EFFICIENT TRANSIENT TOPOLOGY OPTIMIZATION THROUGH DYNAMIC SUBSTRUCTURING

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Summary This paper presents an efficient topology optimization methodology for dynamic structural design problems. The key to the method is to utilize dynamic substructering, or component mode synthesis, i.e. the Craig-Bampton reduction method to speed up the transient analysis. The dynamic substructuring concept is used for both the forward (the state equation) and the backward (adjoint equation). The developed optimization scheme is demonstrated on design problems for linear elastic Mindlin plates subject to impact loads, where the goal is to minimize the deflection in a specified region.

INTRODUCTION AND MOTIVATION

Structural shape and topology optimization for transient problem is known to be computionally expensive and thus less attractive than its steady state counterparts. This is partly due to the fact that a transient analysis is more expensive than a static analysis - especially for implicit time integration schemes - and partly because the sensitivity analysis requires the solution to a terminal value problem. To illustrate the extra computional effort required one may use an example from non-linear static optimization. Here one have to solve a non-linear problem for the state equation, bu the adjoint problem turns out to be linear and thus much cheaper than the state problem [1]. Despite this obvious drawback several works on transient structural topology optimization have been presented and studied in the past decade. The approach has, among others, been used for 1D pulse modulators and filter design in [2], material microstructure design in [3] and design of composites in [4].

The focus of this work is to develop a methodology which is capable of speeding up the computations by orders of magnitude. To realize this goal the dynamic substructering method, namely the Craig-Bampton method, is applied [5], [6]. The main idea of dynamic substructuring, or component mode synthesis, is to partition the computational domain into a set of smaller substructures. The finite element equations on each substructure is then reduced in size by its normal modes while retaining the interface, or boundary degrees of freedom, by so-called constraint modes. The reduced substructure stiffness, mass and damping matrices (also known as super elements) are then assemblied into a reduced global system which can be orders of magnitude smaller than the original, full finite element system. Based on the reduced system, it is now possible to perform the transient analysis very fast which paves the way for a fast transient topology optimization methodology.

THEORY

The mechanical problem considered in this work is based on Mindlin plate theory for which the unknowns are the out of plane displacement w and the two rotations $\boldsymbol{\theta} = \{\theta_x, \theta_y\}$. The governing equations can be stated compactly as follows

$$\nabla^T \boldsymbol{G} \left(\nabla w - \boldsymbol{\theta} \right) = \rho h \frac{\partial^2 w}{\partial t^2} \tag{1}$$

$$\left(\boldsymbol{\alpha}_1 \frac{\partial}{\partial x} + \boldsymbol{\alpha}_2 \frac{\partial}{\partial y} \right)^T \boldsymbol{D} \left(\boldsymbol{\alpha}_1 \frac{\partial}{\partial x} + \boldsymbol{\alpha}_2 \frac{\partial}{\partial y} \right) + \boldsymbol{G} \left(\nabla w - \boldsymbol{\theta} \right) = \frac{\rho h^3}{12} \frac{\partial^2 \boldsymbol{\theta}}{\partial t^2} \tag{2}$$

where h is the thickness, $\boldsymbol{G}$ and $\boldsymbol{D}$ are constitutive matrices and $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are appropriate 3×2 matrices.

When discretized by the standard Galerkin finite element method and including proportional damping, i.e. $\boldsymbol{C} = \alpha \boldsymbol{M} + \beta \boldsymbol{K}$ the matrix equations becomes as follows

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}(t), \quad t \in [0; T] \tag{3}$$

where $\boldsymbol{u} \equiv \boldsymbol{u}(t)$ is a vector collecting both displacement and rotational degrees of freedom and $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ and $\boldsymbol{F}(t)$ are the mass-, damping-, stiffness-matrices and load vector, respectively. The system, together with appropriate boundary and initial conditions, can now be solved by a time integration scheme from 0 to time T. In this work the unconditionally stable Newmark algorithm is used [5].

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Solving Eq. (3) will lead to the aforementioned issues with computional time and we will therefore study a reduced system obtained by the Craig-Bampton method first. For each substructure i in the domain the following reduction is performed

$$\boldsymbol{R}_j = \begin{bmatrix} \boldsymbol{\Phi}_j & \boldsymbol{\Psi}_j \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{4}$$

where $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{\Phi}_j$ is the matrix with n normal modes computed on the interior i of substructure j, i.e. the solution to the eigenvalue problem $(\boldsymbol{K}_j^{ii} - \omega^2 \boldsymbol{M}_j^{ii})\boldsymbol{\Phi}_j = \boldsymbol{0}$ and $\boldsymbol{\Psi}_j$ is the static correction matrix for the interface b degrees of freedom , i.e. $\boldsymbol{\Psi}_j = -\left(\boldsymbol{K}_j^{ii}\right)^{-1} \boldsymbol{K}_j^{ib}$. The super elements are then obtained by

$$\boldsymbol{M}_j^r = \boldsymbol{R}_j^T \boldsymbol{M}_j \boldsymbol{R}_j, \quad \boldsymbol{C}_j^r = \boldsymbol{R}_j^T \boldsymbol{C}_j \boldsymbol{R}_j, \quad \boldsymbol{K}_j^r = \boldsymbol{R}_j^T \boldsymbol{K}_j \boldsymbol{R}_j \; \boldsymbol{F}_j^r \text{and } \boldsymbol{R}_j^T \boldsymbol{F}_j \tag{5}$$

Once assemblied into a system equivalent to Eq. (3), the time response is obtained effeciently by the Newmark algorithm.

The optimization problem investigated in this work can be posed in discrete form as a standard mathematical programme.

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \phi = \textstyle\int_{t_1}^{t_2} \boldsymbol{w}(t)^T \boldsymbol{w}(t) \, \mathrm{d}t \\ \text{s.t.} \quad & \boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}(t) \\ & V(\boldsymbol{x})/V^* - 1 \leq 0 \\ & 0 \leq x_i \leq 1, \;\; i = 1, n \end{aligned} \tag{6}$$

where the objective function is the out of plane displacement evaluated in a specified region from time t_1 to t_2. The first constraint refers to the state equation, the second to a volume constraint and the last to a box constraint on the design variables denoted by $\boldsymbol{x}$. The optimization problem is solved using the Method of Moving Asymptotes [7] and the sensitivities are determined using the adjoint method obtained purely from the discretized state equation and objective. This leads to the solution of a terminal value problem for the Lagrange multiplier $\boldsymbol{\lambda}$.

$$\boldsymbol{M}\ddot{\hat{\boldsymbol{\lambda}}} + \boldsymbol{C}\dot{\hat{\boldsymbol{\lambda}}} + \boldsymbol{K}\hat{\boldsymbol{\lambda}} = \left.\frac{\partial \phi}{\partial \boldsymbol{u}}^T \right|_{T-\tau}, \quad \hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}(T - \tau) \tag{7}$$

here formulated as an initial value problem by a change in variables. The sensitivity can then be obtained from the following

$$\frac{\partial \phi}{\partial x_e} = \int_0^T \frac{\partial(\boldsymbol{w}^T \boldsymbol{w})}{\partial x_e} + \boldsymbol{\lambda}^T \left(-\frac{\partial \boldsymbol{M}}{\partial x_e}\ddot{\boldsymbol{u}} - \frac{\partial \boldsymbol{C}}{\partial x_e}\dot{\boldsymbol{u}} - \frac{\partial \boldsymbol{K}}{\partial x_e}\boldsymbol{u} \right) \mathrm{d}t \tag{8}$$

To reduce memory requirements, the developed implementation uses a check pointing scheme to avoid storing all displacements, velocities and acceleration from the forward problem.

DISCUSSION

The presented study shows that it is possible to reduce the computational effort of transient topology optimization by order(s) of magnitude by use of dynamic substructering. As expected, the price for the obtained speed up is the more involved sensitivity analysis and the bookkeeping associated with the substructuring (i.e. a domain decomposition). The findings presented here can, with some modifications, be applied to e.g. acoustic-structural interaction problems and thus pave the way for new, interesting multi-physical transient topology optimization problems.

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