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Operator representations of sequences and dynamical sampling

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Abstract. This paper is a contribution to the theory of dynamical sampling. Our purpose is twofold. We first consider representations of sequences in a Hilbert space in terms of iterated actions of a bounded linear operator. This generalizes recent results about operator representations of frames, and is motivated by the fact that only very special frames have such a representation. As our second contribution we give a new proof of a construction of a special class of frames that are proved by Aldroubi et al. to be representable via a bounded operator. Our proof is based on a single result by Shapiro & Shields and standard frame theory, and our hope is that it eventually can help to provide more general classes of frames with such a representation.

Key words and phrases : Frames, operator representation, dynamical sampling, Schauder basis, the Carleson condition, Hardy space.

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1. Introduction

Dynamical sampling is a new topic in applied harmonic analysis but has already attracted considerable attention [2, 3, 4, 5, 12, 1, 7, 9, 8]. One of the key questions is how to construct frames $\{f_k\}_{k=1}^{\infty}$ for a separable Hilbert space \mathcal{H} that can be represented on the form $\{T^n \varphi\}_{n=0}^{\infty}$ for a linear operator T and some $\varphi \in \mathcal{H}$. It is known [7] that in an infinite-dimensional Hilbert space, such a representation is available if and only if $\{f_k\}_{k=1}^{\infty}$ is linearly independent; it is also known that it is significantly more complicated to obtain such a representation with a bounded operator T . Various characterizations of boundedness have been reported in [9]; prior to that a class of such frames were constructed in $\ell^2(\mathbb{N})$, based on the so-called Carleson condition [2].

In this paper we extend certain frame results in dynamical sampling to general sequences. This generalization is natural for at least two reasons, one of them being the difficulty of obtaining a representation of a frame in terms of a bounded operator. The second reason is that it turns out that “nice operator theoretical properties” of T typically is unrelated to frame properties of the underlying sequence; we demonstrate this point by several concrete examples. In other words: large classes of standard frames in harmonic analysis are represented by unbounded operators, and sequences that do not form frames might very well have representations in terms of bounded operators.

The second purpose of the paper is to give a detailed analysis of a construction of a class of frames that can be represented via certain diagonal operators; the construction first appeared in [2]. Our proof is based on just a single result by Shapiro and Shields [13] and standard frame theory. We supplement this with a detailed discussion of the Carleson condition, which indeed is the main ingredient in the construction of such frames.

The paper is organized as follows. Section 2 is devoted to the question of how to obtain a representation of a general sequence in terms of a bounded operator. This part of the paper does not even use the Hilbert space structure and holds in general Banach spaces. The Carleson condition and the associated frames for $\ell^2(\mathbb{N})$ are discussed in Section 3.

The standard definitions in frame theory are well-known to the sampling and signal processing community, so we will not repeat them here. The only nonstandard terminology is that we say that a sequence $\{f_k\}_{k=1}^\infty$ in a Hilbert space \mathcal{H} leads to a *frame-like expansion* if there exists a sequence $\{g_k\}_{k=1}^\infty$ in \mathcal{H} such that each $f \in \mathcal{H}$ has an expansion of the type

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k. \quad (1.1)$$

The classical case of a frame-like expansion is obtained by letting $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ be a pair of dual frames, but the more general concept of a frame-like expansion also covers other cases, e.g., a pair of biorthogonal Schauder bases, or a pair of a frame and analysis (or synthesis) pseudo-dual [11, 17].

2. Operator representations of sequences

In this section we consider linearly independent sequences $\{f_k\}_{k=1}^\infty$ in a Hilbert space \mathcal{H} and analyze the existence of a representation $\{T^n f_1\}_{n=0}^\infty$ for a bounded operator $T : \text{span}\{f_k\}_{k=1}^\infty \rightarrow \text{span}\{f_k\}_{k=1}^\infty$. This generalizes known results for frames, proved in [9]. The generalization is motivated by the observation that in general the existence of a representation of a sequence $\{f_k\}_{k=1}^\infty$ of the form $\{T^n f_1\}_{n=0}^\infty$ for a bounded operator T is not closely related with frame properties of the given sequence $\{f_k\}_{k=1}^\infty$. Let us illustrate this by some examples.

Example 2.1. Consider a linearly independent frame $\{f_k\}_{k=1}^\infty$ for an infinite dimensional Hilbert space \mathcal{H} and the associated representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$

in terms of a linear operator $T : \text{span}\{f_k\}_{k=1}^\infty \rightarrow \text{span}\{f_k\}_{k=1}^\infty$. Assume that $\inf_{k \in \mathbb{N}} \|f_k\| > 0$. Consider the sequence $\{\phi_k\}_{k=1}^\infty \subset \mathcal{H}$ given by $\phi_k := 2^k f_k$, $k \in \mathbb{N}$, which leads to a frame-like expansion and satisfies the lower frame condition but fails the upper one. For any $k \in \mathbb{N}$,

$$\phi_{k+1} = 2^{k+1} f_{k+1} = 2^{k+1} T f_k = 2T(2^k f_k) = 2T\phi_k.$$

This shows that $\{\phi_k\}_{k=1}^\infty$ has the representation $\{\phi_k\}_{k=1}^\infty = \{W^n \phi_1\}_{n=0}^\infty$, where $W = 2T$. In particular, the frame $\{f_k\}_{k=1}^\infty$ is represented by a bounded operator if and only if the non-Bessel sequence $\{\phi_k\}_{k=1}^\infty$ is represented by a bounded operator. Consider, e.g., the case where $\{f_k\}_{k=1}^\infty$ is a Riesz basis for \mathcal{H} . Then $\{f_k\}_{k=1}^\infty$ is representable by a bounded operator [7], and hence the Schauder basis $\{\phi_k\}_{k=1}^\infty$ is representable as $\{W^n \phi_1\}_{n=0}^\infty$ via a bounded operator W . Furthermore, taking $\{f_k\}_{k=1}^\infty$ to be a Riesz basis for \mathcal{H} , the family $\{h_k\}_{k=1}^\infty$ given by $h_k := 2^{-k} f_k$ leads to frame-like expansions and it is a Bessel sequence, but does not satisfy the lower frame condition. However, again, $\{h_k\}_{k=1}^\infty$ is representable as $\{W^n h_1\}_{n=0}^\infty$ via a bounded operator W . \square

In the following example we show that we can even have a frame-like expansion for a family of vectors on the form $\{T^n e_1\}_{n=0}^\infty$, where T is bounded and neither the lower nor the upper frame condition is satisfied.

Example 2.2. Using that for all $N \in \mathbb{N} \setminus \{1\}$,

$$1 + 2 + \cdots + (N - 1) = \frac{(N - 1)N}{2}, \quad 1 + 2 + \cdots + N = \frac{N(N + 1)}{2},$$

and denoting

$$I_N := \left\{ \frac{(N - 1)N}{2}, \frac{(N - 1)N}{2} + 1, \dots, \frac{N(N + 1)}{2} - 1 \right\},$$

we see that \mathbb{N} has a splitting into disjoint sets, $\mathbb{N} = \bigcup_{N=2}^\infty I_N$. Note that $|I_N| = N$. Let now $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis for \mathcal{H} , and define the operator T by

$$T e_k := \begin{cases} 2e_{k+1}, & k \in I_N, N \text{ odd}; \\ \frac{1}{2}e_{k+1}, & k \in I_N, N \text{ even}. \end{cases}$$

Clearly T extends to a bounded linear operator on \mathcal{H} . Furthermore,

$$\{T_n e_1\}_{n=0}^\infty = \left\{ e_1, \frac{1}{2} e_2, \frac{1}{4} e_3, \frac{1}{2} e_4, e_5, 2e_6, e_7, \frac{1}{2} e_8, \frac{1}{4} e_9, \frac{1}{8} e_{10}, \dots \right\}.$$

Then $\{T_n e_1\}_{n=0}^\infty$ is a Schauder basis, but neither the lower nor the upper frame condition is satisfied. In fact, $\{T_n e_1\}_{n=0}^\infty$ has a subsequence that tends to infinity in norm, and another subsequence that tends to zero. \square

The following example further confirms that the questions of a sequence $\{f_k\}_{k=1}^\infty$ having “nice frame properties” and “a nice representation” $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ are unrelated in general. Indeed, the considered family $\{f_k\}_{k=1}^\infty$ is not a frame and does not even provide a frame-like expansion, but it has a representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ for a bounded and isometric operator T .

Example 2.3. Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for a Hilbert space \mathcal{H} , and consider the sequence $\{f_k\}_{k=1}^\infty$ given by $f_k := e_k + e_{k+1}$, $k \in \mathbb{N}$. By Example 5.4.6 in [6], $\{f_k\}_{k=1}^\infty$ is a Bessel sequence but not a frame, despite the fact that $\overline{\text{span}}\{f_k\}_{k=1}^\infty = \mathcal{H}$. Since $\{f_k\}_{k=1}^\infty$ is linearly independent, we can consider the operator $T : \text{span}\{f_k\}_{k=1}^\infty \rightarrow \text{span}\{f_k\}_{k=1}^\infty$, $Tf_k := f_{k+1}$, and we clearly have $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$. Then, for any $c_k \in \mathbb{C}$, and any $N \in \mathbb{N}$,

$$\left\| T \sum_{k=1}^N c_k f_k \right\|^2 = \left\| \sum_{k=1}^N c_k (e_{k+1} + e_{k+2}) \right\|^2 = \left\| \sum_{k=1}^N c_k (e_k + e_{k+1}) \right\|^2 = \left\| \sum_{k=1}^N c_k f_k \right\|^2.$$

It follows that T has an extension to an isometric operator $T : \mathcal{H} \rightarrow \mathcal{H}$. \square

Motivated by the above examples we will now consider the question of representability by a bounded operator for general sequences in a Hilbert space. The reader who checks the proof will notice that we do not even use the Hilbert space structure, i.e., the results in this section hold in Banach spaces as well. Thus, the reader who goes for the highest generality can think about \mathcal{H} as a separable Banach space and $\langle f, g \rangle$ as the notation for the action of $g \in \mathcal{H}'$ on $f \in \mathcal{H}$. Given any sequence $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$, define the synthesis operator

$$U : \mathcal{D}(U) \rightarrow \mathcal{H}, \quad U\{c_k\}_{k=1}^\infty := \sum_{k=1}^\infty c_k f_k,$$

where the domain $\mathcal{D}(U)$ is the set of all scalar-valued sequences $\{c_k\}_{k=1}^\infty$ for which $\sum_{k=1}^\infty c_k f_k$ is convergent. Note that in contrast to the usual situation in frame analysis, we do not restrict our attention to sequences $\{c_k\}_{k=1}^\infty$ belonging to $\ell^2(\mathbb{N})$. Consider the right-shift operator \mathcal{T} , which acts on an arbitrary scalar sequence $\{c_k\}_{k=1}^\infty$ by $\mathcal{T}\{c_k\}_{k=1}^\infty := \{0, c_1, c_2, \dots\}$. A vector space V of scalar-valued sequences $\{c_k\}_{k=1}^\infty$ is said to be *invariant under right-shifts* if $\mathcal{T}(V) \subseteq V$.

The following result generalizes one of the key results in [9] to the non-frame case.

Theorem 2.4. Consider a sequence $\{f_k\}_{k=1}^\infty$ in \mathcal{H} which has a representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ for a linear operator $T : \text{span}\{f_k\}_{k=1}^\infty \rightarrow \text{span}\{f_k\}_{k=1}^\infty$. Then the following statements hold.

- (i) If T is bounded, then the domain $\mathcal{D}(U)$ and the kernel N_U of the synthesis operator are invariant under right-shifts; in particular, if $f_k \neq 0$ for all $k \in \mathbb{N}$ then $\left\{ \frac{\|f_{k+1}\|}{\|f_k\|} \right\}_{k=1}^\infty \in \ell^\infty$.

- (ii) T is bounded on $\text{span}\{f_k\}_{k=1}^\infty$ if and only if there is a positive constant K so that

$$\|U\mathcal{T}\{c_k\}_{k=1}^\infty\| \leq K\|U\{c_k\}_{k=1}^\infty\| \text{ for all finite sequences } \{c_k\}_{k=1}^\infty. \quad (2.1)$$

Proof. Throughout the proof, when T is bounded, we let \tilde{T} denote its unique extension to a bounded linear operator on $\overline{\text{span}}\{f_k\}_{k=1}^\infty$.

(i) Assume that T is bounded and consider first a sequence $\{c_k\}_{k=1}^\infty \in \mathcal{D}(U)$. In order to show that $\mathcal{T}\{c_k\}_{k=1}^\infty \in \mathcal{D}(U)$, i.e., that $\sum_{k=1}^\infty c_k f_{k+1}$ is convergent, consider any $M, N \in \mathbb{N}$ with $N > M$; then

$$\begin{aligned} \left\| \sum_{k=1}^N c_k f_{k+1} - \sum_{k=1}^M c_k f_{k+1} \right\| &= \left\| \sum_{k=M+1}^N c_k f_{k+1} \right\| = \left\| T \sum_{k=M+1}^N c_k f_k \right\| \\ &\leq \|T\| \left\| \sum_{k=M+1}^N c_k f_k \right\| \rightarrow 0 \text{ as } M, N \rightarrow \infty. \end{aligned}$$

Thus $\sum_{k=1}^\infty c_k f_{k+1}$ is convergent, i.e., $\mathcal{D}(U)$ is indeed invariant under right-shifts.

In order to prove the invariance of N_U , assume that $\{c_k\}_{k=1}^\infty \in N_U$. The series $\sum_{k=1}^\infty c_k f_{k+1}$ converges by what is already proved, and furthermore

$$\sum_{k=1}^\infty c_k f_{k+1} = \sum_{k=1}^\infty c_k T f_k = \tilde{T} \sum_{k=1}^\infty c_k f_k = 0;$$

this shows that $\mathcal{T}\{c_k\}_{k=1}^\infty \in N_U$, as desired.

Finally, for every $k \in \mathbb{N}$, $\|f_{k+1}\| \leq \|T\| \cdot \|f_k\|$, and thus $\{\frac{\|f_{k+1}\|}{\|f_k\|}\}_{k=1}^\infty \in \ell^\infty$.

- (ii) Assume first that T is bounded. For every $\{c_k\}_{k=1}^\infty \in \mathcal{D}(U)$, we know by (i) that $\mathcal{T}\{c_k\}_{k=1}^\infty \in \mathcal{D}(U)$; furthermore,

$$\|U\mathcal{T}\{c_k\}_{k=1}^\infty\| = \left\| \sum_{k=1}^\infty c_k f_{k+1} \right\| = \left\| \sum_{k=1}^\infty c_k \tilde{T} f_k \right\| \leq \|\tilde{T}\| \cdot \|U\{c_k\}_{k=1}^\infty\|.$$

Clearly this in particular applies to all the finite sequences.

Conversely, assume that there is a constant $K > 0$ so that (2.1) holds. Take an arbitrary $f \in \text{span}\{f_k\}_{k=1}^\infty$, i.e., $f = \sum_{k=1}^N c_k f_k$ for some $N \in \mathbb{N}$ and some $c_1, \dots, c_N \in \mathbb{C}$; letting $c_k = 0$ for $k > N$, we have that

$$\begin{aligned} \|Tf\| &= \left\| \sum_{k=1}^N c_k f_{k+1} \right\| = \left\| \sum_{k=1}^\infty c_k f_{k+1} \right\| = \|U\mathcal{T}\{c_k\}_{k=1}^\infty\| \\ &\leq K\|U\{c_k\}_{k=1}^\infty\| = K\|f\|, \end{aligned}$$

as desired. \square

In Hilbert spaces, the Riesz bases are the “better bases” compared to general Schauder bases, due to the unconditional convergence of the frame decomposition. It is also known that in a Hilbert space the class of Schauder bases that are norm-bounded below and above but do not form Riesz bases, is quite small; thus it does not seem to be worthwhile to consider the general Schauder bases

in Hilbert spaces. However, as already stated the results in the current section also hold if the underlying space is just a Banach space, so it seems natural to generalize some of the results that are known for Riesz bases in Hilbert spaces to Schauder bases in Banach spaces. In Proposition 2.8 we show that for Schauder bases, the invariance of $\mathcal{D}(U)$ under right-shifts in Theorem 2.4(i) actually characterizes the case where the representing operator is bounded. For the special class of Schauder bases that are scaled Riesz bases, Proposition 2.9 will show that boundedness of T is equivalent to boundedness of a certain scalar sequence.

Concerning Theorem 2.4(ii), as one can see in the proof, the boundedness of T implies validity of the inequality in (2.1) not only for the finite sequences, but also for all the elements of $\mathcal{D}(U)$.

Note that when $\{f_k\}_{k=1}^\infty$ is a frame having the form $\{T^n f_1\}_{n=0}^\infty$ for some linear operator T , it is proved in [8] that T is bounded if and only if the kernel N_U is invariant under right-shifts. Under the weaker assumptions in Proposition 2.4, boundedness of T still implies the invariance of N_U under right-shifts, but the converse does not hold, as demonstrated by the following example.

Example 2.5. Let $\{e_k\}_{k=1}^\infty$ be a Riesz basis for \mathcal{H} , and define $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$ by $f_k := k!e_k$. Then $\{f_k\}_{k=1}^\infty$ satisfies the lower frame condition, but not the upper frame condition. Defining $Tf_k = f_{k+1}$, we have $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$; however,

$$Te_k = T\left(\frac{1}{k!}f_k\right) = \frac{1}{k!}f_{k+1} = (k+1)e_k,$$

which shows that T is unbounded. On the other hand, $N_U = \{0\}$, so the kernel is invariant under right-shifts.

Similarly, we can construct a sequence $\{f_k\}_{k=1}^\infty$ satisfying the upper frame condition but not the lower frame condition; for example,

$$\{f_k\}_{k=1}^\infty = \left\{\frac{1}{2}e_1, \frac{1}{3}e_2, \frac{2}{3}e_3, \frac{1}{4}e_4, \frac{3}{4}e_5, \dots\right\}.$$

Representing this sequence on the form $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ again leads to an unbounded operator T , while $N_U = \{0\}$ is invariant under right-shifts.

Notice that in these two examples the domain $\mathcal{D}(U)$ is not invariant under right shifts. \square

We will now show that for sequences $\{f_k\}_{k=1}^\infty$ leading to a frame-like expansion, an extra assumption leads to a characterization of the possibility of representing $\{f_k\}_{k=1}^\infty$ via a bounded operator. This generalizes a result in [9].

Proposition 2.6. Assume that $\{f_k\}_{k=1}^\infty$ leads to a frame-like expansion as in (1.1), and that $\sum_{k=1}^\infty \langle f, g_k \rangle f_{k+1}$ converges for every $f \in \mathcal{H}$. Then $\{f_k\}_{k=1}^\infty$ can be represented as $\{T^n f_1\}_{n=0}^\infty$ for some bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ if and only if

$$f_{j+1} = \sum_{k=1}^{\infty} \langle f_j, g_k \rangle f_{k+1}, \quad \forall j \in \mathbb{N}. \quad (2.2)$$

Proof. If $\{f_k\}_{k=1}^\infty$ can be represented as $\{T^n f_1\}_{n=0}^\infty$ via some bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$, applying T on the frame-like expansion (1.1) leads to $Tf = \sum_{k=1}^\infty \langle f, g_k \rangle f_{k+1}$ for every $f \in \mathcal{H}$; taking $f = f_j$, $j \in \mathbb{N}$ now proves (2.2).

Conversely, assume that (2.2) holds. Consider the operator T defined by

$$Tf := \sum_{k=1}^{\infty} \langle f, g_k \rangle f_{k+1}, \quad f \in \mathcal{H};$$

note that T is well-defined by assumption and bounded by [16, Lemma 2.3]. Now, by (2.2) it follows that $Tf_j = f_{j+1}$, $j \in \mathbb{N}$, which proves that $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$, as desired. \square

Notice that $\sum_{k=1}^\infty \langle f, g_k \rangle f_{k+1}$ may converge for all $f \in \mathcal{H}$ even in cases where $\{f_k\}_{k=1}^\infty$ and/or $\{g_k\}_{k=1}^\infty$ are not frames:

Example 2.7. Let $\{e_k\}_{k=1}^\infty$ be a Riesz basis for \mathcal{H} and consider a sequence $\{m_k\}_{k=1}^\infty$ of nonzero complex numbers such that $|\frac{m_{k+1}}{m_k}| \leq C, \forall k \in \mathbb{N}$, for some positive constant C . Letting $\{f_k\}_{k=1}^\infty = \{m_k e_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty = \{\frac{1}{m_k} e_k\}_{k=1}^\infty$, the series $\sum_{k=1}^\infty \langle f, g_k \rangle f_{k+1}$ converges for all $f \in \mathcal{H}$ and thus the conclusion of Proposition 2.6 holds. Note that unless $\{m_k\}_{k=1}^\infty$ is bounded below and above, $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are not frames. \square

Any Riesz basis $\{f_k\}_{k=1}^\infty$ has a representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ for a linear operator T , which is necessarily bounded [7]. By linear independence, any Schauder basis $\{f_k\}_{k=1}^\infty$ also has an operator representation as $\{T^n f_1\}_{n=0}^\infty$, but not necessarily via a bounded operator, as one can see in Example 2.5. For Schauder bases we will now give a necessary and sufficient condition for the boundedness of T .

Proposition 2.8. Let $\{f_k\}_{k=1}^\infty$ be a Schauder basis for \mathcal{H} and consider the linear operator $T : \text{span}\{f_k\}_{k=1}^\infty \rightarrow \text{span}\{f_k\}_{k=1}^\infty$ such that $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$. Then T is bounded if and only if $\mathcal{D}(U)$ is invariant under right-shifts.

Proof. First assume that $\mathcal{D}(U)$ is invariant under right-shifts and let $\{g_k\}_{k=1}^\infty$ denote the unique biorthogonal sequence associated with $\{f_k\}_{k=1}^\infty$. Let $f \in \mathcal{H}$. Then $f = \sum_{k=1}^\infty \langle f, g_k \rangle f_k$. Therefore $\{\langle f, g_k \rangle\}_{k=1}^\infty \in \mathcal{D}(U)$ and hence, by the right-shift invariance of $\mathcal{D}(U)$, the series $\sum_{k=1}^\infty \langle f, g_k \rangle f_{k+1}$ converges. Furthermore, for every $j \in \mathbb{N}$, $f_{j+1} = \sum_{k=1}^\infty \langle f_j, g_k \rangle f_{k+1}$. It now follows from Proposition 2.6 that T is bounded. The converse implication is given in Theorem 2.4. \square

Recall that Schauder bases might differ from Riesz bases in two aspects: they might not be norm-bounded below and above, and they might lead to frame-like expansions that are conditionally convergent. For the class of Schauder bases consisting of scaled Riesz bases, we can characterize the boundedness of T in a way that is much easier to check compared to the right-shift invariance of $\mathcal{D}(U)$:

Proposition 2.9. *Let $\{f_k\}_{k=1}^\infty = \{m_k e_k\}_{k=1}^\infty$, where $\{e_k\}_{k=1}^\infty$ is a Riesz basis for \mathcal{H} and $\{m_k\}_{k=1}^\infty$ is a sequence of nonzero complex numbers, and consider the operator $T : \text{span}\{f_k\}_{k=1}^\infty \rightarrow \text{span}\{f_k\}_{k=1}^\infty$ such that $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$. Then T is bounded if and only if $\{\frac{\|f_{k+1}\|}{\|f_k\|}\}_{k=1}^\infty \in \ell^\infty(\mathbb{N})$.*

Proof. If T is bounded, the conclusion follows from Theorem 2.4. For the converse part, assume that $\{\frac{\|f_{k+1}\|}{\|f_k\|}\}_{k=1}^\infty \in \ell^\infty$. Then there is a positive constant C such that $\frac{|m_{k+1}|}{|m_k|} \leq C$ for every $k \in \mathbb{N}$. Now let A and B denote Riesz basis bounds of $\{e_k\}_{k=1}^\infty$. For every finite sequence $\{c_k\}$ we have

$$\begin{aligned} \|T \sum c_k e_k\|^2 &= \left\| \sum \frac{c_k}{m_k} f_{k+1} \right\|^2 = \left\| \sum c_k \frac{m_{k+1}}{m_k} e_{k+1} \right\|^2 \\ &\leq B \sum |c_k \frac{m_{k+1}}{m_k}|^2 \leq BC^2 \sum |c_k|^2 \leq \frac{B}{A} C^2 \left\| \sum c_k e_k \right\|^2; \end{aligned}$$

thus the operator T is bounded on $\text{span}\{e_k\}_{k=1}^\infty = \text{span}\{f_k\}_{k=1}^\infty$. \square

We will now show that if a sequence consists of a Schauder basis and a finite and strictly positive number of additional elements, then it can not be represented by a bounded operator T ; this extends a result in [7].

Proposition 2.10. *Let $\{f_k\}_{k=1}^\infty$ be a linearly independent sequence in \mathcal{H} containing an ω -independent subsequence $\{f_k\}_{k=N+1}^\infty$ for some $N \in \mathbb{N}$. Furthermore, assume that at least one f_{i_0} , $i_0 \in \{1, \dots, N\}$, can be written as $\sum_{k=N+1}^\infty c_k f_k$ for some scalar coefficients $c_k \in \mathbb{C}$. Then the operator $T : \text{span}\{f_k\}_{k=1}^\infty \rightarrow \text{span}\{f_k\}_{k=1}^\infty$ such that $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ is unbounded.*

Proof. Assume that T is bounded and extend T by continuity on the closed linear span of $\{f_k\}_{k=1}^\infty$. We split the argument in two cases:

- 1) If $i_0 = N$, then $f_{N+1} = T f_N = T \sum_{k=N+1}^\infty c_k f_k = \sum_{k=N+1}^\infty c_k f_{k+1}$, which contradicts $\{f_k\}_{k=N+1}^\infty$ being ω -independent.
- 2) If $i_0 < N$, then

$$\begin{aligned} f_{i_0+1} &= T f_{i_0} = T \sum_{k=N+1}^\infty c_k f_k = \sum_{k=N+1}^\infty c_k f_{k+1} \\ f_{i_0+2} &= T f_{i_0+1} = T \sum_{k=N+1}^\infty c_k f_{k+1} = \sum_{k=N+1}^\infty c_k f_{k+2} \\ &\vdots \\ f_{N+1} &= T f_N = T \sum_{k=N+1}^\infty c_k f_{k+N-i_0} = \sum_{k=N+1}^\infty c_k f_{k+N-i_0+1} \end{aligned}$$

which contradicts $\{f_k\}_{k=N+1}^\infty$ being ω -independent. \square

Corollary 2.11. *Let $\{f_k\}_{k=1}^\infty$ be a linearly independent sequence in \mathcal{H} containing a Schauder basis $\{f_k\}_{k=N+1}^\infty$ for \mathcal{H} for some $N \in \mathbb{N}$. Then the operator $T : \text{span}\{f_k\}_{k=1}^\infty \rightarrow \text{span}\{f_k\}_{k=1}^\infty$ such that $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ is unbounded.*

3. Frames and the Carleson condition

In this section we consider a class of frames which can be represented via bounded operators. The construction first appeared in Lemma 3.17 in [2]; our purpose is to provide a different proof, which only relies on a single result by Shapiro and Shields [13] and standard frame theory. The key ingredient is the so-called Carleson condition on a sequence $\{\lambda_k\}_{k=1}^\infty$ in the open unit disc, which we discuss first.

3.1. The Carleson condition. Let \mathbb{D} denote the open unit disc in the complex plane. The Hardy space $H^2(\mathbb{D})$ is defined by

$$H^2(\mathbb{D}) := \left\{ f : \mathbb{D} \rightarrow \mathbb{C} \mid f(z) = \sum_{n=0}^{\infty} a_n z^n, \{a_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0) \right\}.$$

The Hardy space $H^2(\mathbb{D})$ is a Hilbert space; given $f, g \in H^2(\mathbb{D})$, $f = \sum_{n=0}^{\infty} a_n z^n$, $g = \sum_{n=0}^{\infty} b_n z^n$, the inner product is defined by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n.$$

Note that $\{z^n\}_{n=0}^{\infty}$ is an orthonormal basis for $H^2(\mathbb{D})$; denoting the canonical basis for $\ell^2(\mathbb{N})$ by $\{\delta_n\}_{n=1}^{\infty}$, the operator $\theta : H^2(\mathbb{D}) \rightarrow \ell^2(\mathbb{N})$ defined by $\theta z^n = \delta_{n+1}$ for $n = 0, 1, \dots$ is a unitary operator from $H^2(\mathbb{D})$ onto $\ell^2(\mathbb{N})$.

Definition 3.1. *A sequence $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{D}$ satisfies the Carleson condition if*

$$\inf_{n \in \mathbb{N}} \prod_{k \neq n} \frac{|\lambda_k - \lambda_n|}{|1 - \overline{\lambda_k} \lambda_n|} > 0. \quad (3.1)$$

Note that at some places in the literature, another terminology is used and a sequence $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{D}$ is said to be *uniformly separated* if (3.1) holds.

For a given sequence $\Lambda = \{\lambda_k\}_{k=1}^\infty \subset \mathbb{D}$, define the sequence-valued operator Φ_Λ by

$$\Phi_\Lambda f = \{f(\lambda_k) \sqrt{1 - |\lambda_k|^2}\}_{k=1}^\infty, \quad f \in H^2(\mathbb{D}). \quad (3.2)$$

Note that the sequence in (3.2) does not necessarily belong to $\ell^2(\mathbb{N})$. In [13], Shapiro and Shields proved the following result.

Proposition 3.2. *A sequence $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{D}$ satisfies the Carleson condition if and only if $\ell^2(\mathbb{N}) = \Phi_\Lambda H^2(\mathbb{D})$; in the affirmative case, Φ_Λ is bounded.*

Corollary 3.3. *The Carleson condition implies that $\sum_{k=1}^\infty (1 - |\lambda_k|^2) < \infty$ and thus $\lim_{k \rightarrow \infty} |\lambda_k| = 1$.*

Proof. Assume that $\Lambda = \{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition. Since the function $f(z) = 1, z \in \mathbb{D}$, is in $H^2(\mathbb{D})$, it follows from Proposition 3.2 that $\Phi_\Lambda f \in \ell^2(\mathbb{N})$. In other words, $\sum_{k=1}^\infty (1 - |\lambda_k|^2) < \infty$. \square

The following result (see, e.g., [10, Thm. 9.2]) gives an easy verifiable criterion for the Carleson condition to hold.

Proposition 3.4. *Let $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{D}$ be a sequence of distinct numbers. If*

$$\exists c \in (0, 1) \text{ such that } \frac{1 - |\lambda_{k+1}|}{1 - |\lambda_k|} \leq c < 1, \quad \forall k \in \mathbb{N}, \quad (3.3)$$

then $\{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition. If $\{\lambda_k\}_{k=1}^\infty$ is positive and increasing, then the condition (3.3) is also necessary for $\{\lambda_k\}_{k=1}^\infty$ to satisfy the Carleson condition.

Corollary 3.5. *For every $\alpha > 1$, the sequence $\{\lambda_k\}_{k=1}^\infty = \{1 - \alpha^{-k}\}_{k=1}^\infty$ satisfies the Carleson condition.*

Proof. Let $\alpha > 1$. The sequence $\{1 - \alpha^{-k}\}_{k=1}^\infty$ is positive and increasing. Furthermore, for every $k \in \mathbb{N}$ we have

$$\frac{1 - \lambda_{k+1}}{1 - \lambda_k} = \frac{\alpha^{-k-1}}{\alpha^{-k}} = \frac{1}{\alpha} < 1.$$

Thus, by Proposition 3.4, $\{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition. \square

The following lemma collects results about modifications on a sequence that preserve the Carleson condition.

Lemma 3.6. *Given any sequence $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{D}$, the following hold:*

- (i) *If $\{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition, then every subsequence of $\{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition.*
- (ii) *If $\lambda_k \neq \lambda_j$ for $k \neq j$ and there is some $n \in \mathbb{N}$ such that $\{\lambda_k\}_{k \geq n}$ satisfies the Carleson condition, then also $\{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition.*
- (iii) *If $\{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition and $\lambda_k \in [0, 1[$ for all $k \in \mathbb{N}$, then for every $\ell \in \mathbb{N}$ the sequence $\{\lambda_k^{1/\ell}\}_{k=1}^\infty$ also satisfies the Carleson condition.*

Proof. (i) is straightforward and (ii) is stated in [5]. To prove (iii), it is enough to show that for any $x, y \in]0, 1[$ we have

$$\frac{x^{1/\ell} - y^{1/\ell}}{1 - x^{1/\ell}y^{1/\ell}} \geq \frac{x - y}{1 - xy}. \quad (3.4)$$

Using the identity $a^\ell - b^\ell = (a - b) \sum_{i=0}^{\ell-1} a^{\ell-i-1} b^i$, we obtain that

$$\frac{x^{1/\ell} - y^{1/\ell}}{1 - x^{1/\ell}y^{1/\ell}} = \frac{x - y}{1 - xy} f(x, y) \quad (3.5)$$

where

$$f(x, y) = \frac{\sum_{i=0}^{\ell-1} (x^{1/\ell} y^{1/\ell})^i}{\sum_{j=0}^{\ell-1} (x^{1/\ell})^{\ell-j-1} (y^{1/\ell})^j}. \quad (3.6)$$

Fixing $x \in]0, 1[$, a direct calculation shows that $\frac{\partial}{\partial y} f(x, y) < 0$ for all $y \in]0, 1[$; thus $f(x, y)$ is decreasing with respect to y . Since $\lim_{y \rightarrow 1} f(x, y) = 1$, we conclude that $f(x, y) \geq 1$ for all $x, y \in]0, 1[$. Using (3.5), this proves that (3.4) holds. \square

3.2. Frame properties and the Carleson condition. In this section we provide an alternative proof of Lemma 3.19 in [2], which yields a construction of a class of operators $T : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ for which $\{T^n h\}_{n=0}^\infty$ is a frame for $\ell^2(\mathbb{N})$ for certain sequences $h \in \ell^2(\mathbb{N})$. We will formulate the result in the setting of a general Hilbert space. The original proof of Lemma 3.19 in [2] uses properties of the Gramian associated with a sequence in the underlying Hilbert space, as well as interpolating sequences in the Hardy space. We will base our proof on the more elementary fact that a Bessel sequence is a frame if and only if the frame operator is surjective, and phrase the interpolation property directly in terms of surjectivity of the operator Φ_Λ in Proposition 3.2.

Consider a sequence $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{D}$ and assume that $\{\sqrt{1 - |\lambda_k|^2}\}_{k=1}^\infty \in \ell^2(\mathbb{N})$. Given any separable Hilbert space \mathcal{H} , choose an orthonormal basis $\{e_k\}_{k=1}^\infty$ and consider the bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ for which $T e_k = \lambda_k e_k$. Let $h := \sum_{k=1}^\infty \sqrt{1 - |\lambda_k|^2} e_k$ and consider the iterated system

$$\{T^n h\}_{n=0}^\infty = \left\{ \sum_{k=1}^\infty \lambda_k^n \sqrt{1 - |\lambda_k|^2} e_k \right\}_{n=0}^\infty. \quad (3.7)$$

We will now state the mentioned result from [2]. Our proof is only based on Proposition 3.2 and standard frame theory.

Theorem 3.7. *Let $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{D}$ and assume that $\{\sqrt{1 - |\lambda_k|^2}\}_{k=1}^\infty \in \ell^2(\mathbb{N})$. Then the sequence $\{T^n h\}_{n=0}^\infty$ in (3.7) is a frame for \mathcal{H} if and only if $\{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition.*

Proof. Define formally the synthesis operator $V : \ell^2(\mathbb{N}_0) \rightarrow \mathcal{H}$ by $V\{a_n\}_{n=0}^\infty = \sum_{n=0}^\infty a_n T^n h$. By Theorem 5.5.1 in [6], the sequence $\{T^n h\}_{n=0}^\infty$ is a frame for \mathcal{H} if and only if the operator V is well-defined and surjective.

First assume that $\{T^n h\}_{n=0}^\infty$ is a frame for \mathcal{H} . For each $\{c_j\}_{j=1}^\infty \in \ell^2(\mathbb{N})$ the surjectivity of V implies that there exists $\{a_n\}_{n=0}^\infty$ in $\ell^2(\mathbb{N}_0)$ such that $\sum_{n=0}^\infty a_n T^n h = \sum_{j=1}^\infty c_j e_j$. It follows that for each $k \in \mathbb{N}$,

$$c_k = \left\langle \sum_{j=1}^\infty c_j e_j, e_k \right\rangle = \sum_{n=0}^\infty a_n \langle T^n h, e_k \rangle = \sum_{n=0}^\infty a_n \lambda_k^n \sqrt{1 - |\lambda_k|^2}. \quad (3.8)$$

Defining $f \in H^2(\mathbb{D})$ by $f(z) = \sum_{n=0}^\infty a_n z^n$, the equation (3.8) turns into $f(\lambda_k) \sqrt{1 - |\lambda_k|^2} = c_k$. Formulated in terms of the operator Φ_Λ in (3.2), this

means that $\ell^2(\mathbb{N}) \subseteq \Phi_\Lambda H^2(\mathbb{D})$. On the other hand, take an arbitrary $f \in H^2(\mathbb{D})$ and choose $\{a_n\}_{n=0}^\infty \in \ell^2(\mathbb{N}_0)$ such that $f(z) = \sum_{n=0}^\infty a_n z^n$. For every $k \in \mathbb{N}$, we have

$$\begin{aligned} \langle V\{a_n\}_{n=0}^\infty, e_k \rangle &= \left\langle \sum_{n=0}^\infty a_n \sum_{j=1}^\infty \lambda_j^n \sqrt{1 - |\lambda_j|^2} e_j, e_k \right\rangle = \sum_{n=0}^\infty a_n \lambda_k^n \sqrt{1 - |\lambda_k|^2} \\ &= \langle \Phi_\Lambda f, \delta_k \rangle. \end{aligned}$$

Therefore, $\Phi_\Lambda f = \{\langle V\{a_n\}_{n=0}^\infty, e_k \rangle\}_{k=1}^\infty \in \ell^2(\mathbb{N})$ and hence, $\Phi_\Lambda H^2(\mathbb{D}) \subseteq \ell^2(\mathbb{N})$. Thus we get $\Phi_\Lambda H^2(\mathbb{D}) = \ell^2(\mathbb{N})$, which by Proposition 3.2 implies that $\{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition.

Conversely, assume that the sequence $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{D}$ satisfies the Carleson condition. We first show that $\sum_{n=0}^\infty a_n T^n h$ is convergent for all $\{a_n\}_{n=0}^\infty \in \ell^2(\mathbb{N}_0)$. Let $\{a_n\}_{n=0}^\infty \in \ell^2(\mathbb{N}_0)$, and define $f \in H^2(\mathbb{D})$ by $f(z) := \sum_{n=0}^\infty a_n z^n$. By Proposition 3.2 we know that $\Phi_\Lambda H^2(\mathbb{D}) = \ell^2(\mathbb{N})$, so $\{f(\lambda_k) \sqrt{1 - |\lambda_k|^2}\}_{k=1}^\infty \in \ell^2(\mathbb{N})$. Now for $N \in \mathbb{N}$, consider the truncated sequence $\{a_n\}_{n=0}^N := \{a_1, a_2, \dots, a_N, 0, 0, \dots\}$ and the associated function $f_N \in H^2(\mathbb{D})$ given by $f_N(z) := \sum_{n=0}^N a_n z^n$. Again, $\{f_N(\lambda_k) \sqrt{1 - |\lambda_k|^2}\}_{k=1}^\infty \in \ell^2(\mathbb{N})$, and since $\Phi_\Lambda : H^2(\mathbb{D}) \rightarrow \ell^2(\mathbb{N})$ is bounded by Proposition 3.2, there is a constant $C > 0$ such that

$$\|\Phi_\Lambda f - \Phi_\Lambda f_N\|^2 \leq C \|f - f_N\|^2 = C \sum_{n=N+1}^\infty |a_n|^2 \rightarrow 0 \text{ as } N \rightarrow \infty.$$

It follows that

$$\begin{aligned} \sum_{n=0}^N a_n T^n h &= \sum_{n=0}^N a_n \sum_{k=1}^\infty \sqrt{1 - |\lambda_k|^2} \lambda_k^n e_k = \sum_{k=1}^\infty \sqrt{1 - |\lambda_k|^2} \sum_{n=0}^N a_n \lambda_k^n e_k \\ &= \sum_{k=1}^\infty \sqrt{1 - |\lambda_k|^2} f_N(\lambda_k) e_k \rightarrow \sum_{k=1}^\infty f(\lambda_k) \sqrt{1 - |\lambda_k|^2} e_k \text{ as } N \rightarrow \infty. \end{aligned}$$

This proves that $\sum_{n=0}^\infty a_n T^n h$ is convergent as claimed, and thus V is well defined from $\ell^2(\mathbb{N}_0)$ into \mathcal{H} . In order to prove that $\{T^n h\}_{n=0}^\infty$ is frame, it is now enough to show that the synthesis operator $V : \ell^2(\mathbb{N}_0) \rightarrow \mathcal{H}$ is surjective. Let $x \in \mathcal{H}$. By Proposition 3.2, there is an $f \in H^2(\mathbb{D})$ such that $f(\lambda_k) \sqrt{1 - |\lambda_k|^2} = \langle x, e_k \rangle$ for all $k \in \mathbb{N}$. Choose $\{a_n\}_{n=0}^\infty \in \ell^2(\mathbb{N}_0)$ such that $f(z) = \sum_{n=0}^\infty a_n z^n$. Then for each $k \in \mathbb{N}$,

$$\begin{aligned} \langle V\{a_n\}_{n=0}^\infty, e_k \rangle &= \left\langle \sum_{n=0}^\infty a_n T^n h, e_k \right\rangle = \sum_{n=0}^\infty a_n \left\langle \sum_{j=1}^\infty \lambda_j^n \sqrt{1 - |\lambda_j|^2} e_j, e_k \right\rangle \\ &= \sum_{n=0}^\infty a_n \lambda_k^n \sqrt{1 - |\lambda_k|^2} = f(\lambda_k) \sqrt{1 - |\lambda_k|^2} = \langle x, e_k \rangle \end{aligned}$$

Therefore $V\{a_n\}_{n=0}^\infty = x$ and thus V is surjective, as desired. \square

Example 3.8. Assume that $\{\lambda_k\}_{k=1}^\infty \subset]0, 1[$ satisfies the Carleson condition. By Theorem 3.7, the sequence $\{T^n h\}_{n=0}^\infty$ defined in (3.7) is a frame for \mathcal{H} . For a fixed $\ell \in \mathbb{N}$, define the operator T_ℓ by $T_\ell e_k = \lambda_k^{1/\ell} e_k$, for all $k \in \mathbb{N}$ and extend it to a bounded operator on \mathcal{H} . Then

$$\{T_\ell^n h\}_{n=0}^\infty = \bigcup_{r=0}^{\ell-1} \{T_\ell^{n\ell+r} h\}_{n=0}^\infty = \bigcup_{r=0}^{\ell-1} \{T_\ell^r T^n h\}_{n=0}^\infty.$$

Since each of the operators T_ℓ^r are surjective and $\{T^n h\}_{n=0}^\infty$ is a frame, it follows that the sequence $\{T_\ell^n h\}_{n=0}^\infty$ is also a frame. \square

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