4-path containing u. Then we let u keep its blue color, but we recolor u, w_1, v in that order so that we obtain a proper 3-coloring in the blue square-graph.

Subcase 1.2.2.2.2: C_d **does not contain** a.

Let $C_d = uz_1z_2 \ldots z_qu$ traversed anticlockwise. The assumption of Subcase 1.2.2.2 implies that z_1 and its third neighbor z_1' are in $int(C)$. All vertices of C_d except u, z_1 and possibly one more, are joined to C' because of the assumption on C_d.

Subcase 1.2.2.2.2.1: u, v **are neighbors.**

In this subcase C_d contains the edge uv. If C_d does not contain b, then, because the other neighbor of v in C_d is in the facial cycle through r_0' guaranteed by (c_9), $C_d : uvz_2z_3z_4u$ where z_3, z_4 are joined to C. The subgraph of G induced by C, C_d, a, r_0' has precisely one face containing vertices of G, by (c_9). We apply induction to that face with z_2 being 4-forbidden. After the induction, we change the colors of a, z_3 to blue, we color z_4 blue as well, and we color u red. If the third neighbor of z_2 is red, then also v is changed to blue. Otherwise it stays red. Now it is easy to color the blue vertices with the three blue colors. If C_d contains b, then either $C_d : uvbz_3z_4z_5z_6u$ where z_3, z_4, z_5, z_6 are joined to C, and r_0', a are the only vertices in $int(C)$ but not in C_d, a configuration which is easy to dispose of, or else $C_d : uvbz_3z_4z_5z_6z_7u$ where four of z_3, z_4, z_5, z_6, z_7 are joined to C. (Again, by (c_1) and Claim 1, there must be either four or five $z_i's$. Also note that b must be the immediate successor of uv on C_d since otherwise we would get a contradiction to (c_1).) Then the subgraph of G induced by C_d, a, r_0' has a unique face with vertices of G and the interior of that face has a vertex joined to some z_i, $3 \leq i \leq 7$. We apply induction to that face with z_i playing the role of r_0 being 4-forbidden.

Subcase 1.2.2.2.2.2: u, v **are not neighbors, but** C_d **contains** b.

Then C_d is of the form $uz_1z_2z_3z_4z_5z_6z_7u$ where $z_3 = b$, and z_4, z_5, z_6, z_7 are joined to C. (It is not possible that b equals z_1 or z_2 because of the connectivity condition (c_1). It is not possible that b equals z_j with $j > 3$ because C_d is dangerous. Note that C_d cannot have length 4 or 5 or 7 because of (c_1). Also note that the 5 consecutive blue vertices on C_d joined to C' does not contradict Claim 1 because only 4 of them are joined to C.)

Now we apply induction to the cycle (in G) $uar'_0bz_2z_1u$ and its interior, where a plays the role of b_0. After the induction we make $b = z_3$ blue, and we color z_3, z_4, z_5, z_6 (in that order) with colors $1, 2, 3$. So, we may assume that C_d does not contain b.

In order to complete Case 1 only the following subcase remains.

Subcase 1.2.2.2.2.3: u, v **are not neighbors, and** C_d **does not contain** b.

As C_d is dangerous when u is made blue, it has at most three vertices not joined to C'. One of these is u. Another is z_1 by the assumption of Subcase 1.2.2.2. As u, v are not neighbors, $z_1 \neq v$. Now C_d does not contain v because of (c_1). Let z'_1 be the neighbor of z_1 not in C_d.

We now form a new graph G'' from G' as follows: We first delete C_d and the vertex a. If $int(C') - V(C_d) - a$ is disconnected we focus on the component containing b and ignore for the moment the other component. Then we add an edge from v to x, and we add an edge from z'_1 to C'. If z_2 has a neighbor z'_2 in the graph under consideration, then we put z_2 back and we add two edges from z_2 to C'. Now we try to use induction with x playing the role of b_0. (There may be one more component of $int(C') - V(C_d) - a$ but that component is easy to dispose of as explained below.)

Let us first consider the case where the induction is possible, that is, G'' has no dangerous cycle and hence satisfies $(c_1) - (c_9)$. Then we give a the blue color of x so that a, b have distinct blue colors. We let u, z_1 be red. It is possible that z_2 has the same blue color as z'_1. In that case we try to make z_2 red. If that fails it is because we create a red facial 4-path in which case z_2 can see only one blue color except that of z'_1. But then z_2 can get another blue color so that it has no color conflict with z'_1. Then we color the blue vertices of C_d which is possible because we only have to watch the colors of a and z_2 or z'_2.

If z_2 is joined to C (or to a component of $int(C') - V(C_d) - a$ not containing b), and there is a z_i ($i \geq 2$) which is not joined to C, then we argue similarly except that we now have to apply induction (with z_i playing the role of b_0) to the component of $int(C') - V(C_d) - a$ not containing b.

So we may assume that it is not possible to use induction to G''. This means that we create a dangerous cycle C'_d in G'' when z'_1 becomes blue. The cycle C'_d is disjoint from C_d. However, C'_d may contain v or b or both.

Subcase 1.2.2.2.2.3.1: **The edge** $z_1 z_1'$ **is a bridge in** $int(C') - a$.

We consider three subcases.

Subcase 1.2.2.2.2.3.1.1: **Each of** z_2, z_q **is joined to** C

We form G'' as in Subcase 1.2.2.2.2.3 above except that we now delete a and $C_d - z_1$. Again, we add the edge vx and let x play the role of b_0. But now we add two edges from z_1 to C forcing z_1 to be blue. Then we use induction. After the induction we give a the blue color of x, and we make u red. There may be a vertex z_i ($i \geq 3$) on C_d such that the edge $z_i z_i'$ from z_i leaving C_d is a bridge in $int(C)$. We then apply induction to the component Q of $int(C) - z_i z_i'$ not containing a, b and with z_i playing the role of r_0 and being 4-forbidden.

After the induction z_1, a may have the same blue color. We try to make z_1 red. If this is possible, it is easy to complete the coloring. So assume that it is not possible, that is, we create a facial 4-path containing z_1. So z_1' and one of its two other neighbors are red. Hence we can recolor z_1 so that it has a blue color that does not conflict with any other blue color. Then we color $z_2, \ldots z_q$ (except z_i which is red) using colors $1, 2, 3$, ignoring z_i'. Then z_i' is the only vertex having color conflicts with other blue vertices. We avoid this by permuting the blue colors in Q.

Subcase 1.2.2.2.2.3.1.2: z_2 **is not joined to** C

Since the vertices $z_3, \ldots, z_q$ are all joined to $C - r_0$, the path with these vertices has at most 4 vertices, by Claim 1. As C_d is dangerous (when u is made blue) it has length 5, that is, $q = 4$. We form G'' as in Subcase 1.2.2.2.2.1.1 above, that is, we delete a and $C_d - z_1$. We add the edge vx and let x play the role of b_0. We add two edges from z_1 to C forcing z_1 to be blue. Then we use induction. After the induction we give a the blue color of x, and we make u red. Let z_2' be the neighbor of z_2 not in C_d. We apply induction to the component Q of $int(C) - z_2 z_2'$ not containing a, b and with z_2 playing the role of r_0 and being 4-forbidden. After the induction b, a, z_1, z_3, z_4 are blue, and u, r_0', z_2 are red. The vertices z_3, z_4 do not yet have a color $1, 2, 3$, and z_1, a may have the same blue color. We try to make z_1 red and recolor z_2, Q such that z_2 now plays the role of b_0. If this is possible, it is easy to complete the coloring. So assume that it is not possible, that is, we create a red facial 4-path containing z_1. So z_1' and one of its two other neighbors are red. Hence we can recolor z_1 so that it has a blue color that does not

conflict with any other blue color except that of the neighbor z_2' of z_2 (which we ignore at the moment). Then we color z_4, z_3 using colors $1, 2, 3$, again ignoring z_2'. Then z_2' is the only vertex having color conflicts with others. We avoid this by permuting the blue colors in Q.

Subcase 1.2.2.2.2.3.1.3: z_q **is not joined to** C

Again, $q = 4$. In this case we form a graph G''' from G' as follows: We delete C_d and focus on the component of $int(C') - V(C_d)$ containing a. We add an additional edge from a to C forcing a to become blue. We also add an edge from z_1' to a vertex of C which we call z_1 (with a slight abuse of notation). Then we apply induction to the resulting graph with z_1 playing the role of r_0 and being 4-forbidden. We also apply induction to the other component of $int(C') - V(C_d)$ joined to $z_q = z_4$ such that z_4 plays the role of r_0 and is 4-forbidden. Now a is blue and z_1, z_4 are red. We also make u blue. Recall that r_0' is red, and b is blue. In order to avoid the possible color conflict between a, b we try to make a red. This is possible unless we create a red facial 4 path starting at r_0' or a. In either case a can see only one blue vertex except b and u. So we can give a a blue color so that it has no color conflict (as u does not yet have a color $1, 2, 3$). We now give the blue vertex u a color $1, 2, 3$ such that it has no blue color conflict except possibly with the neighbor z_4' of z_4 outside C_d. This is possible because we only need to watch a or v (but not both) and z_1'. Then we color the blue vertices z_2, z_3 again ignoring the neighbor z_4' of z_4 outside C_d. That vertex z_4' can be disposed of by permuting the blue colors in the component of $int(C) - z_4 z_4'$ not containing a, b.

Subcase 1.2.2.2.2.3.2: **The edge** $z_1 z_1'$ **is not a bridge in** $int(C') - a$.

Since $int(C') - a - z_1 z_1'$ is connected it has a path from C_d to C_d'. Hence C_d has length 2 modulo 3, and hence length 5 by Claim 1 and the path must leave C_d at z_2 and enter C_d' at a vertex w_1 not joined to C.

Subcase 1.2.2.2.2.3.2.1: b **is not contained in** C_d'.

Since $int(C) - a$ is connected it has a path from C_d' to b. That path must leave C_d' at a vertex not joined to C. From these observations it follows that also C_d' has length 2 modulo 3. We may assume that C_d, C_d' each has length 5 by Claim 1. (Otherwise we replace an appropriate blue 3-path by an edge and use induction.) So $C_d = u z_1 z_2 z_3 z_4 u$, and $C_d' = z_1' w_1 w_2 w_3 w_4 z_1'$ where

w_2, w_3 are joined to C, and the above-mentioned path from C'_d to b leaves C'_d at w_4. In particular, C'_d does not contain v.

We form a new graph G'' as follows. We delete from G' the vertex a and the vertices of C_d. We add the edge vx and we add an edge from the neighbor z'_2 of z_2 outside C_d to z'_1. Then we apply induction where x plays the role of b_0.

We may assume that there is no dangerous cycle in G'' containing the new edge $z'_1 z'_2$. For then G would contain a cycle $z'_1 z_1 z_2 z'_2 z''_2 z'''_2 w_1 z'_1$ where z'_2, z''_2, z'''_2 are joined to C. Then we form a new graph from G' by deleting $a, u, z_4, z_3, z_2, z'_2, z''_2$ and adding the edge vx and two edges from each of z'''_2, z_1 to C. After the induction (where x plays the role of b_0) we give a the blue color of x, we make z_2, u red and z_4, z_3, z'_2, z''_2 blue. If z'_1 is blue, we change the color of z_1 to red. If w_1 is blue we change it to red. Now it is easy to color the blue vertices $1, 2, 3$: We first ignore the colors $1, 2, 3$ of z_1, z'''_2. Then we assign colors $1, 2, 3$ to z_1 (if it is blue), $z_4, z_3, z'_2, z'''_2, z''_2$ in that order or (if z'_1 is blue) $z'''_2, z''_2, z'_2, z_3, z_4$ in that order. So assume that there is no dangerous cycle.

By similar arguments, we may also assume that the new edge $z'_1 z'_2$ is not part of a double edge. Also, G'' satisfies the connectivity condition (c_1). So we can apply induction to G''.

After the induction a will get the blue color of x, and we shall color C_d as follows. z_3, z_4 are blue, and u is red. If z'_1 is red, then z_1 is colored blue and z_2 is colored red. If z'_1, z'_2 are blue, then then z_1, z_2 are colored red. If z'_1, z'_2 are blue and red, respectively, then we make z_1 is blue and z_2 red unless we create a red facial 4-path. In that case we make z_2 blue and z_1 red. Now it is easy to color the blue vertices of C_d by the colors $1, 2, 3$ unless z_1, z'_2 are blue, and z'_1 is red. (It is here useful to note that w_2, w_3 are blue, and only one more vertex of C'_d can be blue.) Below we dispose of that case.

In order to color z_1, z_3, z_4 we investigate the Kempe chains containing z'_2. Assume that z'_2 has blue color 1. We claim that the Kempe chain of colors $1, 2$ (respectively $1, 3$) containing z'_2 does not contain b or v or any blue neighbor of v. We show this by focussing at C'_d. Recall that z'_1 is red. Then at most one of w_1, w_4 is blue because w_1, w_2, w_3, w_4 cannot all be blue as they are in a 5-cycle. If w_1, say, is blue, then we may assume that the third neighbor w'_1 of w_1 is red since otherwise, we make w_1 red. There now may be a Kempe chain from w_1 to the third neighbor of w_4 which we denote w'_4. But then w_1, w'_4 have the same color, and we switch colors of w_2, w_3 so that we prevent the Kempe chain under consideration to connect w_1, w'_4.

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We shall now color z_1, z_3, z_4. We may assume that one of w_1, w_4 is blue since otherwise it is easy to color z_1, z_3, z_4 (possibly after switching colors in a Kempe chain so that z_2', a have distinct colors). In the discussion below we assume it is w_1. (Note that only one of w_1, z_1', w_4 can be blue because there cannot be four blue vertices in a 5-cycle.)

If z_2', w_1, a are colored $1, 1, 2$ or $1, 2, 2$, then we give z_1, z_3, z_4 colors $3, 2, 1$.

If z_2', w_1, a are colored $1, 2, 3$, then we switch colors of the Kempe chain with colors $2, 3$ containing w_1 so that z_2', w_1, a are colored $1, 3, 3$, a previous case. (If w_4 is blue and w_1 is red we consider instead the Kempe chain with colors $1, 2$ containing z_2'.)

If z_2', w_1, a are colored $1, 2, 1$, then we switch colors of the Kempe chain with colors $1, 3$ containing z_2' so that z_2', w_1, a are colored $3, 2, 1$, a previous case.

If z_2', w_1, a are colored $1, 1, 1$, then we switch colors of a Kempe chain so that z_2', w_1, a are not colored with the same color, a previous case.

Subcase 1.2.2.2.2.3.2.2: b **is contained in** C_d'.

By (c_1), b is a neighbor of v on C_d'. Then $w_3 = b, w_4 = v$. We do induction as in Subcase 1.2.2.2.2.3.2.1 except that we need not use Kempe chains. (Also, we may assume that we do not create a dangerous cycle containing the edge $z_1' z_2'$ as $V(G)$ then consists of the vertices of C, the vertices r_0', a, the vertices of C_d, C_d' and the vertices of the new dangerous cycle. It is then easy to color G directly.) In Subcase 1.2.2.2.2.3.2.1 we used Kempe chains only in the case where, after the induction, a, z_1, z_2' are blue and u, z_2, z_1' are red. z_3, z_4, z_1 do not yet have colors $1, 2, 3$. We ignore the colors $1, 2, 3$ of $a, w_2, w_3 = b$. Possible w_1 or its third neighbor w_1' is blue. But they are not both blue since, otherwise, we make w_1 red. Now we can assign colors $1, 2, 3$ to the vertices $z_1, z_3, z_4, a, w_3, w_2$ in that order.

This completes the proof of Case 1.

Below we consider the Case 2 where b_0 exists, and finally Case 3 where r_0 exists and is 4-forbidden. We already note here that Case 3 is almost the same as Case 2. In fact Case 3 is much easier, because much of the reasoning in Case 2 is about coloring the vertex b_0. This difficulty does not exist in Case 3.

Case 2: b_0 **exists.**

If all vertices in $int(C)$ are joined to $C - b_0$, then $G - V(C)$ is a path whose vertices are blue. It is part of a path starting at b_0. We traverse this path from b_0 and color its vertices $1, 2, 3, 1, 2, 3, \ldots$.

So assume that some vertex in $int(C)$ has no neighbor in $C - b_0$. Then there is a unique path $b_0 b_1 \ldots b_k b'_0$ such that each of $b_0, b_1, \ldots, b_k$ has a neighbor in $C - b_0$ but b'_0 has no neighbor in $C - b_0$. Possibly $k = 0$. Let H be the graph obtained from G by deleting the vertices $b_1, b_2, \ldots, b_k$ and adding the edge $b_0 b'_0$. Let x, y be the two other neighbors of b'_0 such that the path $b_0 b'_0 x$ turns sharp right at b'_0.

We may assume that G has no dangerous cycle. For such a cycle would have the form $C' = b_0 b_1 \ldots b_k b'_0 x b'_1 b'_2 \ldots b'_m a b_0$ where b'_0, a are the only two non-blue vertices. If $G - (V(C) \cup V(C'))$ is disconnected, then the two components are joined to b'_0, b'_m, respectively. We apply induction to each component where b'_0 plays the role of r_0 and is 4- forbidden, and b'_m plays the role of b_0 (and a facial cycle of $C \cup C'$ plays the role of C). If $G - (V(C) \cup V(C'))$ is connected and is joined to only b'_0 in C', then we again apply induction to $G - (V(C) \cup V(C'))$ where b'_0 plays the role of r_0 and is 4-forbidden. Finally, if $m = 0$ and $G - (V(C) \cup V(C'))$ is joined to each of b'_0, x by an edge, then the dangerous cycle C' has the form $b_0 b_1 \ldots b_k b'_0 x a b_0$. Since $int(C)$ has no 5-path with five blue vertices, by Claim 1, it follows that $C' = b_0 b_1 b'_0 x a b_0$. (Note that a dangerous cycle cannot have 6 or 7 vertices. It may have 8 vertices, but then it either has a blue 5-path, which contradicts Claim 1, or else the two non-blue vertices have distance at least 3 on the cycle.) Let C'' be the cycle obtained from $C \cup C'$ by deleting the edge $b_0 a$. We apply, if possible, induction to C'' and its interior such that x plays the role of b_0. If the induction is possible, it is easy to color b_1, b_0. So we may assume that induction is not possible, that is, $int(C'')$ has a dangerous cycle S containing the neighbor y of b'_0 in $int(C'')$. If y has a neighbor y' on S which is joined to C (and therefore blue), we apply induction to the graph G' obtained from $G - b'_0 - b_1$ by adding an edge from y to b_0 and an additional edge from x to C. Then we apply induction so that b_0 plays the role of r_0 which is either right-forbidden or left-forbidden. We make the choice so that y' is forced to be blue by that choice. However, as y' is already blue because it is joined to C, we do not create a new dangerous cycle and therefore induction is possible. As y becomes red, it has no color conflict with the blue vertex x. Now b'_0 is made red, and it is easy to color the blue vertices b_1, b_0. So we may assume that y has no neighbor on S which is joined to C. So $S = y z_1 z_2 z_3 z_4 y$ where z_2, z_3 are joined to C. Now we apply induction to each of the two

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components H_1, H_2 of $int(C) - b_1 - b_0' - y - z_2 - z_3$ where H_1 contains z_1 and H_2 contains z_4, x. (If z_4, x are in distinct components, the argument is similar and easier.) Before induction we identify z_i with a vertex on the outer cycle (and add an additional edge from x to the outer cycle). We let z_i play the role of r_0 and let it be 4-forbidden for $i = 1, 4$. Then we make y blue and b_0' red. Finally we assign colors $1, 2, 3$ to y, z_3, z_2, b_1, b_0 in that order ignoring the neighbor of z_1 in H_1. If that neighbor has the same color as y or z_2, then we permute colors in H_1. This shows that G has no dangerous cycle.

Subcase 2.1: $H - V(C) - b_0'$ is disconnected with components M_x containing x and M_y containing y.

For $z \in \{x, y\}$, we let H_z be the subgraph of G induced by $V(C) \cup V(M_z)$, together with the edge zb_0. We apply induction to each of the graphs H_x, H_y. In each case C is the outer cycle and b_0 is blue. If there is one, let x' be the only other blue vertex, other than either b_1 and b_2 or x and y, that has distance at most 2 from b_0 in G. Assume without loss of generality that x' is in M_x. Apply the induction to H_x and H_y. Suppose that b_0' can be made red without violating Conclusion (i). Notice that x and y and x' are the only possible neighbours of b_k in the blue square-graph that are already colored with one of the blue colors. We temporarily ignore the color of y and color in this order $b_k, \ldots, b_1$ and recolor b_0 with the three blue colors. There are at least two ways to do this; for one of these colorings, b_0 has a color different from that of x'. If y is blue, we complete the coloring by permuting the blue colors in M_y so that y gets a color different from those of x and b_k. If x is red, then we color b_0' with the blue color of b_0 in H_x, and, again ignoring the color of y, color $b_k, \ldots, b_1$ and recolor b_0. As in the preceding paragraph, one of the two possibilities will give b_0 a color different from that of x'. If y is blue, appropriately permuting the blue colors of M_y will give y a blue color different from those of b_0 and b_k, as required. Thus, we may assume that:

Fact y: x is blue, y is red, and making b_0 red introduces a red facial 4-path $b_0 y z_1 z_2$, with z_1, z_2 in M_y.

Subcase 2.1.1: $M_y - y$ is disconnected with components M_1, M_2.

Then we apply induction to M_1, M_2. Before induction, we add an edge from the neighbor of y in M_1 (respectively M_2) to b_0. We get a red-blue coloring of M_y with no facial red 4-path, and, by permuting colors in M_1 we may assume that we get a proper coloring in the blue vertices in the square-

graph of M_y such that the color of y is blue. However, this contradicts Fact y. So in Subcase 2.1 we are left with

Subcase 2.1.2: $M_y - y$ is connected.

Subcase 2.1.2.1: y has a neighbor y' in M_y such that, if y' is made blue, then we create no dangerous cycle in M_y.

In this case we can apply induction to M_y as follows. We let y be joined to a vertex on the outer cycle which will play the role of r_0 and which will be either right-forbidden or left-forbidden depending on whether y' is reached by a right turn or left turn on the path r_0yy'. Note that condition (c_9) is satisfied because of the assumption of Subcase 2.1.2: $M_y - y$ is connected. After this induction y is red. We now combine this coloring of M_y with the coloring of M_x. Recall that x is blue in this coloring by a remark immediately before Fact y. We let b_0' be red which creates no red facial 4-path in M_y. Then we color $b_k, b_{k-1}, \ldots, b_0$ blue and we color them $1, 2, 3$ in that order.
So in Subcase 2.1.2 and hence also in Subcase 2.1 we are left with

Subcase 2.1.2.2: If a neighbor y' of y in M_y is made blue, we create a dangerous cycle C'. If the third neighbor of y, say y'', is made blue, we create a dangerous cycle C'''

Assume first that C' does not contain y.
We applied earlier induction to M_y where y is joined to the vertex playing the role of b_0. We concluded (Fact y) that y would be red and that there would be a red facial 3-path starting at y. Let us assume that in some such coloring the red facial 3-path starting at y turns sharp right. The case where it turns sharp left is treated in the same way. There may be another coloring where it turns sharp left, but we shall not use that. Let y' be the neighbor of y in this path.
We combine the two colorings obtained by applying induction to M_x, M_y where each of x, y is joined to a vertex playing the role of b_0. Then we make b_0' red, and we obtain a facial 4-path $b_0'yy'u$. Each of y', u is in C'. There is at most one more vertex v in C' which is not joined to C because C' becomes dangerous if we make y' blue. The vertex v must exist because $M_y - y$ is connected (since otherwise, we would get a contradiction to Fact y). Also, v must be a neighbor of y'. Possibly v is blue.
As $M_y - y$ is connected, it is not possible that both neighbors of y'' in C''

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are joined to C. So, y'' has a neighbor y''' on C''' which is not joined to C. If we make y'' blue, we make C''' dangerous. If we make y'', y''' blue we make C''' forbidden. So one of y'', y''' is red in the coloring of M_y. Hence, y has at most two blue neighbors (in the square-graph) in C'''. Because of Fact y, we conclude that y must have a third blue neighbor in the square-graph. That blue vertex must be v (because y', u are red). And we shall get a problem if we make v red (since otherwise, we could color y blue, contradicting Fact y.) Summarizing, if we let v, y interchange colors, then y has no conflict with its blue neighbors (in the square-graph) in C'''. Moreover, we must create a facial 4-path containing v. This implies that v has only one blue neighbor (in the square-graph) not in C'. We now recolor as follows: We first give y a blue color such that it has no color conflict with a blue vertex in C'''. Then we keep v as blue, but we color v such that it has no color conflict with the blue vertices outside C'. We keep the blue color of the vertices of C' joined to C but we recolor them (using colors $1, 2, 3$) such that they have no color conflict with v or the neighbor u' of u outside C'. This coloring contradicts Fact y.

In the argument above we assume that C' does not contain y. So, we need to comment on the case where C' contains y. In that case $C'' = C'$, and y, y', y'' are the only vertices of C' that are not joined to C. We apply induction to each of the two components of $H_y - V(C')$. We think of y' (respectively y'') as a vertex on the outer cycle playing the role of r_0 and being 4-forbidden. Then we can let y be blue and obtain a contradiction to Fact y.

This completes the discussion of Subcase 2.1.

So, to complete the proof of Theorem 1 in Case 2, there only remains Subcase 2.2 below.

Subcase 2.2: $H - V(C) - b_0'$ **is connected.**

Subcase 2.2.1: **If x is made blue, we create no dangerous cycle in $G - b_0$.**

We have already discussed the case where r_0 exists and is either right-forbidden or left-forbidden. We shall therefore apply Theorem 1 to H where b_0 plays the role of r_0 and is right-forbidden. This is possible because of the assumptions in Subcases 2.2 and 2.2.1. After we we have applied Theorem 1, x is blue, and b_0' is red. If also y is red, or if $k > 0$, (or both), then we can color $b_k, b_{k-1}, \ldots, b_0$ blue, and we color them $1, 2, 3$ in that order. So, we

may assume that $k = 0$ (and hence $H = G$) and also the following:

Fact x : If, in Subcase 2.2.1, we apply Theorem 1 to G such that b_0 plays the role of r_0 (and it is either right-forbidden or left-forbidden), then b'_0 is red, and x, y are blue. Moreover, there is a blue vertex x' of distance 2 to b_0, and x', x, y have distinct blue colors.

The existence of x' in Fact x combined with the assumption of Subcase 2.2 implies that not both of x, y are joined to C. If y is joined to C, then we let x, y interchange roles. So, we may assume that y is not joined to C.

Let ab_0d be a 3-path of C traversed anticlockwise. We now form a new graph H' from G by deleting b_0, b'_0 and adding two new vertices b, c, and also adding the path $abcd$ and the edges by, cx. We now try to use the induction hypothesis to H' with b playing the role of r_0 being left-forbidden or right-forbidden. (Note that H' has more vertices than G but fewer edges inside the outer cycle so it makes sense to try induction.) If this induction works, then y will become red. We use this coloring to G except that we let b'_0 be red and b_0 blue. Now give b_0 a color distinct from those of x, x'. This coloring would satisfy the conclusion of Theorem 1, and therefore, we may assume that we cannot apply induction. This implies that we must have one of the following subcases 2.2.1.1 or 2.2.1.2 below.

Subcase 2.2.1.1: $G - V(C) - y$ **is disconnected.**

Note that then we do not create a dangerous cycle by making y blue. This means that x, y can interchange role unless x is joined to C. So, we may assume that x' belongs to the component of $G - V(C) - y$ not containing x. (This is clear if x is joined to C. And if x is not joined to C, then we let x, y interchange roles. If $G - V(C) - x$ is connected we proceed to Subcase 2.2.1.2.)

Let us now focus on a coloring discussed in Fact x. (Now x and y have their original roles.) In this coloring x and y are blue, but y cannot have two blue neighbors (in G). For if that were the case we could make y red and we would have a contradiction to Fact x. Let the blue colors of x, y, x' be $1, 2, 3$, respectively. We now interchange colors in the Kempe chain with colors $1, 3$ in the square-graph containing x that is, the connected component containing x in the subgraph of G^2 consisting of the vertices of color $1, 3$. Then x changes color from 1 to 3. As y has color 2 and may have a neighbor (in G) of color 1 or 3, but not both, x' keeps its color 3. This contradicts the last statement

in Fact x and completes the discussion of Subcase 2.2.1.1.

Subcase 2.2.1.2: $G - V(C) - y$ **is connected. If a neighbor of** y **distinct from** b_0' **is made blue, then we create a dangerous cycle in** H'.

Let the two other neighbors of y be denoted y', y''. Let C' be the dangerous cycle that arises if we make y' blue. Let C''' be the dangerous cycle that arises if we make y'' blue. Then all vertices of C' are joined to C except y' and one or two more vertices. Again, we consider two subcases.

Subcase 2.2.1.2.1: $C' = C'''$.

In this case C' contains the non-blue vertices y', y'' and possibly a third non-blue vertex. Because of the connectivity conditions of G, that third non-blue vertex must exist and it must be y. So we may assume that $C' = yy'z_1z_2y''y$ traversed clockwise, and $int(C) - V(C')$ is disconnected with components H_1 joined to y' and H_2 joined to y'' (because z_1, z_2 are joined to C.) Let us focus on a coloring discussed in Fact x. In this coloring x and y are blue. Also z_1, z_2 are blue. Then y', y'' are red. Let the colors of x', x, y be $1, 3, 2$, respectively. In G^2 we consider the Kempe chain with colors $1, 3$ containing x'. We may assume that this Kempe chain also contains x. If x' is in H_1 this implies that the neighbors of y', y'' outside C' have colors $1, 3$, respectively. If x' is in H_2, then we switch colors in the Kempe chain with colors $1, 2$ containing y so that x', x, y get colors $1, 3, 1$, a contradiction unless that Kempe chain contains x' which implies that the neighbor of y'' outside C' has blue color 1. In either case, the neighbor of y'' outside C' is blue. Now we interchange colors of y, y'. Then y' is blue of color 2. This may create a color conflict is with a vertex in H_1. However, then we recolor H_1 by applying the induction to H_1 with y' playing the role of b_0.

Subcase 2.2.1.2.2: $C' \neq C'''$.

In this case C', C''' are disjoint. We earlier introduced the facial path ab_0d on C traversed anticlockwise. One of a, d is joined to x' in $int(C)$. We may assume that a is joined to x'. For if d is joined to x' in $int(C)$, then we interchange the roles of x, y. (This is possible because the existence of the disjoint cycles C', C''' imply that we do not create a new forbidden or dangerous cycle if we make y blue.) We choose the notation such that y' is obtained from a right turn at the edge $b_0'y$. Now all vertices of C' are joined

to C except y' and one or two more vertices.

Recall that x is blue in H'. However, it is not possible that x is one of the blue vertices on C'. For if that were the case, then x would be a neighbor of y' by the connectivity condition (c_1). Then we let x, y interchange roles and now we get a contradiction to the assumption of Subcase 2.2.1.2 because making y' blue now does not create a new dangerous cycle. So, x is not on C'. Clearly, x' is not in C' because of C'' and the connectivity condition (c_1).

As $G - V(C) - y$ is connected (since otherwise we are in Case 2.2.1.1 which we have disposed of) it has a path from x' to x. This path contains two vertices z_1, z_2 which are in C' and distinct from y' and the blue vertices on C'.

Let us again focus on a coloring discussed in Fact x. Let the blue colors of x, y, x' be $1, 2, 3$. respectively. One of the vertices z_1, y', z_2 may be blue, but two of them cannot be blue. In G^2 we consider the Kempe chain with colors $1, 3$ containing x'. We may assume that this Kempe chain contains a path P from x' to x since otherwise we can interchange colors $1, 3$ in such a way that x, x' have the same color which implies that we can color b_0 and complete Subcase 2.2.1.2 and hence Subcase 2.2.1.

As only one of z_1, z_2 can be blue, the path P cannot use the edge $z_1 z_2$ in G^2. If y' is blue, it has color 1 or 3, and then P contains a path $z_1' y' z_2'$ where z_1', z_2' are neighbors of z_1, z_2, respectively, not in C'. In that case we make y' red. So assume y' is red. Then P enters C' in z_1 (or from the neighbor z_1' of z_1 outside C') and leaves C' from z_2 (or goes to a neighbor z_2' of z_2 outside C'). In between, P uses only vertices of C' which are joined to C. We may assume that only one of z_i, z_i' is blue for $i = 1, 2$ since otherwise, we make z_i red. Then we change colors of the blue colors of C' joined to C. Now the Kempe chain with colors $3, 1$ which contains x' does not contain x and hence b_0 can be colored. This completes the discussion of Subcase 2.2.1.

Subcase 2.2.2: **If x is made blue, we create a dangerous cycle C_x in $G - b_0$. Similarly, if y is made blue, we create a dangerous cycle C_y in $G - b_0$.**

We may assume that C_x, C_y are disjoint. For if C_x, C_y share precisely one edge, then they have length 5 and $H - V(C)$ consists only of C_x, C_y, b_0' and two edges incident with b_0', and it is easy to color $int(C)$. If C_x, C_y share at least two edges, then $C_x = C_y$. If they do not contain b_0', then $C_x = C_y = yxb_0''uvy$ where u, v are joined to C. In this case we apply

induction to $H - b_0' - x - y - v - u$ with b_0'' playing the role of b_0, and it is easy to extend the resulting coloring to G. Finally, if $C_x = C_y$, and C_x contains b_0', we apply induction to the two components of $int(C) - V(C_x)$ such that each of x, y plays the role of r_0 and is 4-forbidden. It is then easy to color all the other vertices of C_x and also the vertices $b_0', b_k, \ldots, b_0$ blue and also color them by colors $1, 2, 3$. So we may assume that C_x, C_y are disjoint.

Recall that x' denotes the unique vertex in $int(C)$ which is a neighbor of b_0 in the square-graph. (We may assume that x' exists since we otherwise replace an appropriate path on C starting at b_0 by a single edge.) Since $H - V(C) - b_0'$ is connected, we may choose the notation such that, every path in $H - V(C) - b_0'$ from x' to x intersects C_y. (This includes the case where x' is in C_y.) Let y_1, y_2 be the neighbors of y in C_y such that the path $b_0'yy_1$ turns left at y. Since $H - V(C) - b_0'$ is connected, y_2 is not joined to C. Possibly, $y_1 = x'$.

Subcase 2.2.2.1: y_1 is not joined to C.

In this subcase y, y_1, y_2 are the only vertices of C_y which are not joined to C. Hence $G - V(C) - V(C_y)$ has a component Q_1 containing x' and a vertex y_1' joined to y_1. Let Q_2 denote the other component of $G - V(C) - V(C_y)$ having a vertex y_2' joined to y_2. We apply induction to each of Q_1, Q_2. When we apply induction to Q_1, Q_2 we add an edge from y_i' to a vertex of C for $i = 1, 2$. With a slight abuse of notation we call that vertex y_i. When we apply induction, y_i will play the role of r_0 and we call y_i 4-forbidden. We also add one more edge from b_0' to C in order to get a cubic graph. After the induction we make y blue. Note that b_0' is already blue and also has a color $1, 2, 3$. y_1, y_2 are now red. We then color y and the vertices of C_y joined to C by the colors $1, 2, 3$. Finally, we color the vertices $b_k, \ldots, b_0$ by the colors $1, 2, 3$. We may assume that precisely one of b_0', x is blue since otherwise, we make b_0' red. We now argue why this coloring of the blue vertices is possible. We first color y. We ignore the blue colors in Q_1 at the moment. So, here we have to avoid the blue colors of b_0', x, y_2'. However, if both b_0', x are blue, then we recolor b_0' so that it becomes red. So we can color y. Then we color the other blue vertices of C_y which is possible because we only have to watch y_2', y. Again, we ignore the blue colors in Q_1. After this we permute the blue colors in Q_1 such that the color of y_1' has no conflict with a blue color in C_y. Finally, we color the vertices $b_k, \ldots, b_0$. There is only one problem, namely that we cannot color b_0 because of the color of x' and two other vertices. We

consider the case where these two other vertices are y, x or y, b_0'. (In other words, we consider the case $k = 0$. If $k \geq 1$, the argument is similar.) So, we may assume that $k = 0$ and that x, y, x' (or b_0', y, x') have colors $1, 2, 3$, respectively. We may assume that y_1' has color 3 since otherwise, y_1' has color 1 (or is red), and we switch the two blue colors $2, 3$ in Q_1. Then the neighbor u_1 of y_1 on C_y joined to C must have color 1. The number of vertices of C_y joined to C is 2 modulo 3. By Claim 1, we may assume that it is 2 since otherwise, replace 4 edges of C_y with a single edge and use induction. Again, we consider the Kempe chain with colors $1, 3$ containing x'. As this must contain x (or b_0'), and y_2 is red, we conclude that y_2' is blue with color 1. But then we can make y red and color b_0 with the color 2. (Note that the recoloring of y from blue to red is because we want to give b_0 a color $1, 2, 3$. In Case 3 below we shall encounter a similar situation with r_0 instead of b_0. Recoloring y might give a red facial 4-path. However, we need not recolor y in Case 3 below because we shall not give r_0 a color $1, 2, 3$.)

Subcase 2.2.2.2: y_1 is joined to C. That is, $y_1 = x'$.

We may assume that $C_y = yy_2y_3y_4y_1y$ (labelled anticlockwise) where y_1 is joined to C.

Thus we consider the case where C_y has length 5. (If C_y has length 4, that is, $C_y = yy_2y_3y_1y$ where y_1, y_3 are joined to C, then the proof is similar and easier. If C_y is longer, then C_y has three consecutive vertices joined to C. We replace these three vertices by a single edge and use induction. So only the case where C_y has length 5 needs consideration.)

By a similar argument, we may assume that $C_x = xx_2x_3x_4x_1x$ (labelled clockwise) where two of x_1, x_4, x_3 are joined to C.

If y_4 is not joined to C, then y_3 is joined to C. Then we apply induction to the two components of $int(C) - y_4$. In one of the components we add the edge y_1y_3 before we use induction. In the other component we let y_4 play the role of r_0 and we call it 4-forbidden. So we may assume that y_4 is joined to C. Now we delete from H the vertices y, y_1, y_4, y_3 and use induction. Before the induction we also add an edge from the third neighbor y_3' of y_3 to C. That neighbor on C is also called y_3 with a slight abuse of notation. This y_3 will play the role of r_0 and is called 4-forbidden. We add an edge from y_2 to C and also an edge between y_2, b_0'. If the interior of C is disconnected (after the deletion of y, y_1, y_4, y_3), then the neighbor b_0 of b_0' on C will play the role of b_0. (If the interior of C is connected, which is the most difficult

case, then b_0 is just a red vertex on C.) We do not create a dangerous cycle (because such a cycle would have to contain the edge between y_2, b_0', and all vertices on this cycle outside C_x would have to be joined to C. Then x_2, y_2 have a common neighbor y_2' joined to C. In this case it is easy to complete the proof by induction.) We then apply induction. After the induction we make y red. If y_3' is red, it is easy to color the blue vertices $y_1, y_4, b_k, \ldots, b_0$ by $1, 2, 3$ in that order. (Before we do that we make b_0' red in case x is blue.) So assume that y_3' is blue and has (possibly) the same color as y_2.

Now we try to make y_2 red. If we do not create a red facial 4-path, then it is easy to color the blue vertices $b_k, \ldots, b_0, y_1, y_4$ by $1, 2, 3$ in that order. So assume that we create a red facial 4-path if we make y_2 red. We keep y_2 blue, but give it a color $1, 2, 3$ distinct from its blue neighbors in G^2 except possibly b_0'. So now y_2, b_0' may have the same blue color. If we can make b_0' red we have finished. So assume that this is not possible, that is, either

(i) there is a red facial path xx_1x_1' or else

(ii) the vertices x, x_2 are red.

In case (i) x_3, x_4 are joined to C. If we can make x_2 red, it is easy to color the blue vertices. So, x_2' and one of its two other neighbors are red. But then ignore the colors $1, 2, 3$ of x_3, x_4 and we color the blue vertices $y_4, y_1, b_0', \ldots, b_0, x_2, x_3, x_4$ in that order. So assume that we have (ii): x, x_2 are red.

Now we can give b_0' a color $1, 2, 3$ distinct from that of y_2 and x_1. If possible, we make the color of b_0' distinct from the color of y_3'. We color the blue vertices $y_4, y_1, b_k, \ldots, b_0$ by $1, 2, 3$ in that order. This works unless y_3', y_2, x_1 have distinct blue colors, say $1, 2, 3$, respectively. We uncolor b_0' and try to switch colors in the Kempe chain with colors $2, 3$ containing x_1. We may assume that then y_2 changes color. Then we try to switch colors in the Kempe chain with colors $1, 3$ containing x_1. We may assume that then y_3' changes color. But then x_1, x_4, x_3, x_3' (where x_3' is the third neighbor of x_3) are blue and of colors $3, 2, 1, 3$ (or $3, 1, 2, 3$), respectively. We try to make x_3 red. If this is possible, then we complete the proof using the afore-mentioned Kempe chain with colors $1, 3$. So assume that the vertices in the facial 3-path $x_2x_2'x_2''$ are red. We now make x_2 blue and x_3, b_0' red. We ignore colors $3, 2$ of x_1, x_4 and color now the blue vertices $x_2, x_4, x_1, y_4, y_1, b_0, \ldots, b_k$ in that order.

Case 3: r_0 **exists and is** 4**-forbidden.**

This case is similar to and much easier than Case 2. We use the same

notation except that now we have a vertex r_0 instead of b_0.

If $k > 0$, then Case 3 follows immediately from Case 2 by first letting r_0 play the role of b_0 in Case 2. After we obtain the desired coloring in Case 2 we just change the color of r_0 from blue to red. So assume that $k = 0$, that is, $H = G$.

The Subcase 3.1 corresponding to Subcase 2.1 is trivial: Just apply induction to M_x and M_y so that b_0' becomes blue. In Subcase 3.2 (where $H - V(C) - b_0' = G - V(C) - b_0'$ is connected) the Subcase 3.2.1 is trivial because it reduces to Case 1. So only Subcase 3.2.2 needs attention. We repeat word for word the proof of Subcase 2.2.2. In Subcase 2.2.2 we use an inductive argument in which b_0' becomes blue. If b_0' remains blue throughout the proof of Subcase 2.2.2, then there is no problem in letting r_0 be red. In Subcase 2.2.2.1 we make, at some stage, b_0' red in order to be able to color b_0. This is not necessary in Subcase 3.2.2.1. We also at some stage make b_0' red if x is blue. Also this is not a problem in Subcase 3.2.2.1. The only case where an additional argument is needed is in Subcase 2.2.2.2 just before statements (i),(ii). At this stage y_3', y_2, b_0' are blue. y_3', y_2 have distinct blue colors, say $1, 2$. The only color conflict is that y_2, b_0' have the same color. We would like to change the color of b_0' to either 3 or red because then it is possible to color y_4, y_1 by colors $3, 2$, respectively. So, we try to make b_0' red. If this works, then we have finished in Subcase 2.2.2.2. But in Subcase 3.2.2.2 we now need to consider a possible red facial path $r_0 b_0' x x_1$. As two vertices of C_y are joined to C, we conclude that x_3, x_4 are joined to C. We may assume that x_2 is blue since otherwise we can give y_4, y_1, b_0' colors $3, 1, 3$, respectively. So, x_2 is blue of color 3. If we can make x_2 red we have finished. So, we may assume that the third neighbor x_2' of x_2 is red and that one of the two other neighbors of x_2' is red. Now we can recolor x_2, x_3, x_4 with colors $1, 2, 3$ or $2, 1, 3$, respectively, so that the only possible color conflict is that x_4 has the same color as the third neighbor x_1' of x_1. That color conflict can be eliminated by permuting colors $1, 2, 3$ in the component of $int(C) - x_1$ containing x_1'.

5   Wegner's conjecture

Theorem 1 is similar to Conjecture 1 except that the 3-coloring of the blue graph is not obtained from Brooks' theorem. Also, Theorem 1 is very close to Wegner's conjecture when restricted to planar cubic 2-connected graphs.

Indeed, such a graph G has a facial cycle of length at most 5. We may assume that this is the outer cycle. We select a vertex on this cycle which we call b_0. We insert a vertex d_0 of degree 2 on the outer cycle such that b_0 is adjacent to the vertex d_0 of degree 2 in order to satisfy condition (c_4) in Theorem 1. Then we apply Theorem 1. The red square-graph is planar except for a pair of crossing edges in the outer face. The blue square-graph is 3-colorable except that b_0 may have a blue neighbor in the blue square-graph when we ignore d_0. Thus, the square of G can be colored in 7 colors such that only two edges join vertices of the same color, a slight weakening of Wegner's conjecture.

To obtain the full version of Wegner's conjecture we need additional arguments. In this reasoning we shall use the classical result of Kotzig [8] that every planar triangulation of minimum degree at least 4 has a so-called *light edge*, that is an edge such that the sum of degrees of its ends is at most 11.

Theorem 2 *Let G be a planar graph of maximum degree at most 3. Then G^2 is 7-colorable.*

Proof of Theorem 2. The proof is by induction on the number of vertices. The basis of the induction is trivial so we proceed to the induction step. Assume (reductio ad absurdum) that Theorem 2 is false, and let G be a counterexample with the smallest number of edges. Clearly, G has more than 7 vertices.

Claim (1): G is cubic and 2-connected.

Proof of Claim (1). If G has a vertex of degree < 3, then it has degree at most 6 in G^2. We delete this vertex. If the vertex has two neighbors and they are nonadjacent, we add an edge between its neighbors. Then we use induction. (The reason that we add an edge between the neighbors is that they are adjacent in G^2 and should therefore receive distinct colors when we use induction.) So, G is cubic.

Clearly, G is connected. If G has a cut-edge e, then we delete e and apply the induction hypothesis to the connected components of the resulting graph. By permuting the colors in one of the components, if necessary, we obtain a 7-coloring of G^2. So, G is cubic and 2-connected. This proves Claim (1).

Claim (2): G is 3-connected.

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Proof of Claim (2). Suppose (reductio ad absurdum) that G contains two edges x_1x_2, y_1y_2 such that $G - x_1x_2 - y_1y_2$ has two components G_1, G_2 such that G_i contains x_i, y_i for $i = 1, 2$. By choosing G_1 to be minimal we may assume that x_1, y_1 are not joined by an edge in G. If x_2, y_2 are not joined by an edge in G, then we apply induction to (the square of) $G_i + x_iy_i$ for $i = 1, 2$. By permuting colors we may assume that x_1, y_2 have the same color, and x_2, y_1 have the same color. This results in a 7-coloring of G^2, a contradiction. So assume that x_2, y_2 are joined by an edge in G, and let their third neighbors be x_3, y_3, respectively. Then we apply induction to (the square of) $G_1 + x_1y_1$ and to $G_2 - x_2 - y_2 + x_3y_3$. By permuting colors we may assume that x_1, y_3 have the same color, and x_3, y_1 have the same color. By permuting the remaining colors we may assume that all colors of vertices adjacent (in G) to y_3 (except x_3) are also adjacent (in G) to y_1. Hence y_2 has 3 available colors among the 7 colors used for coloring G^2, and x_2 has at least one available color. Now we can color first x_2 and then y_2 and obtain a contradiction which proves Claim (2).

Claim (3): G has no edge xy which is contained in two distinct cycles C_1, C_2 such that C_1 has length 3 and C_2 has length at most 5.

Proof of Claim (3). Suppose (reductio ad absurdum) that xy, C_1, C_2 exist. As G is 3-connected, C_1, C_2 are facial cycles, and C_2 has length 4 or 5. (Clearly, C_1 is facial, and clearly C_2 cannot have length 3. If C_2 is nonfacial, then it has two vertices whose deletion makes the graph disconnected.) If C_2 has length 4 we contract C_1, C_2 into a vertex v and use induction. If v has color 1 and the neighbors have colors $2, 3, 4$, then the vertices x, y can receive two of the colors $2, 3, 4$, a third vertex of C_2 can be colored 1, and now it is easy to color the two other vertices of $C_1 \cup C_2$ as well. So assume that $C_2 = xx_1x_2x_3yx$ and $C_1 = yy_1xy$. Delete the edge xy and draw $G - xy$ such that the outer cycle is $C = xx_1x_2x_3yy_1x$. Let x_1', x_2', x_3', y_1' be the neighbors of x_1, x_2, x_3, y_1, respectively, inside C. We may assume that x_1', x_2', x_3', y_1' are distinct. For if two of them are identical, then we contract that vertex and C into a single vertex (of degree 3). We apply induction, and then it is easy to modify the coloring of the contracted graph to a 7-coloring of G^2, a contradiction. Now we try to apply Theorem 1 to $G - xy$ where y_1 plays the role of r_0 and is right-forbidden. We also try to apply Theorem 1 to $G - xy$ where y_1 plays the role of r_0 and is left-forbidden. If one of these attempts works, then we change the colors of x, y to blue. As y_1' is

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red, it is easy to give x, y two colors $1, 2, 3$. So we may assume that it is not possible to apply Theorem 1. Because G is 3-connected, $int(C) - y_1'$ is connected so $(c_6), (c_9)$ hold. The rest of $(c_1) - (c_9)$ all trivially hold, except (c_8). So we may assume that we create a dangerous or forbidden cycle C' in $int(C)$ when make a second neighbor u_1 of y_1' blue. Possibly, C' does not contain u_1. Similarly we create a dangerous or forbidden cycle C'' when make the third neighbor u_2 of y_1' blue. Possibly, C'' does not contain u_2. As a dangerous cycle has at least three blue vertices and there are only 4 blue vertices when we apply induction, it follows that each of C', C'' has length 4 or 5. Hence they are facial cycles. As C', C'' have at least one of x_1', x_2', x_3' in common, and there is only one facial cycle containing x_i' and not intersecting C it follows that $C' = C''$. As G is 3-connected, it follows that either $C' = C'' = u_1 u_2 x_1' x_2' x_3' u_1$ in which case G has 12 vertices and G^2 has chromatic number 6, or else $C' = C'' = y_1' x_1' x_2' x_3' y_1'$ in which case G has 10 vertices and G^2 has chromatic number 6, or else $C' = C'' = x_1' x_2' x_3' z_1 z_2 x_1'$ where z_1, z_2 are distinct from y_1', u_1, u_2. In this case we contract all vertices of C_1, C_2, C' into a single vertex and apply the induction hypothesis to the square of the resulting graph. The resulting 7-coloring can easily be modified to a 7-coloring of G^2.

Claim (4): G has no triangle.

Proof of Claim (4). Suppose (reductio ad absurdum) that G has a triangle $x_1 x_2 x_3 x_1$ which can be chosen to be the outer triangle. We now apply Theorem 1 where x_1 plays the role of r_0 and is 4-forbidden. There is no dangerous cycle, as every dangerous cycle has at least three blue vertices. We may create a red facial path $x_2 x_1 x_1' x_2'$ or a red facial path $x_3 x_1 x_1' x_3'$ or both. In that case x_2, x_2' and x_3, x_3' are non-neighbors in the square-graph, by Claim (3). So, we identify x_2, x_2' or x_3, x_3' before we apply the 4-Color Theorem to the red square-graph. This contradiction proves Claim (4).

Claim (5): G has no non-facial cycle of length < 6.

Proof of Claim (5). Suppose (reductio ad absurdum) that C is a non-facial cycle of length < 6. By Claim (3), C has no chord. Hence each edge not in C but incident with a vertex of C joins C to a vertex inside or outside C. So precisely one or two edges join C to its interior or exterior. This contradiction to Claim (2) proves Claim (5).

Claim (6): G is cyclically 4-edge-connected, that is, if E is a set of three

edges such that $G - E$ *is disconnected, then* E *consists of three edges incident with the same vertex.*

Proof of Claim (6).

Suppose (reductio ad absurdum) that G has a set E of three edges x_1x_2, y_1y_2, z_1z_2 such that $G - E$ has two components G_1, G_2 such that G_i contains x_i, y_i, z_i for $i = 1, 2$ and such that none of G_1, G_2 is a single vertex. By Claim (4), each of G_1, G_2 has more than three vertices. We consider four new graphs G_1', G_1'', G_2', G_2''. G_i' is obtained from G_i by adding a vertex g_i joined to x_i, y_i, z_i for $i = 1, 2$. G_i'' is obtained from G_i by adding three vertices x_i', y_i', z_i' forming a triangle and also adding the edges $x_ix_i', y_iy_i', z_iz_i'$ for $i = 1, 2$. We first apply induction to G_1'' and G_2''. If x_1, y_1, z_1 get distinct colors in G_1'', and x_2, y_2, z_2 get distinct colors in G_2'', then it is easy to combine the two colorings to get a coloring of G^2, a contradiction. If x_1, y_1, z_1 get the same color in G_1'', then we apply induction to G_2', and again it is easy to combine the two colorings. So we may assume that in G_1'' the vertices $x_1, y_1, z_1, x_1', y_1', z_1'$ have colors $1, 1, 2, 3, 4, 5$, respectively. We may assume that in G_2' the vertices g_2, x_2, y_2, z_2 have colors $1, 3, 4, 5$, respectively. Now we try to combine the two colorings. The only possible conflict is that z_1 (which has color 2 in G_1) can see a neighbor of z_2 in G_2 which also has color 2. In G_1 we may switch colors $2, 6$. We may also switch colors $2, 7$. One of these two color switches will result in a proper coloring of G^2, a contradiction which proves Claim (6).

Claim (7): G does not contain two distinct 4-cycles having an edge in common.

Proof of Claim (7). Suppose (reductio ad absurdum that $C_1 : xx_1x_2yx$ and $C_2 : xyx_3x_4x$ are 4-cycles. We delete the edge xy and think of $C_1 \cup C_2 - xy$ as a 4-cycle. We apply induction to the resulting cubic graph. After the induction it is easy to color x, y. We may argue as follows: Let x_i' be the third neighbor of x_i for $i = 1, 2, 3, 4$. After the induction we may assume that x_i has color i for $i = 1, 2, 3, 4$. We can now give x one of the colors $5, 6, 7$. Similarly for y. If the possible colors for x, y are distinct, we have finished. So assume that x_1', x_2', x_3', x_4' have colors $5, 5, 6, 6$ or $5, 6, 5, 6$, respectively. If we can change the color of x_1 to 3, then we can give x, y the colors $1, 7$, respectively. So assume that the color 3 is present at a neighbor of x_1'. Similarly, a neighbor of x_2' (respectively x_3', respectively x_4') has color 4 (respectively 1, respectively 2). If we can change the colors of x_1, x_3 to 7, then we can give x, y the colors

$1, 3$, respectively. So, we may assume that the color 7 is present at a neighbor of x_1' and also at a neighbor of either x_2' or x_4' or both. Assume that 7 is present at a neighbor of x_2'. (If 7 is present at a neighbor of x_4', the proof is similar.) We switch colors of x_1, x_2. We may assume that the color 2 is present at a neighbor of x_3' since otherwise, we could change the color of x_3 to 2 and complete the proof. Similarly the color 1 is present at a neighbor of x_4'. Now the vertices x_1, x_2, x_3, x_4, x, y are colored $4, 3, 7, 3, 2, 1$, respectively.

The dual version of Kotzig's result on light edges in triangulations implies that G has two facial cycles C_1, C_2 of length k_1, k_2 respectively, such that C_1, C_2 have an edge xy in common and such that $k_1 \leq k_2, k_1 + k_2 \leq 11$. Hence $k_1 \leq 5$. By Claim (4), $k_1 \geq 4$. By Claim (7), $k_2 \geq 5$.

We choose C_1, C_2 such that $k_1 + k_2$ is minimum. We delete the edge xy and draw G such that the outer cycle C is $C_1 \cup C_2 - xy$. This cycle can be described as $C : xx_1x_2 \ldots x_{k_2-2}yy_1y_2 \ldots y_{k_1-2}x$. Let the third neighbors of $x_1, x_2, \ldots, y_{k_1-2}$ be denoted $x_1', x_2', \ldots, y_{k_1-2}'$, respectively. As G is 3-connected, and there are at most 7 edges from C to its interior (since $k_1 + k_2 \leq 11$), it follows from Claim (6) that $G - V(C)$ is connected.

We now apply Theorem 1 to $G - V(C)$ where one of $x_1, x_{k_2-2}, y_1, y_{k_1-2}$ plays the role of r_0 and all other vertices of C are red. We call r_0 either left-forbidden or right-forbidden in order to prevent that there is a red facial 4-path containing an edge of C and starting at the neighbor of r_0 on C distinct from x, y. (We shall later make x, y blue so that a red facial path cannot start at x or y.) We divide the argument into two cases.

Consider first the case where $k_2 \leq 6$. We apply Theorem 1 to $G - xy$ where we let x_1 (or x_{k_2-2} or y_1 or y_{k_1-2}) play the role of r_0. Before we show that we can apply Theorem 1, we explain how this will complete the proof. After the application of Theorem 1 we make x, y blue, and we can extend the 3-coloring of the blue square-graph to first y and then x because x is adjacent (in G^2) to at most one blue vertex inside C. To justify the last statement, the statement (iii) in Theorem 1 implies that the neighbor of x_1 inside C is red, and therefore it is possible to give x a blue color and also a color $1, 2, 3$. We then apply the 4-Color Theorem to the red square-graph. The only problem is that there may be a facial 4-path when $k_2 = 6$, namely $x_1x_2x_3x_4$. Note that the vertices x_1, x_4 are not neighbors in the square-graph because of Claim (5). So, before we apply the 4-Color Theorem we identify x_1, x_4. After this identification the red square-graph is planar.

We now explain why we can apply Theorem 1 to $G - xy$. Claims (4),(5),(6)

and the minimality of $k_1 + k_2$ imply that the vertices $x_1', x_2', \ldots, y_{k_1-2}'$ are distinct. We claim that condition (c_9) is satisfied. That is, $int(C) - x_1'$ is connected. For suppose that $int(C) - x_1'$ has two components H_1, H_2. As G is cyclically 4-edge-connected, $k_1 + k_2 = 11$, that is $k_2 = 6, k_1 = 5$, and the notation can be chosen such that H_1 contains y_3', y_2', y_1', and H_2 contains x_2', x_3', x_4'. But then the edges yy_1, xy_3 and the edge from x_1' to H_1 separate G, a contradiction to Claim (6). Similarly, $int(C) - x'$ is connected whenever x is one of $x_1, x_{k_2-2}, y_1, y_{k_1-2}$.

So, the only problem in the case $k_2 \le 6$ is that there may be a forbidden or dangerous cycle C_1' when we try to apply Theorem 1 with x_1 (or one of $x_{k_2-2}, y_1, y_{k_1-2}$) playing the role of r_0 and being right-forbidden or left-forbidden. Then C_1' is disjoint from C, and C_1' contains at least three blue vertices. At least two of these are joined to C. We also try to apply Theorem 1 with one of $x_{k_2-2}, y_1, y_{k_1-2}$ playing the role of r_0 (and being right-forbidden or left-forbidden). Assume C_2', C_3', C_4' are the resulting forbidden or dangerous cycles. Then each C_i' is disjoint from C, and C_i' contains at least three blue vertices. At least two of these are joined to C.

We claim that precisely two vertices of C_i' are joined to C. For, if three vertices of C_1', say, are joined to C, then C_1' has length 5 and shares an edge with a 4-cycle. Hence $k_1 = 4, k_2 \le 5$. As we have noted earlier that $k_2 \ge 5$, the notation can be chosen such that $C_1' = x_2' x_3' y_1' z_1 z_2 x_2'$. But then there is no dangerous cycle when we think of x_3 as r_0 and let it be left-forbidden. This contradiction shows that C_i' is joined to precisely two vertices of C for each $i = 1, 2, 3, 4$.

Consider the graph H induced by C and C_1' and x_1'. As C_1' has at most two non-blue vertices, there are at most 7 edges from H to vertices not in H. As G is cyclically 4-edge-connected, at most one face of H is not a face of G. Now it is easy to see that some facial cycle of H is a 4-cycle and hence $k_1 \le 4$. If C_1' does not contain x_1', then C_1' contains a path $x_1'' x_2' x_3'$ where x_1'' is a neighbor of x_1'. Then the edge $x_2 x_2'$ is contained in a facial 4-cycle and a facial 5-cycle implying that $k_2 \le 5$. But now we can apply Theorem 1 to $G - xy$ with x_3 playing the role of r_0 and being right-forbidden or left-forbidden (because now there cannot be a dangerous cycle). So we may assume that C_1' contains x_1'. But then at least three vertices of C_1' are joined to C, a contradiction to an earlier claim.

This completes the case $k_2 < 7$.

Consider finally the case where $k_2 = 7$. Then $k_1 = 4$.

We try to apply Theorem 1 where one of x_1, x_5 plays the role b_0. If this is not possible, then $G - V(C)$ contains a cycle which contains three of the vertices $x_2', x_3', x_4', y_1', y_2'$ and one or two more vertices. This is easily seen to contradict the assumption $k_2 = 7$ and that G is 3-connected and $G - V(C)$ is connected. (Consider for example the case where the dangerous cycle contains y_2', x_2', x_3'. Then the dangerous cycle must be of the form $y_2' r_1 r_2 x_2' x_3' y_2'$ because G is 3-connected. But then $G - V(C)$ is disconnected, a contradiction.) So we assume that we can apply Theorem 1 where x_1 plays the role b_0.

If the neighbor x_1' of x_1 inside C is red, then we make x blue and we can give it a color $1, 2, 3$. Then we focus on the red square-graph. We first delete y and then identify x_2, x_5, y_2. Then we apply the 4-Color Theorem. We can extend the 4-coloring to include y because y can see only the colors of x_4, x_5, y_1.

If the neighbor x_1' of x_1 inside C is blue, then we make x, y red. Then we focus on the red square-graph. We first delete y, x and then identify x_2, x_5, y_2 as before. Then we apply the 4-Color Theorem. We can extend the 4-coloring to include y, x because y can see only the colors of x_4, x_5, y_1, and x can see only the colors of y, y_1, y_2.

This contradiction completes the proof of the theorem.

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