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Criteria for generalized translation-invariant frames

by

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Abstract. This paper provides new sufficient and necessary conditions for the frame property of generalized translation-invariant systems. The conditions are formulated in the Fourier domain and consist of estimates involving the upper and lower frame bound. In contrast to known conditions of a similar nature, the estimates take the phase of the generating functions into consideration and not only their modulus. The possibility of phase cancellations makes these estimates optimal for tight frames. The results on generalized translation-invariant systems will be proved in the setting of locally compact abelian groups, but even for Euclidean space and the special case of wavelet and shearlet systems the results are new.

1. Introduction. Deriving sufficient and necessary conditions for the frame property of structured function systems has a long history in time-frequency and time-scale analysis. In this paper we study a class of structured function systems known as generalized translation-invariant systems. These offer a common framework for discrete and continuous structured function systems such as Gabor systems, wave packet systems, wavelets, shearlets, and curvelets, but are also of independent interest.

1.1. Overview and contributions. The paper aims to derive necessary and sufficient conditions for the frame and the Bessel property of generalized translation-invariant systems that are based on properties of the generating functions in the Fourier domain. The first results similar in nature go back to the very beginning of modern frame theory and the influential papers by Daubechies [19] and Daubechies, Grossmann and Meyer [21]. In [19], Daubechies provides general conditions on the generators and parameters of Gabor and wavelet systems to form a Bessel system or a frame for $L^2(\mathbb{R})$.

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These fundamental results attracted the attention of several groups of researchers [12, 15, 16, 27, 36, 45, 48, 51] and led to improvements and generalizations over the subsequent decades, whose precise nature is discussed in Section 1.2.

For clarity, in the remainder of this introduction, we focus on a subclass of *discrete* function systems, called generalized shift-invariant systems. In the setting of a locally compact abelian group G , written additively, a *generalized shift-invariant system* in $L^2(G)$ is a countable union of the form

$$\bigcup_{j \in J} \{g_j(\cdot - \gamma) : \gamma \in \Gamma_j\}$$

for a family $\{\Gamma_j\}_{j \in J}$ of discrete, co-compact subgroups in G and a family $\{g_j\}_{j \in J}$ of functions in $L^2(G)$. Further, a generalized shift-invariant system $\bigcup_{j \in J} \{g_j(\cdot - \gamma)\}_{\gamma \in \Gamma_j}$ in $L^2(G)$ is called a *frame* for $L^2(G)$ whenever there exist constants $A, B > 0$, called *frame bounds*, such that

$$(1.1) \quad A\|f\|_2^2 \leq \sum_{j \in J} \sum_{\gamma \in \Gamma_j} |\langle f, g_j(\cdot - \gamma) \rangle|^2 \leq B\|f\|_2^2$$

for all $f \in L^2(G)$. A family $\bigcup_{j \in J} \{g_j(\cdot - \gamma)\}_{\gamma \in \Gamma_j}$ satisfying the upper frame bound is called a *Bessel sequence*, and a frame for which the frame bounds can be chosen equal is called *tight*.

Frames in $L^2(G)$ are of interest in applications in, e.g., signal analysis and functional analysis, as they guarantee unconditionally L^2 -convergent and stable expansions of functions in $L^2(G)$. Indeed, for a given frame $\bigcup_{j \in J} \{g_j(\cdot - \gamma)\}_{\gamma \in \Gamma_j}$ in $L^2(G)$, there exists a system $\bigcup_{j \in J} \{\tilde{g}_{j,\gamma}\}_{\gamma \in \Gamma_j}$ such that every function $f \in L^2(G)$ has an expansion of the form

$$(1.2) \quad f = \sum_{j \in J} \sum_{\gamma \in \Gamma_j} \langle f, \tilde{g}_{j,\gamma} \rangle g_j(\cdot - \gamma) = \sum_{j \in J} \sum_{\gamma \in \Gamma_j} \langle f, g_j(\cdot - \gamma) \rangle \tilde{g}_{j,\gamma}$$

with unconditional norm convergence.

Verifying the frame inequalities (1.1) directly is often an impossible task. However, for many special cases simple sufficient and necessary conditions for the frame property are known. The new criteria presented in this paper will be derived under two unconditional convergence properties (called p -UCP for $p = 1, \infty$) that will be introduced in Definition 2.2. The p -UCP is a mild convergence property that guarantees almost periodicity of an auxiliary function w_f , essential to our analysis, that will be introduced in Section 2.2. The ∞ -UCP will simply be assumed throughout the remainder of this subsection; here, for simplicity, we also assume that the Lebesgue differentiation theorem holds on \hat{G} . The frame criteria are phrased as estimates involving

the functions $t_\alpha : \widehat{G} \rightarrow \mathbb{C}$, $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$, defined, whenever convergent, as

$$t_\alpha(\omega) = \sum_{\{j \in J : \alpha \in \Gamma_j^\perp\}} \frac{1}{d(\Gamma_j)} \hat{g}_j(\omega) \overline{\hat{g}_j(\omega + \alpha)} \quad \text{for a.e. } \omega \in \widehat{G},$$

where Γ_j^\perp is the dual lattice of Γ_j , and $d(\Gamma_j)$ denotes the covolume of Γ_j .

Now, the sufficient condition presented in Theorem 3.5 implies that a system $\bigcup_{j \in J} \{g_j(\cdot - \gamma)\}_{\gamma \in \Gamma_j}$ forms a frame for $L^2(G)$ with bounds A_1 and B_1 if

$$(1.3) \quad B_1 := \operatorname{ess\,sup}_{\omega \in \widehat{G}} \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} |t_\alpha(\omega)| < \infty,$$

$$(1.4) \quad A_1 := \operatorname{ess\,inf}_{\omega \in \widehat{G}} \left(t_0(\omega) - \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)| \right) > 0.$$

The subscript p is used in the constants A_p and B_p to indicate the relation to the ℓ_p -norm of the sequence $\{t_\alpha(\omega)\}_\alpha$ for a.e. $\omega \in \widehat{G}$. In contrast to previously known sufficient conditions for generalized shift-invariant systems, the estimates (1.3) and (1.4) take the phase of the generating functions into consideration and not only their modulus. To be more precise, the previously known sufficient conditions are not based on the functions $t_\alpha : \widehat{G} \rightarrow \mathbb{C}$, but on the non-negative functions

$$\widehat{G} \ni \omega \mapsto \sum_{\{j \in J : \alpha \in \Gamma_j^\perp\}} \frac{1}{d(\Gamma_j)} |\hat{g}_j(\omega) \hat{g}_j(\omega + \alpha)| \in [0, \infty[.$$

By considering the phase of the generating functions, the estimates (1.3) and (1.4) allow, for each $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$, for phase cancellations in the sum over $\{j \in J : \alpha \in \Gamma_j^\perp\}$ and therefore often lead to improvements of the known estimates; see, e.g., Example 3.9 for an orthonormal basis demonstrating this. In fact, the estimates (1.3) and (1.4) are optimal for tight frames in the sense that they recover precisely the frame bound.

The necessary condition presented in Theorem 3.2 implies that if a system $\bigcup_{j \in J} \{g_j(\cdot - \gamma)\}_{\gamma \in \Gamma_j}$ is a Bessel sequence in $L^2(G)$ with upper frame bound B , then

$$(1.5) \quad B \geq B_2 := \operatorname{ess\,sup}_{\omega \in \widehat{G}} \left(\sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} |t_\alpha(\omega)|^2 \right)^{1/2}.$$

Combining this with the known fact that for a frame $\bigcup_{j \in J} \{g_j(\cdot - \gamma)\}_{\gamma \in \Gamma_j}$ for $L^2(G)$ with lower bound $A > 0$ necessarily $A_\infty := \operatorname{ess\,inf}_{\omega \in \widehat{G}} t_0(\omega) \geq A$, we deduce that $A \leq A_\infty \leq B_2 \leq B$ is necessary for a frame $\bigcup_{j \in J} \{g_j(\cdot - \gamma)\}_{\gamma \in \Gamma_j}$ with bounds A and B .

For the applicability of the frame expansions (1.2), it is not only essential to verify the frame inequalities (1.1), but also to provide good estimates of the frame bounds. The necessary and sufficient conditions obtained yield together *frame bound estimates* for generalized shift-invariant systems. Indeed, for a Bessel system $\bigcup_{j \in J} \{g_j(\cdot - \gamma)\}_{\gamma \in \Gamma_j}$ with optimal upper bound $B > 0$, the bound B can be estimated by the snug bounds

$$(1.6) \quad B_2 \equiv \operatorname{ess\,sup}_{\omega \in \widehat{G}} \|\{t_\alpha(\omega)\}_\alpha\|_{\ell^2} \leq B \leq \operatorname{ess\,sup}_{\omega \in \widehat{G}} \|\{t_\alpha(\omega)\}_\alpha\|_{\ell^1} \equiv B_1.$$

If, furthermore, $\bigcup_{j \in J} \{g_j(\cdot - \gamma)\}_{\gamma \in \Gamma_j}$ is a frame in $L^2(G)$ with optimal lower bound $A > 0$, then

$$(1.7) \quad A_1 \leq A \leq A_\infty.$$

The results presented hold not only for discrete frames and systems as discussed above, but also for their continuous and semicontinuous counterparts. To summarize, the main contributions of the paper are new necessary and sufficient conditions for the frame property and the Bessel property of generalized translation-invariant systems that are (i) derived under minimal assumptions, (ii) optimal for tight frames, (iii) verifiable and computable, and that provide (iv) snug frame bound estimates which collapse to equality for tight frames.

1.2. Related work. For shift-invariant systems, i.e., generalized shift-invariant systems with a single, fixed translation lattice Γ , the Bessel and frame properties can be *characterized* in terms of (bi-infinite) matrix-valued functions, known as *dual Gramian matrices*, as introduced by Ron and Shen [44] (see also Janssen [35]). Consequently, the aforementioned necessary and sufficient conditions in (1.3)–(1.5) can be derived ⁽¹⁾ from simple norm estimates of bi-infinite Hermitian matrices $M = (m_{i,j})_{i,j \in \Gamma^\perp}$ on $\ell^2(\Gamma^\perp)$ (see [44, Section 1.6]). In particular, the estimates (1.6) and (1.7) are known for separable Gabor systems [34, 48]; these estimates are the best-known improvement of Daubechies' Gabor frame bound estimates [19]. Furthermore, the dual Gramian characterization has, in a fiberization formulation [4], been extended to the setting of locally compact abelian groups [8, 11, 31]. Hence, for shift-invariant systems, or more generally, translation-invariant systems on such groups, conditions (1.3)–(1.5) follow from these characterizations and should not be considered new.

For function systems that are not shift-invariant, the fiberization characterization breaks down. In spite of this, Ron and Shen [49] obtained dual

⁽¹⁾ For example, the necessary condition in (1.5) for shift-invariant systems follows from the norm estimate $\|M\| \geq (\sum_{j \in \Gamma^\perp} |m_{0,j}|^2)^{1/2}$ by noticing that the 0th column of the dual Gramian matrix at $\omega \in \widehat{G}$ is $\{t_\alpha(\omega)\}_{\alpha \in \Gamma^\perp}$.

Gramian-type ⁽²⁾ characterizations for special types of generalized shift-invariant systems in $L^2(\mathbb{R}^d)$. For example, for generalized shift-invariant systems satisfying the *finite intersection (FI) condition* (i.e., the intersection of any finite subfamily of the lattices $\{\Gamma_j\}_{j \in J}$ is a full-rank lattice), the Bessel property can be characterized by the norm of the dual Gramian matrices since the FI condition essentially reduces the analysis to standard dual Gramian analysis. On the other hand, many generalized shift-invariant systems violate the FI condition, e.g., systems with both rational and non-rational lattices. For lower frame bound characterizations by dual Gramian analysis additional restrictions on the lattices and generators are needed, most notably the small tail assumption. To handle wavelet systems associated with expansive, but not necessarily integer, matrix dilations, other assumptions on the family of lattices and generators are made such as temperateness and roundedness in generalized shift-invariant systems (see [49] for definitions). However, we stress that none of the assumptions used in [49] are weak enough to allow for dual Gramian characterization of wavelet frames associated with arbitrary real, expansive dilations.

An alternative route for deriving necessary and sufficient conditions for wavelet frames with integer, expansive dilations goes through quasi-affine systems [18, 46, 47]. This link is known to generalize to rational, expansive dilations [6], although one has to consider a family of quasi-affine systems to capture the frame property of the given wavelet system. Since quasi-affine systems are shift-invariant, sufficient and necessary conditions for rational wavelet systems are readily available. We stress that such estimates differ slightly from the ones given in this paper. The estimates given in [45] for wavelet systems with integer, expansive dilations utilize the quasi-affine route, but they ignore the phase of the wavelet generator and are therefore not optimal for tight frames.

The approach we follow relies on a connection between the frame properties of generalized shift-invariant systems and an associated almost periodic auxiliary function [28, 33, 41, 51]. Our methods are closely related to the work of Hernández, Labate and Weiss [28], but while [28] is concerned with tight frame characterization using uniqueness of the coefficients of almost periodic Fourier series, we focus on non-tight frames by bounding the Fourier series. The connection with Fourier analysis is valid under the 1-UCP, which is weak enough to provide sufficient and necessary conditions for wavelet frames in $L^2(\mathbb{R}^d)$ associated with any real, expansive dilation matrix and any translation lattice. The 1-UCP is even weak

⁽²⁾ For generalized shift-invariant systems the direct link using *one* dual Gramian matrix for each fiber has to be replaced by a less direct link of infinite families of finite matrices for each fiber.

enough to handle every choice of real, invertible dilation (not necessarily expansive) almost surely with respect to the Haar measure on $\mathrm{GL}_d(\mathbb{R})$ (see Section 4.2).

To wrap up the discussion, no necessary or sufficient conditions, optimal for tight frames, are currently known for wavelet systems associated with expansive, real dilations. In fact, the lack of optimal frame bound estimates for such systems led Christensen [14] to ask whether sufficient conditions as in (1.3) and (1.4) can also be obtained for wavelet systems with non-integer dilations. The sufficient conditions found in this paper answer this question in the affirmative.

1.3. Outline. The paper is organized as follows. In Section 2 we introduce generalized translation-invariant systems and the 1-UCP condition in the setting of locally compact abelian groups. The main results on generalized translation-invariant systems are presented in Section 3. Necessary and sufficient conditions for generalized translation-invariant frames are contained in Sections 3.1 and 3.2, respectively. In Section 3.3 we compare the frame bound estimates obtained with known estimates. Section 4 is devoted to applications and examples. Gabor systems and wavelet systems are considered in Sections 4.1 and 4.2, respectively. Finally, we consider composite wavelet and cone-adapted shearlet systems in Section 4.3, and we derive new frame characterizations of the continuous ℓ th order α -shearlet transform in Section 4.4.

2. Generalized translation-invariant systems. Throughout this paper, G will denote a second countable locally compact abelian group. The character group of G is denoted by \widehat{G} and forms a second countable locally compact abelian group itself. The group operation in both G and \widehat{G} is written additively as $+$ and the identity element is denoted by 0 . The Haar measure on G will be denoted by μ_G . It is assumed that the Haar measure on G is given and that the Haar measure on \widehat{G} is the Plancherel measure. The subset $\Gamma \subseteq G$ will be a closed, co-compact subgroup of G , i.e., the quotient space G/Γ is compact. In this case, the annihilator Γ^\perp of Γ is the countable, discrete subgroup $\Gamma^\perp := \{\omega \in \widehat{G} : \omega(x) = 0, \forall x \in \Gamma\}$. It is assumed that the Haar measure on Γ is given and that the Haar measure on G/Γ is the unique quotient measure provided by Weil's integral formula. Using this quotient measure $\mu_{G/\Gamma}$ on G/Γ , the *covolume* or *size* of the subgroup $\Gamma \subseteq G$ is defined as $d(\Gamma) := \mu_{G/\Gamma}(G/\Gamma)$.

2.1. Generalized translation-invariant frames. The function systems defined next form the central object of this paper. Here, the translate of a function $f \in L^2(G)$ by $y \in G$ is denoted as $T_y f := f(\cdot - y)$.

DEFINITION 2.1. Let J be a countable index set. For each $j \in J$, let $\Gamma_j \subseteq G$ be a closed, co-compact subgroup, and let P_j be an arbitrary (countable or uncountable) index set. For a given family $\bigcup_{j \in J} \{g_{j,p}\}_{p \in P_j} \subset L^2(G)$ of functions, the collection

$$\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$$

of translates is called a *generalized translation-invariant (GTI) system* in $L^2(G)$.

We stress that the translation subgroups $\Gamma_j \subset G$, $j \in J$, are not assumed to be discrete, but merely closed and co-compact. In $G = \mathbb{R}^d$ (and $\widehat{G} = \mathbb{R}^d$), each closed, co-compact subgroup Γ_j is of the form $C(\mathbb{Z}^k \times \mathbb{R}^{d-k})$ for some $0 \leq k \leq d$ and $C \in \text{GL}_d(\mathbb{R})$. The (discrete) annihilator Γ_j^\perp is then given by $(C^T)^{-1}(\mathbb{Z}^k \times \{0\}^{d-k})$.

Following [32], it is assumed that the generating functions of a generalized translation-invariant systems satisfy the following three *standing hypotheses*. For each $j \in J$:

- (I) The triple $(P_j, \Sigma_{P_j}, \mu_{P_j})$ forms a σ -finite measure space.
- (II) The mapping $p \mapsto g_{j,p}$ from (P_j, Σ_{P_j}) into $(L^2(G), \mathcal{B}_{L^2(G)})$ is Σ_{P_j} -measurable, where $\mathcal{B}_{L^2(G)}$ denotes the Borel σ -algebra on $L^2(G)$.
- (III) The mapping $(p, x) \mapsto g_{j,p}(x)$ from $(P_j \times G, \Sigma_{P_j} \otimes \mathcal{B}_G)$ into $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ is $(\Sigma_{P_j} \otimes \mathcal{B}_G)$ -measurable, where \mathcal{B}_G denotes the Borel σ -algebra on G .

For most applications in this paper, it suffices to take $\{P_j\}_{j \in J}$ to be countable index sets equipped with the counting measure (cf. Section 4). In this case, the three standing hypotheses (I)–(III) are automatically satisfied.

A generalized translation-invariant system $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ will be called a *generalized translation-invariant frame* for $L^2(G)$, with respect to $\{L^2(P_j \times \Gamma_j) : j \in J\}$, whenever there exist constants $A, B > 0$, called the *frame bounds*, such that

$$(2.1) \quad A\|f\|^2 \leq \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} |\langle f, T_\gamma g_{j,p} \rangle|^2 d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p) \leq B\|f\|^2$$

for all $f \in L^2(G)$. A generalized translation-invariant system satisfying the upper frame bound is called a *Bessel system* or a *Bessel family* in $L^2(G)$. For such a Bessel system, the frame operator $S : L^2(G) \rightarrow L^2(G)$ associated with $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is defined weakly by equating $\langle Sf, f \rangle$ with the central term of (2.1).

In order to check whether a generalized translation-invariant system forms a Bessel system or a frame for $L^2(G)$, it suffices to check the frame condition on a dense subspace of $L^2(G)$. Let \mathcal{E} denote the set of all closed

Borel sets $E \subseteq \widehat{G}$ satisfying $\mu_{\widehat{G}}(E) = 0$. For a fixed $E \in \mathcal{E}$, define the dense subspace $\mathcal{D}_E(G)$ of $L^2(G)$ as

$$\mathcal{D}_E(G) = \{f \in L^2(G) : \hat{f} \in L^\infty(\widehat{G}) \text{ and} \\ \exists K \subset \widehat{G} \setminus E \text{ compact with } \hat{f}\mathbf{1}_K = \hat{f} \text{ a.e.}\},$$

where \hat{f} denotes the Fourier transform of $f \in L^2(G)$ and $\mathbf{1}_K$ the characteristic function on K . Every function $f \in \mathcal{D}_E(G)$ has the property that \hat{f} is zero almost everywhere on $E + B(0, \delta)$ for some $\delta > 0$, where $B(0, \delta)$ denotes the open ball of radius δ , since $\text{supp } \hat{f}$ is compact, E is closed, and $\text{supp } \hat{f} \cap E = \emptyset$.

We consider the set $E \in \mathcal{E}$, called the *blind spot*, as fixed, but arbitrary. The actual choice of E depends on the application considered. For wavelet systems, the blind spot E is usually chosen to be the complement \mathcal{O}^c of the dual orbit $\mathcal{O} \subseteq \widehat{G}$ of the dilation group [23], whereas for Gabor systems it usually suffices to take $E = \emptyset$.

2.2. Unconditional convergence property and Fourier analysis. This section is devoted to the 1-UCP condition and to generalized Fourier series of almost periodic functions essential to our study of generalized translation-invariant frames. Recall that an almost periodic function is the uniform limit of trigonometric polynomials. The space of almost periodic functions is denoted by $\text{AP}(G)$.

Given a generalized translation-invariant system $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and a function $f \in \mathcal{D}_E(G)$, define, for $j \in J$, the map $w_{f,j} : G \rightarrow [0, \infty]$ by

$$(2.2) \quad w_{f,j}(x) = \int_{P_j} \int_{\Gamma_j} |\langle T_x f, T_\gamma g_{j,p} \rangle|^2 d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p).$$

By the standing hypotheses (I)–(III), the integrals in (2.2) are well-defined. Throughout this section, we will further assume that

$$(2.3) \quad \int_{P_j} |\hat{g}_{j,p}(\cdot)|^2 d\mu_{P_j}(p) \in L^1_{\text{loc}}(\widehat{G}).$$

It can be shown, using arguments from [32, 38], that the integrability condition (2.3) ensures that each $w_{f,j}$ is a trigonometric polynomial, and in particular it is Γ_j -periodic, continuous, and bounded.

Next, we define $w_f : G \rightarrow [0, \infty]$ as the following sum of non-negative, trigonometric polynomials:

$$w_f(x) = \sum_{j \in J} w_{f,j}(x) = \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} |\langle T_x f, T_\gamma g_{j,p} \rangle|^2 d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p).$$

The function $w_f : G \rightarrow [0, \infty]$ is well-defined, but it might attain the

value of positive infinity without any further assumptions on the generalized translation-invariant system.

Under a suitable regularity condition on $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$, the function w_f becomes almost periodic. The connection between the almost periodicity of w_f and generalized translation-invariant frames was first used by Laugesen [40, 41] for wavelet systems and extended to arbitrary generalized shift-invariant (GSI) systems in $L^2(\mathbb{R}^d)$ by Hernández, Labate and Weiss [28].

The following regularity condition, sufficiently weak for our purposes, was introduced in [24].

DEFINITION 2.2. Let $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ be a generalized translation-invariant system.

- (i) The generalized translation-invariant system is said to satisfy the *1-unconditional convergence property* (1-UCP) with respect to $E \in \mathcal{E}$ whenever, for all $f \in \mathcal{D}_E(G)$, the function $w_f : G \rightarrow \mathbb{C}$ is almost periodic and the series

$$(2.4) \quad w_f = \sum_{j \in J} w_{f,j}$$

converges unconditionally with respect to the mean $M : \text{AP}(G) \rightarrow \mathbb{C}$,

$$M(|f|) = \lim_{n \rightarrow \infty} \frac{1}{\mu_G(H_n)} \int_{H_n} |f(x)| d\mu_G(x),$$

where $(H_n)_{n \in \mathbb{N}}$ is any increasing sequence of open, relatively compact subsets $H_n \subseteq G$ with $G = \bigcup_{n \in \mathbb{N}} H_n$ and such that

$$\lim_{n \rightarrow \infty} \frac{\mu_G((x + H_n) \cap (G \setminus H_n))}{\mu_G(H_n)} = 0$$

for all $x \in G$.

- (ii) If (2.4) holds with uniform convergence, the generalized translation-invariant system is said to satisfy the *∞ -UCP* with respect to $E \in \mathcal{E}$.

The 1-UCP is the weakest known assumption under which Fourier analysis of w_f can be exploited to study the frame properties of the associated system. Note that the almost periodicity of w_f is assumed in 1-UCP, while in ∞ -UCP it follows from the uniform convergence of (2.4).

Both 1-UCP and ∞ -UCP are automatically satisfied whenever the generalized translation-invariant system $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the local integrability condition (LIC) or the weaker α -local integrability condition (α -LIC) introduced in [28] and [32], respectively. A generalized translation-invariant system is said to satisfy the *α -LIC* with respect to E if

$$\sum_{j \in J} \frac{1}{\mathfrak{d}(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\widehat{G}} |\hat{f}(\omega) \hat{f}(\omega + \alpha) \hat{g}_{j,p}(\omega + \alpha) \hat{g}_{j,p}(\omega)| d\mu_{\widehat{G}}(\omega) d\mu_{P_j}(p) < \infty$$

for all $f \in \mathcal{D}_E(G)$.

The generalized Fourier series of w_f is stated in the next result, adapted from [24].

PROPOSITION 2.3. *Suppose $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the 1-UCP with respect to $E \in \mathcal{E}$. Moreover, suppose that*

$$(2.5) \quad \sum_{j \in J} \frac{1}{\mathfrak{d}(\Gamma_j)} \int_{P_j} |\hat{g}_{j,p}(\cdot)|^2 d\mu_{P_j}(p) \in L^1_{\text{loc}}(\widehat{G} \setminus E).$$

Then, for all $f \in \mathcal{D}_E(G)$, the Fourier–Bohr transform of $w_f : G \rightarrow \mathbb{C}$ at $\alpha \in \widehat{G}$ is

$$\widehat{w}_f(\alpha) := M(w_f \cdot \bar{\alpha}) = \sum_{j \in J} \widehat{w}_{f,j}(\alpha)$$

with absolute convergence. The generalized Fourier series of $w_f : G \rightarrow \mathbb{C}$ is given by

$$(2.6) \quad w_f = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} c_\alpha \alpha,$$

where the Fourier coefficients are given by

$$c_\alpha = \int_{\widehat{G}} \hat{f}(\omega) \overline{\hat{f}(\omega + \alpha)} \sum_{\{j \in J: \alpha \in \Gamma_j^\perp\}} \frac{1}{\mathfrak{d}(\Gamma_j)} \int_{P_j} \overline{\hat{g}_{j,p}(\omega)} \hat{g}_{j,p}(\omega + \alpha) d\mu_{P_j}(p) d\mu_{\widehat{G}}(\omega).$$

Furthermore, if the ∞ -UCP holds, then w_f agrees pointwise with its generalized Fourier series (2.6).

Proof. The result, except for the specific form of the Fourier coefficients, can be found in [24]. Indeed, by [24, Proposition 3.10], the generalized Fourier series of $w_f : G \rightarrow \mathbb{C}$ is $w_f = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} c_\alpha \alpha$ with coefficients

$$c_\alpha = \sum_{\{j \in J: \alpha \in \Gamma_j^\perp\}} \int_{\widehat{G}} \hat{f}(\omega) \overline{\hat{f}(\omega + \alpha)} \frac{1}{\mathfrak{d}(\Gamma_j)} \int_{P_j} \overline{\hat{g}_{j,p}(\omega)} \hat{g}_{j,p}(\omega + \alpha) d\mu_{P_j}(p) d\mu_{\widehat{G}}(\omega).$$

For fixed $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$, the assumption (2.5), together with Cauchy–Schwarz' inequality, yields

$$\sum_{\{j \in J: \alpha \in \Gamma_j^\perp\}} \frac{1}{\mathfrak{d}(\Gamma_j)} \int_{P_j} |\hat{g}_{j,p}(\cdot) \hat{g}_{j,p}(\cdot + \alpha)| d\mu_{P_j}(p) \in L^1_{\text{loc}}(\widehat{G} \setminus E),$$

which in turn implies that

$$\sum_{\{j \in J: \alpha \in \Gamma_j^\perp\}} \int_{\widehat{G}} \left| \hat{f}(\omega) \hat{f}(\omega + \alpha) \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega + \alpha) d\mu_{P_j}(p) \right| d\mu_{\widehat{G}}(\omega) < \infty.$$

Thus, by Fubini–Tonelli’s theorem, the series and integral defining c_α can be interchanged to obtain the desired form in the proposition. ■

To ease notation, we define the following functions appearing implicitly in the Fourier coefficients in Proposition 2.3.

DEFINITION 2.4. Let $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ be a generalized translation-invariant system satisfying

$$(2.7) \quad \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} |\hat{g}_{j,p}(\omega)|^2 d\mu_{P_j}(p) < \infty$$

for $\mu_{\widehat{G}}$ -a.e. $\omega \in \widehat{G}$. The associated *auto-correlation functions* $\{t_\alpha\}_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp}$ are $\mu_{\widehat{G}}$ -a.e. defined by

$$t_\alpha : \widehat{G} \rightarrow \mathbb{C}, \quad \omega \mapsto \sum_{\{j \in J: \alpha \in \Gamma_j^\perp\}} \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p).$$

Phrased in terms of the auto-correlation functions, the assumptions (2.5) and (2.7) require $t_0 : \widehat{G} \rightarrow \mathbb{C}$ to be locally integrable and uniformly bounded, respectively. Any generalized translation-invariant system forming a Bessel family with upper bound $B > 0$ satisfies

$$(2.8) \quad t_0(\omega) \leq B$$

for $\mu_{\widehat{G}}$ -a.e. $\omega \in \widehat{G}$, as shown in [28, 32, 38]. The assumptions of any of our results in Section 3 imply (2.7), hence the auto-correlation functions t_α will always be well-defined. In particular, the assumption (2.3) is satisfied.

3. Sufficient and necessary conditions for the frame property.

This section contains the main results of the paper. Section 3.1 provides a necessary condition for the Bessel and the frame property of a generalized translation-invariant system. Sufficient conditions are presented in Section 3.2. In Section 3.3 the sufficient conditions are compared to known frame bound estimates.

3.1. Necessary conditions. Inequality (2.8) is a necessary condition for the Bessel property of a generalized translation-invariant system. Under a weak regularity assumption, Theorem 3.2 below presents a much stronger necessary condition for a generalized translation-invariant system to form a Bessel family.

The proof of Theorem 3.2 makes use of a differentiation process for integrals on locally compact groups as in [22, Section 2]. In order to apply this, the following notion is useful (cf. [22, Definition 2.1]).

DEFINITION 3.1. Let G be a locally compact group with Haar measure μ_G . A decreasing sequence $(U_k)_{k \in \mathbb{N}}$ of finite measure Borel sets is called a D' -sequence in G if every neighborhood of 0 contains some U_k , and if there exists a constant $C > 0$ such that

$$0 < \mu_G(U_k - U_k) \leq C\mu_G(U_k)$$

for all $k \in \mathbb{N}$, where $U_k - U_k := \{u - v : u, v \in U_k\}$.

Given a D' -sequence $(U_k)_{k \in \mathbb{N}}$ for G and an $f \in L^1_{\text{loc}}(G)$, a point $x_0 \in G$ satisfying

$$(3.1) \quad \lim_{k \rightarrow \infty} \frac{1}{\mu_G(U_k)} \int_{x_0 + U_k} f(x) d\mu_G(x) = f(x_0)$$

is called a *Lebesgue point* of f . Lebesgue's differentiation theorem [29, Theorem 44.18] asserts that the set of Lebesgue points of any $f \in L^1_{\text{loc}}(G)$ has full measure, or equivalently that (3.1) holds for a.e. $x_0 \in G$.

In [8], it is shown that any compactly generated abelian Lie group G admits a D' -sequence. As a consequence, any locally compact abelian group G of the form $G = \mathbb{R}^d \times \mathbb{T}^m \times \mathbb{Z}^n \times F$, where $d, m, n \in \mathbb{N}$ and F is finite, has a D' -sequence. On the other hand, Bownik and Ross [8] also show that some infinite-dimensional locally compact abelian groups, e.g., the tubby torus $\mathbb{T}^{\mathbb{N}_0}$, do not admit a D' -sequence.

THEOREM 3.2. *Let G be a locally compact abelian group such that \widehat{G} admits a D' -sequence. Let $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ be a generalized translation-invariant system satisfying the 1-UCP such that the Fourier series of w_f converges unconditionally pointwise to $w_f(x_0)$ for some $x_0 \in G$. Suppose $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ forms a Bessel system in $L^2(G)$ with Bessel bound B . Then*

$$(3.2) \quad B \geq B_2 := \operatorname{ess\,sup}_{\omega \in \widehat{G}} \left(\sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} |t_\alpha(\omega)|^2 \right)^{1/2}.$$

If moreover $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ forms a frame with lower frame bound $A > 0$, then $A_\infty := \operatorname{ess\,inf}_{\omega \in \widehat{G}} t_0(\omega) \geq A$.

Proof. The “moreover” part is a consequence of [24, Theorem 3.13]. The remainder of the proof is divided into three steps:

STEP 1: *Rewriting the frame operator.* Let $S : L^2(G) \rightarrow L^2(G)$ denote the frame operator associated with the Bessel family $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$. The map $(f_1, f_2) \mapsto \langle S f_1, f_2 \rangle$ is a well-defined, bounded sesquilinear form on

$L^2(G) \times L^2(G)$ with

$$(3.3) \quad |\langle Sf_1, f_2 \rangle| \leq B \|f_1\|_2 \|f_2\|_2$$

for all $f_1, f_2 \in L^2(G)$. The frame operator is related to the function $w_f : G \rightarrow \mathbb{C}$ introduced in (2.4) by $w_f(x) = \langle ST_x f, T_x f \rangle$, in particular $w_f(0) = \langle Sf, f \rangle$ for $f \in \mathcal{D}_E(G)$. By translation invariance of $\mathcal{D}_E(G)$, we can assume $x_0 = 0$, that is, pointwise convergence of the Fourier series of w_f at the origin to $w_f(0)$. By Proposition 2.3, it then follows that

$$(3.4) \quad \langle Sf, f \rangle = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} \int_{\widehat{G}} \widehat{f}(\omega) \overline{\widehat{f}(\omega + \alpha)} \overline{t_\alpha(\omega)} d\mu_{\widehat{G}}(\omega)$$

for all $f \in \mathcal{D}_E(G)$. Here, each auto-correlation function $t_\alpha : \widehat{G} \rightarrow \mathbb{C}$ is well-defined with $\|t_\alpha\|_\infty \leq B$ by (2.8). The identity (3.4), together with an application of the polarization identity for sesquilinear forms and the bound (3.3), therefore gives

$$(3.5) \quad \left| \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} \int_{\widehat{G}} \widehat{f}_1(\omega) \overline{\widehat{f}_2(\omega + \alpha)} \overline{t_\alpha(\omega)} d\mu_{\widehat{G}}(\omega) \right| \leq B \|f_1\|_2 \|f_2\|_2$$

for all $f_1, f_2 \in \mathcal{D}_E(G)$. For brevity, we define

$$c_\alpha := \int_{\widehat{G}} \widehat{f}_1(\omega) \overline{\widehat{f}_2(\omega + \alpha)} \overline{t_\alpha(\omega)} d\mu_{\widehat{G}}(\omega) \quad \text{for } \alpha \in \Lambda := \bigcup_{j \in J} \Gamma_j^\perp.$$

Then (3.5) simply reads $|\sum_\alpha c_\alpha| \leq B \|f_1\|_2 \|f_2\|_2$.

STEP 2: Construction of test functions f_1, f_2 . First, we assume that $E = \emptyset$ in the 1-UCP assumption. Let $\omega_0 \in \widehat{G}$ be a common Lebesgue point of $|t_\alpha|^2 \in L^\infty(\widehat{G}) \subset L^1_{\text{loc}}(\widehat{G})$ for all $\alpha \in \Lambda$. The dual group \widehat{G} is second countable, hence metrizable. The metric $d_{\widehat{G}}$ inducing the given topology on \widehat{G} can be chosen to be translation-invariant. Let $\sigma : \mathbb{N} \rightarrow J$ be a bijection and define $A_{m,n} := \bigcup_{i=1}^n \Gamma_{\sigma(i)}^\perp \cap B(0, m)$, where $B(0, m)$ denotes the open ball, relative to $d_{\widehat{G}}$, with radius $m > 0$ and center $0 \in \widehat{G}$. Then, given any $\alpha \in \Lambda$, there exists $m, n \in \mathbb{N}$ such that $\alpha \in A_{m,n}$.

For $m, n \in \mathbb{N}$, we set $\delta_{m,n} = \min\{d_{\widehat{G}}(\alpha, \alpha') : \alpha, \alpha' \in A_{m,n} \text{ with } \alpha \neq \alpha'\}$. Note that $\delta_{m,n} > 0$ since $A_{m,n}$ is a finite set. Let $(U_k)_{k \in \mathbb{N}}$ be a D' -sequence in \widehat{G} . The sets U_k lie eventually inside an arbitrary neighborhood of $0 \in \widehat{G}$. Thus, by local compactness of \widehat{G} , we can assume without loss of generality that U_1 is relatively compact. Moreover, we let $K \in \mathbb{N}$ be such that $U_k \subset B(0, \delta_{m,n}/2)$ for all $k \geq K$. Then, for all $k \geq K$,

$$(3.6) \quad \mu_{\widehat{G}}((\alpha + U_k) \cap (\alpha' + U_k)) = 0 \quad \text{for all } \alpha, \alpha' \in A_{m,n} \text{ with } \alpha \neq \alpha'.$$

Define $f_1 \in \mathcal{D}_\emptyset(G)$ by $\widehat{f}_1 := \mu_{\widehat{G}}(U_k)^{-1/2} \mathbf{1}_{\omega_0 + U_k}$. For $k \geq K$, define $h : \widehat{G} \rightarrow \mathbb{C}$

on $\omega_0 + U_k + \Lambda_{m,n}$ by

$$h(\omega + \alpha) = \overline{t_\alpha(\omega)} \quad \text{for a.e. } \omega \in \omega_0 + U_k$$

for each $\alpha \in \Lambda_{m,n}$ and by $h(\omega) = 0$ for $\omega \in \widehat{G} \setminus (\omega_0 + U_k + \Lambda_{m,n})$. The property (3.6) of U_k guarantees that h is well-defined. Let $\widehat{f}_2 := \|h\|_2^{-1}h$. Then $f_2 \in \mathcal{D}_0(G)$ with $\|f_2\|_2 = 1$. A direct calculation shows that

$$\begin{aligned} \sum_{\alpha \in \Lambda_{m,n}} c_\alpha &= \sum_{\alpha \in \Lambda_{m,n}} \int_{\widehat{G}} \widehat{f}_1(\omega) \overline{\widehat{f}_2(\omega + \alpha)} \overline{t_\alpha(\omega)} d\mu_{\widehat{G}}(\omega) \\ &= \|h\|_2^{-1} \sum_{\alpha \in \Lambda_{m,n}} \frac{1}{\mu_{\widehat{G}}(U_k)^{1/2}} \int_{\omega_0 + U_k} |t_\alpha(\omega)|^2 d\mu_{\widehat{G}}(\omega) \\ &= \frac{1}{\mu_{\widehat{G}}(U_k)^{1/2}} \left(\sum_{\alpha \in \Lambda_{m,n}} \int_{\omega_0 + U_k} |t_\alpha(\omega)|^2 d\mu_{\widehat{G}}(\omega) \right)^{-1/2} \\ &\quad \cdot \left(\sum_{\alpha \in \Lambda_{m,n}} \int_{\omega_0 + U_k} |t_\alpha(\omega)|^2 d\mu_{\widehat{G}}(\omega) \right) \\ &= \left(\sum_{\alpha \in \Lambda_{m,n}} \frac{1}{\mu_{\widehat{G}}(U_k)} \int_{\omega_0 + U_k} |t_\alpha(\omega)|^2 d\mu_{\widehat{G}}(\omega) \right)^{1/2} \end{aligned}$$

for any $m, n \in \mathbb{N}$ and $k \geq K$. An application of Lebesgue's differentiation theorem [29, Theorem 44.18] next gives

$$\lim_{k \rightarrow \infty} \left(\sum_{\alpha \in \Lambda_{m,n}} \frac{1}{\mu_{\widehat{G}}(U_k)} \int_{\omega_0 + U_k} |t_\alpha(\omega)|^2 d\mu_{\widehat{G}}(\omega) \right)^{1/2} = \left(\sum_{\alpha \in \Lambda_{m,n}} |t_\alpha(w_0)|^2 \right)^{1/2}$$

for any $m, n \in \mathbb{N}$.

STEP 3: *An ε -argument.* For arbitrary $\varepsilon > 0$, there exist $M, N \in \mathbb{N}$ such that, for all $m \geq M$ and $n \geq N$,

$$(3.7) \quad \left| \sum_{\alpha \in \Lambda} c_\alpha - \sum_{\alpha \in \Lambda_{m,n}} c_\alpha \right| \leq \varepsilon.$$

From (3.5) and (3.7) it now follows by the triangle inequality that

$$\left(\sum_{\alpha \in \Lambda_{m,n}} |t_\alpha(w_0)|^2 \right)^{1/2} \leq B + \varepsilon$$

for $m \geq M$ and $n \geq N$. Since $\sum_{\alpha \in \Lambda_{m,n}} |t_\alpha(w_0)|^2$ is a bounded and increasing sequence in n and m , the limit

$$\left(\sum_{\alpha \in \Lambda} |t_\alpha(w_0)|^2 \right)^{1/2} = \sup_{m \geq M, n \geq N} \left(\sum_{\alpha \in \Lambda_{m,n}} |t_\alpha(w_0)|^2 \right)^{1/2}$$

exists.

Since $\varepsilon > 0$ was taken arbitrary, it follows that

$$(3.8) \quad \left(\sum_{\alpha \in \Lambda} |t_\alpha(\omega_0)|^2 \right)^{1/2} \leq B.$$

Being a countable union of null sets, the complement of the set of common Lebesgue points of $|t_\alpha|^2$, $\alpha \in \Lambda$, is a null set. We conclude that (3.8) holds for $\mu_{\widehat{G}}$ -a.e. $\omega_0 \in \widehat{G}$, which completes the proof for the case $E = \emptyset$.

Lastly, consider the case when $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the 1-UCP with respect to an arbitrary $E \in \mathcal{E}$. In this case the Fourier supports of f_1 and f_2 might intersect E . If this happens, f_1 and f_2 should be approximated by functions from $\mathcal{D}_E(G)$ (see [32, Remark 4] for the details). ■

REMARK 3.3. (i) In Theorem 3.2, the assumption of 1-UCP with pointwise unconditional convergence of the Fourier series of w_f can be replaced by the simpler, but stronger, assumption that the generalized translation-invariant system satisfies the ∞ -UCP.

(ii) While the existence of a D' -sequence in \widehat{G} is sufficient for the differentiation process on Lebesgue's integrals [29, Theorem 44.18], it may not be necessary. In fact, it is an open problem whether Lebesgue's differentiation theorem holds on all second countable locally compact abelian groups [8, Section 7.1].

The 1-UCP assumption in Theorem 3.2 cannot be dropped, as demonstrated by Example 3.4 below. The construction of the generalized shift-invariant system follows [9, Example 3.2].

EXAMPLE 3.4. Let $G = \mathbb{Z}$. Let $N \in \mathbb{N}$ be such that $N \geq 2$. Define the lattices $\Gamma_j = N^j \mathbb{Z}$ for $j \in \mathbb{N}$. Let $\tau_1 = 0$ and define τ_j , $j \geq 2$, inductively as the smallest $t \in \mathbb{Z}$ in absolute value satisfying

$$(3.9) \quad \left(\bigcup_{i=1}^{j-1} (\tau_i + N^i \mathbb{Z}) \right) \cap (t + N^j \mathbb{Z}) = \emptyset.$$

In case t and $-t$ both are minimizers, pick τ_j to be positive. Then

$$\mathbb{Z} = \bigcup_{j \in \mathbb{N}} (\tau_j + N^j \mathbb{Z}),$$

with the union being disjoint (see also [24, Lemma 4.4]). Define the generators $g_j = \mathbf{1}_{\tau_j} \in \ell^2(\mathbb{Z})$ for $j \in \mathbb{N}$. By construction, the generalized shift-invariant system $\bigcup_{j \in \mathbb{N}} \{T_\gamma g_j\}_{\gamma \in \Gamma_j}$ is the canonical basis $\{\mathbf{1}_k\}_{k \in \mathbb{Z}}$ for $\ell^2(\mathbb{Z})$ and thus it forms, in particular, a frame for $\ell^2(\mathbb{Z})$ with frame bounds $A = B = 1$. The system $\bigcup_{j \in \mathbb{N}} \{T_\gamma g_j\}_{\gamma \in \Gamma_j}$ satisfies the 1-UCP only for the case $N = 2$ as shown in [24, Example 3.1]. We now show that the bound (3.2) fails for $N \geq 3$ in spite of the Bessel property.

Observe that $\hat{g}_j \in L^2(\mathbb{T})$ is given by $\hat{g}_j(\omega) = e^{2\pi i \tau_j \omega}$ for $\omega \in [0, 1)$. Hence, for any $\alpha \in \bigcup_{j=1}^{\infty} N^{-j}\mathbb{Z}$,

$$t_\alpha(\omega) = \sum_{\{j \in \mathbb{N}: \alpha \in N^{-j}\mathbb{Z}\}} \frac{1}{d(\Gamma_j)} \hat{g}_j(\omega) \overline{\hat{g}_j(\omega + \alpha)} = \sum_{\{j \in \mathbb{N}: \alpha \in N^{-j}\mathbb{Z}\}} N^{-j} e^{-2\pi i \alpha \tau_j}$$

for all $\omega \in [0, 1)$. Since t_α is independent of the variable $\omega \in [0, 1)$, we fix an arbitrary $\omega \in [0, 1)$ in the following calculations.

We rewrite $\bigcup_{j=1}^{\infty} N^{-j}\mathbb{Z}$ as the following disjoint union:

$$\bigcup_{j=1}^{\infty} N^{-j}\mathbb{Z} = \mathbb{Z} \cup \bigcup_{m=1}^{\infty} N^{-m}(\mathbb{Z} \setminus N\mathbb{Z}).$$

For $\alpha \in \mathbb{Z}$, a direct calculation gives

$$t_\alpha(\omega) = \sum_{j=1}^{\infty} N^{-j} = \frac{1}{N-1}.$$

On the other hand, writing $\alpha \notin \mathbb{Z}$ as $\alpha = kN^{-m}$ with $k \in \mathbb{Z} \setminus N\mathbb{Z}$ and $m \in \mathbb{N}$ yields

$$t_\alpha(\omega) = \sum_{j=m}^{\infty} N^{-j} e^{-2\pi i k N^{-m} \tau_j}.$$

Hence

$$|t_\alpha(\omega)| = N^{-m} \left| \sum_{\ell=0}^{\infty} N^{-\ell} e^{-2\pi i k N^{-m} \tau_{\ell+m}} \right|.$$

Next, we claim that it then follows that

$$|t_\alpha(\omega)| \geq N^{-m} \left(1 - \sum_{\ell=1}^{\infty} N^{-\ell} \right) = N^{-m} \frac{N-2}{N-1}.$$

To see this, let $z_\ell \in \mathbb{T}$, $\ell \in \mathbb{N}_0$. By the triangle inequality, we have

$$\begin{aligned} \left| \sum_{\ell=0}^{\infty} N^{-\ell} z_\ell \right| &\geq |z_0| - \left| z_0 - \sum_{\ell=0}^{\infty} N^{-\ell} z_\ell \right| = 1 - \left| \sum_{\ell=1}^{\infty} N^{-\ell} z_\ell \right| \\ &\geq 1 - \sum_{\ell=1}^{\infty} N^{-\ell} = \frac{N-2}{N-1}, \end{aligned}$$

which proves the claim.

Let $N \geq 3$. Since for each $m \in \mathbb{N}$ there are infinitely many $k \in \mathbb{Z} \setminus N\mathbb{Z}$ with $|t_{kN^{-m}}(\omega)| > N^{-m}/2$, we see that (3.2) is violated as $B_2 = \infty$. In fact, for any $p \in [1, \infty)$, we have

$$\sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} |t_\alpha(\omega)|^p = \infty$$

for all $\omega \in [0, 1)$. Thus, no ℓ^p -norm of $\{t_\alpha(\omega)\}_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp}$, with $\omega \in [0, 1)$, can be finite.

The discreteness of the group $G = \mathbb{Z}$ in Example 3.4 is not crucial. In fact, the construction is easily transferred to $L^2(\mathbb{R})$, e.g., by starting from the Gabor-like orthonormal basis $\{T_k e^{2\pi i m \cdot} \mathbb{1}_{[0,1)}\}_{k,m \in \mathbb{Z}}$.

3.2. Sufficient conditions. The following result, Theorem 3.5, provides a sufficient condition and estimates of the frame bounds for generalized translation-invariant frames. The proof is based on the simple estimate that a (generalized) Fourier series of an almost periodic function is bounded from above by the sum of the modulus of its coefficients and is bounded from below by the absolute value of its constant term minus the sum of the other terms in modulus.

THEOREM 3.5. *Let $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ be a generalized translation-invariant system satisfying the 1-UCP.*

(i) *Suppose the system satisfies*

$$(3.10) \quad B_1 := \operatorname{ess\,sup}_{\omega \in \widehat{G}} \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} |t_\alpha(\omega)| < \infty.$$

Then it is a Bessel system in $L^2(G)$ with Bessel bound B_1 .

(ii) *Suppose the system satisfies (3.10) and*

$$(3.11) \quad A_1 := \operatorname{ess\,inf}_{\omega \in \widehat{G}} \left(t_0(\omega) - \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)| \right) > 0.$$

Then it is a frame for $L^2(G)$ with lower bound A_1 and upper bound B_1 .

Proof. Suppose $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies condition (3.10) and the 1-UCP with respect to $E \in \mathcal{E}$. By definition of w_f and the fact that $\mathcal{D}_E(G)$ is dense in $L^2(G)$, it suffices in (ii) to show that $A_1 \|f\|^2 \leq w_f(0) \leq B_1 \|f\|^2$ for all $f \in \mathcal{D}_E(G)$, while in (i) it suffices to prove the upper bound.

As a consequence of Proposition 2.3, we find that the auxiliary function $w_f : G \rightarrow \mathbb{C}$ has the generalized Fourier series $\sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} c_\alpha \alpha$, where $c_\alpha := \int_{\widehat{G}} \widehat{f}(\omega) \overline{\widehat{f}(\omega + \alpha)} t_\alpha(\omega) d\mu_{\widehat{G}}(\omega)$. It will first be shown that w_f coincides pointwise with its generalized Fourier series. In order to do so, we show that the generalized Fourier series is uniformly convergent. An application of Beppo Levi's theorem and Young's inequality for products gives

$$\begin{aligned}
(3.12) \quad \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |c_\alpha| &\leq \int_{\widehat{G}} \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |\hat{f}(\omega) \hat{f}(\omega + \alpha) t_\alpha(\omega)| d\mu_{\widehat{G}}(\omega) \\
&\leq \frac{1}{2} \int_{\widehat{G}} |\hat{f}(\omega)|^2 \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)| d\mu_{\widehat{G}}(\omega) \\
&\quad + \frac{1}{2} \int_{\widehat{G}} \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |\hat{f}(\omega + \alpha)|^2 |t_\alpha(\omega)| d\mu_{\widehat{G}}(\omega) \\
&= \int_{\widehat{G}} |\hat{f}(\omega)|^2 \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)| d\mu_{\widehat{G}}(\omega),
\end{aligned}$$

where equality follows by the change of variable $\omega \mapsto \omega - \alpha$ and $\overline{t_\alpha(\omega - \alpha)} = t_{-\alpha}(\omega)$. Since $\sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} |c_\alpha| < \infty$, an application of the Weierstrass M-test shows that the generalized Fourier series of w_f converges uniformly to an almost periodic function. By uniqueness of Fourier coefficients, it follows that $w_f(x) = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} c_\alpha \alpha(x)$ pointwise for all $x \in G$.

Setting $x = 0$ in the Fourier series representation of w_f and using (3.12) now gives

$$w_f(0) = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} c_\alpha \leq \int_{\widehat{G}} |\hat{f}(\omega)|^2 \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} |t_\alpha(\omega)| d\mu_{\widehat{G}}(\omega) \leq B_1 \|f\|_2^2$$

for all $f \in \mathcal{D}_E(G)$. This shows (i). Assume now also that the assumption in (ii) is satisfied. Then from the triangle inequality and (3.12) it follows that $f \in \mathcal{D}_E(G)$ for all

$$\begin{aligned}
w_f(0) &= \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} c_\alpha \geq c_0 - \left| \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} c_\alpha \right| \\
&\geq \int_{\widehat{G}} |\hat{f}(\omega)|^2 \left(t_0(\omega) - \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)| \right) d\mu_{\widehat{G}}(\omega) \geq A_1 \|f\|_2^2
\end{aligned}$$

as desired. ■

The frame bound estimates of Theorem 3.5 are optimal for tight frames. That is, for a generalized translation-invariant system $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ that satisfies the 1-UCP and forms a tight frame, the estimates in Theorem 3.5 recover precisely the frame bound of the given frame. This simple observation is stated as the next result.

PROPOSITION 3.6. *Let $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ be a generalized translation-invariant system satisfying the 1-UCP. Suppose the system is a tight frame for $L^2(G)$ with frame bound $A > 0$. Then $A = A_1 = B_1$.*

Proof. Suppose $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a tight frame for $L^2(G)$ with bound $A > 0$. By [32, Theorem 3.4] and [24, Theorem 3.11], for any α in $\bigcup_{j \in J} \Gamma_j^\perp$ we have

$$t_\alpha(\omega) = A\delta_{\alpha,0}$$

for $\mu_{\widehat{G}}$ -a.e. $\omega \in \widehat{G}$. Hence $\sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)| = 0$ almost everywhere on \widehat{G} , and the conclusion follows. ■

3.3. Comparison of frame bound estimates. In this section we compare the frame bound estimates provided by Theorem 3.5 with known estimates. For this, we state the following result [32, Proposition 3.7].

PROPOSITION 3.7. *Let $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ be a generalized translation-invariant system.*

(i) *Suppose the system satisfies*

$$(3.13) \quad B' := \operatorname{ess\,sup}_{\omega \in \widehat{G}} \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} |\hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega + \alpha)| \, d\mu_{P_j}(p) < \infty.$$

Then it is a Bessel family in $L^2(G)$ with Bessel bound B' .

(ii) *Suppose the system satisfies (3.13) and*

$$A' := \operatorname{ess\,inf}_{\omega \in \widehat{G}} \left(\sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} |\hat{g}_{j,p}(\omega)|^2 \, d\mu_{P_j}(p) - \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp \setminus \{0\}} |\hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega + \alpha)| \, d\mu_{P_j}(p) \right) > 0.$$

Then it is a frame for $L^2(G)$ with lower bound A' and upper bound B' .

In [32], the term *absolute CC-condition* was used for condition (3.13). The important difference between the CC-condition (3.10) and the absolute CC-condition (3.13) is the placement of the absolute sign in the summand. In the CC-condition, it is possible to have phase cancellations within each auto-correlation function, while the absolute CC-condition prohibits such cancellations. It is a simple observation that a generalized translation-invariant system satisfying the absolute CC-condition also satisfies the CC-condition.

LEMMA 3.8. *Suppose $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the absolute CC-condition (3.13). Then it also satisfies the CC-condition (3.10).*

Proof. Suppose the system satisfies the absolute CC-condition (3.13). Then an application of Beppo Levi's theorem gives

$$\begin{aligned} \sum_{j \in J} \sum_{\alpha \in \Gamma_j^\perp} \left| \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p) \right| \\ \leq \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} |\hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega + \alpha)| d\mu_{P_j}(p) < \infty \end{aligned}$$

for $\mu_{\widehat{G}}$ -a.e. $\omega \in \widehat{G}$. By the absolute convergence of the series, a reordering of the summation does not affect the convergence. Thus

$$\operatorname{ess\,sup}_{\omega \in \widehat{G}} \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} \left| \sum_{\{j \in J: \alpha \in \Gamma_j^\perp\}} \frac{1}{d(\Gamma_j)} \int_{P_j} \hat{g}_{j,p}(\omega) \overline{\hat{g}_{j,p}(\omega + \alpha)} d\mu_{P_j}(p) \right| < \infty,$$

as required. ■

Jakobsen and the first named author [32] showed that a generalized translation-invariant system satisfying the absolute CC-condition automatically satisfies the α -LIC. Thus the 1-UCP is implicitly assumed in the estimate (3.13).

The generalized translation-invariant system in Example 3.4 with $N = 2$ satisfies the frame bound estimates based on the CC-condition, but dramatically fails the estimates based on the absolute CC-condition as demonstrated in the next example.

EXAMPLE 3.9. Let $G = \mathbb{Z}$. Consider the system $\bigcup_{j \in \mathbb{N}} \{T_\gamma g_j\}_{\gamma \in \Gamma_j}$ in $\ell^2(\mathbb{Z})$ with $\Gamma_j = 2^j \mathbb{Z}$ and $g_j = \mathbf{1}_{\tau_j}$, where $(\tau_j)_{j \in \mathbb{N}} \subset \mathbb{Z}$ is chosen as in (3.9). This system is a frame for $\ell^2(\mathbb{Z})$ with frame bounds $A = B = 1$. For $\omega \in [0, 1)$, a direct calculation gives

$$\begin{aligned} \sum_{j \in \mathbb{N}} \frac{1}{d(\Gamma_j)} \sum_{\alpha \in \Gamma_j^\perp} |\hat{g}_j(\omega) \hat{g}_j(\omega + \alpha)| &= \sum_{j \in \mathbb{N}} \frac{1}{2^j} \sum_{\alpha \in \Gamma_j^\perp} |e^{2\pi i \tau_j \alpha}| \\ &= \sum_{j \in \mathbb{N}} \frac{1}{2^j} \#(\Gamma_j^\perp) = \sum_{j \in \mathbb{N}} 1 = \infty. \end{aligned}$$

Thus $\bigcup_{j \in \mathbb{N}} \{T_\gamma g_j\}_{\gamma \in \Gamma_j}$ fails the estimate (3.13). On the other hand, it follows from Proposition 3.6 that $\bigcup_{j \in \mathbb{N}} \{T_\gamma g_j\}_{\gamma \in \Gamma_j}$ satisfies (3.10) and (3.11) with $A_1 = B_1 = 1$.

The discrepancy between the frame bound estimates in Proposition 3.7 and the estimates in Theorem 3.5 might occur even for well-known orthonormal bases. Indeed, the Meyer wavelet is an example of an orthonormal basis in $L^2(\mathbb{R})$ for which the estimates based on the absolute CC-condition give the poor estimates $A' = -1$ and $B' = 3$ (see [19, p. 984]). However,

the frame bound estimates in Theorem 3.5 give the correct frame bounds, namely $A_1 = B_1 = 1$. For a direct verification of the characterizing equations $t_\alpha = \delta_{\alpha,0}$ for the Meyer wavelet, the interested reader is referred to Daubechies' book [20, Section 4.2.1].

Finally, we remark that Casazza, Christensen and Janssen [13] give an example of a Gabor system which is a Bessel system in $L^2(\mathbb{R})$, but where $\sum_\alpha |t_\alpha(\omega)| = \infty$ for a.e. $\omega \in \mathbb{R}$. This demonstrates that for Bessel systems both the CC-condition and the absolute CC-condition can fail even though the LIC and thus the α -LIC and the 1-UCP hold.

4. Applications and examples. In this section the sufficient conditions given in Theorem 3.5 will be considered for special types of generalized translation-invariant systems. The focus will be on explicit formulas for the auto-correlation functions $t_\alpha : \widehat{G} \rightarrow \mathbb{C}$ and the associated *remainder function*

$$R : \widehat{G} \rightarrow [0, \infty], \quad R(\omega) = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}} |t_\alpha(\omega)|,$$

which are the main ingredients in the estimates of Theorem 3.5. Here, it should be understood that the (formal) expression for t_α might only be well-defined once we impose the CC-condition. Deducing from these formulas the necessary condition in Theorem 3.2 in each special case is straightforward and is left to the reader.

4.1. Gabor systems. Given a countable index set J , let $\{g_j\}_{j \in J} \subset L^2(G)$. Let $\Gamma \subseteq G$ be a closed, co-compact subgroup and let $\Lambda \subseteq \widehat{G}$ be such that equipping it with a σ -algebra Σ_Λ and a measure μ_Λ gives a measure space $(\Lambda, \Sigma_\Lambda, \mu_\Lambda)$ satisfying the standard hypotheses. The (semi-) co-compact *Gabor system* associated with the pair (Γ, Λ) is the collection of functions

$$\{M_\lambda T_\gamma g_j\}_{\lambda \in \Lambda, \gamma \in \Gamma, j \in J} = \{\lambda(\cdot) g_j(\cdot - \gamma)\}_{\lambda \in \Lambda, \gamma \in \Gamma, j \in J},$$

where $M_\lambda f(x) = \lambda(x)f(x)$ denotes the modulation operator on $L^2(G)$. The Gabor system $\{M_\lambda T_\gamma g_j\}_{\lambda \in \Lambda, \gamma \in \Gamma, j \in J}$ cannot be expressed as a generalized translation-invariant system. However, since $|\langle f, M_\lambda T_\gamma g_j \rangle| = |\langle f, T_\gamma M_\lambda g_j \rangle|$ for all $f \in L^2(G)$, the Gabor system $\{M_\lambda T_\gamma g_j\}_{\lambda \in \Lambda, \gamma \in \Gamma, j \in J}$ is a Bessel system or a frame if, and only if, the corresponding (generalized) translation-invariant system $\{T_\gamma M_\lambda g_j\}_{\lambda \in \Lambda, \gamma \in \Gamma, j \in J}$ with $g_{j,\lambda} = M_\lambda g_j$ is a Bessel system or a frame. The auto-correlation functions t_α associated with the system $\{T_\gamma M_\lambda g_j\}_{\lambda \in \Lambda, \gamma \in \Gamma, j \in J}$ can (formally) be written as

$$t_\alpha(\omega) = \sum_{j \in J} \int_\Lambda \hat{g}_j(\omega - \lambda) \overline{\hat{g}_j(\omega - \lambda - \alpha)} d\mu_\Lambda(\lambda)$$

for $\alpha \in \Gamma^\perp$.

Any translation-invariant system satisfying the CC-condition also satisfies the α -LIC. Thus an application of Theorem 3.5 gives the frame bound estimates (3.10) and (3.11), where the 0th auto-correlation function t_0 is given by

$$(4.1) \quad t_0(\omega) = \sum_{j \in J} \int_{\Lambda} |\hat{g}_j(\omega - \lambda)|^2 d\mu_{\Lambda}(\lambda)$$

and the remainder function $R : \widehat{G} \rightarrow [0, \infty]$ by

$$(4.2) \quad R(\omega) = \sum_{\alpha \in \Gamma^{\perp} \setminus \{0\}} \left| \sum_{j \in J} \int_{\Lambda} \hat{g}_j(\omega - \lambda) \overline{\hat{g}_j(\omega - \lambda - \alpha)} d\mu_{\Lambda}(\lambda) \right|.$$

The frame bound estimates associated with (4.1) and (4.2) allow for phase cancellations over the modulation parameter $\lambda \in \Lambda$. Moreover, if Λ is a closed subgroup, we only need to take the essential supremum and infimum in (3.10) and (3.11), respectively, over a fundamental domain of Λ in \widehat{G} . For singly generated Gabor frames in $L^2(\mathbb{R}^d)$ associated with a pair of full-rank lattices (Λ, Γ) , the frame bound estimates (3.10) and (3.11) using (4.1) and (4.2) recover precisely the frame bound estimates by Ron and Shen [44, 48].

The sufficient conditions for Gabor frames are often formulated in the time domain. To do this, we switch the roles of Γ and Λ and consider the Gabor system $\{M_{\lambda}T_{\gamma}g_j\}_{\lambda \in \Lambda, \gamma \in \Gamma, j \in J} = \{T_{\lambda}\mathcal{F}^{-1}T_{\gamma}g_j\}_{\lambda \in \Lambda, \gamma \in \Gamma, j \in J}$, where \mathcal{F}^{-1} denotes the inverse Fourier transform. In this way, one obtains auto-correlation functions $s_{\alpha} : G \rightarrow \mathbb{C}$, $\alpha \in \Lambda^{\perp}$, given by

$$s_{\alpha}(x) := \sum_{j \in J} \int_{\Gamma} \overline{g_j(x - \gamma - \alpha)} g_j(x - \gamma) d\mu_{\Gamma}(\gamma),$$

provided the series converges. Hence, if

$$B_1 := \operatorname{ess\,sup}_{x \in G} \sum_{\alpha \in \Lambda^{\perp}} |s_{\alpha}(x)| < \infty,$$

$$A_1 := \operatorname{ess\,inf}_{x \in G} \left(s_0(x) - \sum_{\alpha \in \Gamma^{\perp} \setminus \{0\}} |s_{\alpha}(x)| \right) > 0,$$

then $\{M_{\lambda}T_{\gamma}g_j\}_{\lambda \in \Lambda, \gamma \in \Gamma, j \in J}$ is a frame for $L^2(\mathbb{R}^d)$ with bounds A_1 and B_1 . For singly generated Gabor frames in $L^2(\mathbb{R}^d)$ associated with a pair of full-rank lattices (Λ, Γ) , these estimates precisely recover [25, Proposition 6.5.5].

4.2. Wavelet systems. Let $\operatorname{Aut}(G)$ denote the collection of all bi-continuous group homomorphisms on G . For an automorphism $a \in \operatorname{Aut}(G)$, let $|a|$ denote its *modulus*, i.e., the unique positive constant such that

$$\int_G f(a(x)) d\mu_G(x) = |a| \int_G f(x) d\mu_G(x)$$

for all $f \in L^1(G)$. Denote by $D_a f(x) := |a|^{1/2} f(a(x))$ the *unitary dilation operator* on $L^2(G)$.

Let J and L be countable index sets, let $\{\psi_\ell\}_{\ell \in L} \subset L^2(G)$, let $\mathcal{A} := \{a_j\}_{j \in J} \subset \text{Aut}(G)$ and let $\Gamma \subseteq G$ be a closed, co-compact subgroup. The *wavelet system* in $L^2(G)$ associated with the pair (\mathcal{A}, Γ) is the collection of functions

$$(4.3) \quad \{D_a T_\gamma \psi_\ell\}_{a \in \mathcal{A}, \gamma \in \Gamma, \ell \in L} = \{|a_j|^{1/2} \psi_\ell(a_j(\cdot) - \gamma)\}_{j \in J, \gamma \in \Gamma, \ell \in L}.$$

By considering the commutation relation $D_a T_\gamma = T_{a^{-1}(\gamma)} D_a$ for $a \in \mathcal{A}$ and $\gamma \in \Gamma$, the wavelet system (4.3) can be written as the generalized translation-invariant system $\bigcup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j}$ with $\Gamma_j = a_j^{-1}(\Gamma)$ and $g_{j,p} = D_{a_j} \psi_\ell$ for $j \in J$ and $p = \ell \in P$ with $P = L$ equipped with the counting measure.

The *adjoint* of an automorphism $a \in \text{Aut}(G)$ is the automorphism $\hat{a} : \widehat{G} \rightarrow \widehat{G}$ defined by $\hat{a}(\omega) = \omega \circ a$ for $\omega \in \widehat{G}$. With this notion, the annihilators Γ_j^\perp of Γ_j for $j \in J$ can be written as $\Gamma_j^\perp = (a_j^{-1}(\Gamma))^\perp = \hat{a}_j(\Gamma^\perp)$ (cf. [8, Proposition 6.5]). For $\alpha \in \bigcup_{j \in J} \hat{a}_j(\Gamma^\perp)$, the auto-correlation function $t_\alpha : \widehat{G} \rightarrow \mathbb{C}$ can be formally written as

$$\begin{aligned} t_\alpha(\omega) &= \sum_{\ell \in L} \sum_{j \in \kappa(\alpha)} \frac{1}{d(\Gamma_j)} (\widehat{D_{a_j} \psi_\ell})(\omega) \overline{(\widehat{D_{a_j} \psi_\ell})(\omega + \alpha)} \\ &= \sum_{\ell \in L} \sum_{j \in \kappa(\alpha)} \frac{|a_j|}{d(\Gamma_j)} \hat{\psi}_\ell(\hat{a}_j^{-1}(\omega)) \overline{\hat{\psi}_\ell(\hat{a}_j^{-1}(\omega + \alpha))}, \end{aligned}$$

where $\kappa(\alpha) := \{j \in J : \alpha \in \hat{a}_j(\Gamma^\perp)\}$. Observe that $\kappa(0) = J$. Therefore, for wavelet systems satisfying the 1-UCP, an application of Theorem 3.5 yields the frame bound estimates as in (3.10) and (3.11), where

$$(4.4) \quad t_0(\omega) = \sum_{\ell \in L} \sum_{j \in \kappa(\alpha)} \frac{|a_j|}{d(\Gamma_j)} |\hat{\psi}_\ell(\hat{a}_j^{-1}(\omega))|^2$$

is the Calderón sum, and the remainder function $R : \widehat{G} \rightarrow [0, \infty]$ takes the form

$$(4.5) \quad R(\omega) = \sum_{\alpha \in \bigcup_{j \in J} \hat{a}_j(\Gamma^\perp) \setminus \{0\}} \left| \sum_{\ell \in L} \sum_{j \in \kappa(\alpha)} \frac{|a_j|}{d(\Gamma_j)} \hat{\psi}_\ell(\hat{a}_j^{-1}(\omega)) \overline{\hat{\psi}_\ell(\hat{a}_j^{-1}(\omega + \alpha))} \right|.$$

Thus for all generators and for all scales in $\kappa(\alpha)$, we have the possibility of cancellations in the estimates for each $\alpha \in \bigcup_{j \in J} \hat{a}_j(\Gamma^\perp) \setminus \{0\}$. This is in contrast to known sufficient conditions and frame bound estimates for wavelet systems based on the absolute CC-condition. The latter conditions use the remainder function $\hat{R} : \widehat{G} \rightarrow [0, \infty]$ given by

$$(4.6) \quad \tilde{R}(\omega) = \sum_{j \in J} \frac{|a_j|}{d(T_j)} \sum_{\alpha \in \hat{a}_j(\Gamma^\perp) \setminus \{0\}} \sum_{\ell \in L} |\hat{\psi}_\ell(\hat{a}_j^{-1}\omega) \overline{\hat{\psi}_\ell(\hat{a}_j^{-1}(\omega + \alpha))}|,$$

in which only the moduli of the generating functions are considered. To wrap up the discussion, we state the following result.

THEOREM 4.1. *Given countable index sets J, L , let $\{\psi_\ell\}_{\ell \in L} \subset L^2(G)$, let $\{a_j\}_{j \in J} \subset \text{Aut}(G)$ and let $\Gamma \subseteq G$ be a closed, co-compact subgroup. Suppose the system $\{D_{a_j} T_\gamma \psi_\ell\}_{j \in J, \gamma \in \Gamma, \ell \in L}$ satisfies the 1-UCP and*

$$(4.7) \quad b_1 := \text{ess sup}_{\omega \in \hat{G}} (t_0(\omega) + R(\omega)) < \infty,$$

$$(4.8) \quad a_1 := \text{ess inf}_{\omega \in \hat{G}} (t_0(\omega) - R(\omega)) > 0,$$

where t_0 and R are given in (4.4) and (4.5), respectively. Then the system $\{D_{a_j} T_\gamma \psi_\ell\}_{j \in J, \gamma \in \Gamma, \ell \in L}$ is a frame for $L^2(G)$ with bounds a_1 and b_1 .

The wavelet system in (4.3) is defined with respect to an arbitrary family $\mathcal{A} \subseteq \text{Aut}(G)$. For such general systems, the LIC, and hence α -LIC and 1-UCP, are not necessarily satisfied whenever the system satisfies the CC-condition. However, under additional assumptions on $\mathcal{A} \subseteq \text{Aut}(G)$, simple sufficient conditions and characterizations for the LIC are known. For example, for a family $\{a_j\}_{j \in J} \subset \text{Aut}(G)$ for which the adjoints $\{\hat{a}_j\}_{j \in J}$ are *expanding* in the sense of [1, Definition 18], the LIC is automatically satisfied for any system satisfying the CC-condition. In particular, for a cyclic group $\mathcal{A} = \langle a \rangle$ generated by $a \in \text{Aut}(G)$, several simple sufficient conditions for the LIC are known. For example, if the underlying group G has a compact open subgroup, the dilation group can be assumed to be expanding in the sense of [2, 3]. For systems on such groups, the LIC admits a simple characterization [38]. For a general locally compact abelian group G , it is shown in [38] that the LIC for wavelet systems associated to $\mathcal{A} = \langle a \rangle$ is equivalent to local integrability of the Calderón sum t_0 , provided that the adjoint automorphisms are expansive in the sense of [38, Proposition 4.9]. See also [6, Proposition 2.7] for the same result on $G = \mathbb{R}^d$. In the latter setting, the characterization of the LIC holds in fact for any wavelet system satisfying the so-called *lattice counting estimate*. In [7], Bownik and the first named author show that the lattice counting estimate holds for all dilations $A \in \text{GL}_d(\mathbb{R})$ with $|\det A| \neq 1$ and for almost every translation lattice Γ with respect to an invariant probability measure on the set of lattices. As a consequence, Theorems 3.2 and 3.5 are applicable for almost all wavelet systems in $L^2(\mathbb{R}^d)$ in the probabilistic sense of [7].

The remainder of this subsection is devoted to two examples: one for which phase cancellations in (4.5) can occur and another for which such cancellations cannot be expected. Both examples take place in $L^2(\mathbb{R}^d)$. In

this setting, any automorphism is given by $x \mapsto Ax$ for some $A \in \mathrm{GL}_d(\mathbb{R})$. For such an automorphism, the modulus reads $|\det A|$ and the adjoint is A^T . A discrete, co-compact subgroup $\Gamma \subseteq \mathbb{R}^d$ is a full-rank lattice in \mathbb{R}^d , i.e., $\Gamma = C\mathbb{Z}^d$ for some $C \in \mathrm{GL}_d(\mathbb{R})$. The annihilator Γ^\perp of a full-rank lattice $\Gamma \subseteq \mathbb{R}^d$ can be identified with the dual lattice $\Gamma^* = C^\sharp \mathbb{Z}^d$, where $C^\sharp := (C^T)^{-1}$.

EXAMPLE 4.2. Let $A \in \mathrm{GL}_d(\mathbb{R})$, let $B := A^T$ and let $\Gamma = C\mathbb{Z}^d$ be a full-rank lattice in \mathbb{R}^d satisfying $\Gamma^* \cap B^j \Gamma^* = \{0\}$ for all $j \in \mathbb{Z} \setminus \{0\}$. Examples of such pairs (A, Γ) are $B = \beta I$ with I denoting the identity matrix, $\Gamma = \mathbb{Z}^d$, and $\beta \in \mathbb{R}$ being such that $\beta^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z} \setminus \{0\}$. Now, since $B^j \Gamma^*$, $j \in \mathbb{Z}$, are disjoint outside the origin, the set $\kappa(\alpha)$ is a singleton for each $\alpha \in \bigcup_{j \in \mathbb{Z}} B^j \Gamma^* \setminus \{0\}$. Therefore, the remainder function $R : \mathbb{R}^d \rightarrow [0, \infty]$ takes the form

$$\begin{aligned} R(\omega) &= \frac{1}{|\det C|} \sum_{\alpha \in \bigcup_{j \in \mathbb{Z}} B^j \Gamma^* \setminus \{0\}} \left| \sum_{\ell \in L} \hat{\psi}_\ell(B^{-j}\omega) \overline{\hat{\psi}_\ell(B^{-j}(\omega + \alpha))} \right| \\ &= \frac{1}{|\det C|} \sum_{j \in \mathbb{Z}} \sum_{k \in \Gamma^* \setminus \{0\}} \left| \sum_{\ell \in L} \hat{\psi}_\ell(B^{-j}\omega) \overline{\hat{\psi}_\ell(B^{-j}\omega + k)} \right|. \end{aligned}$$

Consequently, phase cancellation between scales cannot occur in the estimates in Theorem 4.1. This observation fits precisely with a result of Laugesen [42], who proved that for wavelet systems in $L^2(\mathbb{R})$ with transcendental dilations $a > 0$ and integer translates, which in particular implies that $\bigcap_{j \in \mathbb{Z}} a^j \mathbb{Z} = \{0\}$, no cancellations between scales can happen for *any kind* of frame bound estimate based on $w_f(x)$. Note that despite the fact that no phase cancellations can occur, the estimate is still optimal for tight frames. This phenomenon is due to the fact that the characterizing equations for tight wavelet systems with expansive dilation A satisfying $\bigcap_{j \in \mathbb{Z}} B^j \Gamma^* = \{0\}$ are very restrictive properties of ψ_ℓ . For example, Riesz bases having this property must be combined MSF wavelets [5, 10, 17].

In the previous example it was assumed that the lattices $B^j \Gamma^*$, $j \in \mathbb{Z}$, are disjoint outside the origin. The next example assumes that the lattices involved are nested.

EXAMPLE 4.3. Let $A \in \mathrm{GL}_d(\mathbb{R})$, let $B := A^T$ and let $\Gamma = C\mathbb{Z}^d$ be a full-rank lattice in \mathbb{R}^d satisfying $B\Gamma^* \subset \Gamma^*$. In the case $\Gamma = \mathbb{Z}^d$, this assumption is equivalent with A being integer-valued. The union $\bigcup_{j \in \mathbb{Z}} B^j \Gamma^* \setminus \{0\}$ can be rewritten as the *disjoint* union $\bigcup_{m \in \mathbb{Z}} B^m (\Gamma^* \setminus B\Gamma^*)$. For $\alpha = B^m q$, where $m \in \mathbb{Z}$ and $q \in \Gamma^* \setminus B\Gamma^*$, we have $\kappa(\alpha) = \{j \in \mathbb{Z} : j \leq m\}$. Therefore, the remainder function $R : \mathbb{R}^d \rightarrow [0, \infty]$ takes the form

$$\begin{aligned}
(4.9) \quad R(\omega) &= \frac{1}{|\det C|} \sum_{m \in \mathbb{Z}} \sum_{q \in \Gamma^* \setminus B\Gamma^*} \left| \sum_{j=-\infty}^m \sum_{\ell \in L} \hat{\psi}_\ell(B^{-j}\omega) \overline{\hat{\psi}_\ell(B^{-j}(\omega + B^m q))} \right| \\
&= \frac{1}{|\det C|} \sum_{m \in \mathbb{Z}} \sum_{q \in \Gamma^* \setminus B\Gamma^*} \left| \sum_{n=0}^{\infty} \sum_{\ell \in L} \hat{\psi}_\ell(B^{n+m}\omega) \overline{\hat{\psi}_\ell(B^n(B^m\omega + q))} \right|.
\end{aligned}$$

Since the functions t_0 and R are B -dilation periodic, i.e., $t_0(B\omega) = t_0(\omega)$ and $R(B\omega) = R(\omega)$ for a.e. $\omega \in \mathbb{R}^d$, the estimates (4.7) and (4.8) read

$$(4.10) \quad b_1 = \operatorname{ess\,sup}_{\omega \in B(\Omega) \setminus \Omega} \left(\frac{1}{|\det C|} \sum_{\ell \in L} \sum_{j \in \mathbb{Z}} |\hat{\psi}_\ell(B^j\omega)|^2 + R(\omega) \right),$$

$$(4.11) \quad a_1 = \operatorname{ess\,inf}_{\omega \in B(\Omega) \setminus \Omega} \left(\frac{1}{|\det C|} \sum_{\ell \in L} \sum_{j \in \mathbb{Z}} |\hat{\psi}_\ell(B^j\omega)|^2 - R(\omega) \right),$$

where $\Omega := B(0, 1)$ is the unit ball in \mathbb{R}^d . For univariate wavelets with $A = B = 2$ and $\Gamma = c\mathbb{Z}$, $c > 0$, these estimates coincide ⁽³⁾ with Tchamitchian's estimates as communicated by Daubechies [19, 20]

To show that the frame bound estimates from Theorem 4.1 improve the sufficient condition based on the remainder function (4.6), note that in this special case, (4.6) simply reads

$$\tilde{R}(\omega) = \frac{1}{|\det C|} \sum_{j \in \mathbb{Z}} \sum_{\alpha \in \Gamma^* \setminus \{0\}} \sum_{\ell \in L} |\hat{\psi}_\ell(B^j\omega) \overline{\hat{\psi}_\ell(B^j\omega + \alpha)}|.$$

Now to see that $R(\omega) \leq \tilde{R}(\omega)$ for a.e. $\omega \in \mathbb{R}^n$, one simply uses the triangle inequality and notes that there is a bijection between the indices $(m, n, q) \in (\mathbb{Z}, \mathbb{N}, \Gamma^* \setminus B\Gamma^*)$ and the indices $(j, \alpha) \in \mathbb{Z} \times \Gamma^* \setminus \{0\}$ given by

$$(m, n, q) \mapsto (j, \alpha), \quad \text{where } \alpha = B^n q \text{ and } j = n + m.$$

For $\Gamma = \mathbb{Z}^d$, the above two examples show the two extremes on the possible phase cancellations of Theorem 3.5 that happen for integer dilations and certain irrational dilations. For a rational dilation matrix $A \in \operatorname{GL}_d(\mathbb{Q})$, frame bound estimates with phase cancellations in (4.5) over infinitely many scales are clearly also possible. In fact, Laugesen [42] remarked that this would be possible for rational dilations in dimension one, such dilations being necessarily expansive. Recall that the analysis in the present paper does not require that the dilation is expansive, only that the 1-UCP is satisfied.

4.3. Composite wavelets and shearlet systems. Consider the Cartesian product $I \times J$ of two countable index sets I and J . Let $A_i, B_j \in \operatorname{GL}_d(\mathbb{R})$

⁽³⁾ The frame bound estimates (4.10) and (4.11) are slightly improved versions of the estimates that occur in [19, Theorem 2.9]. The improvement boils down in essence to a change of variables and taking suprema and infima differently than in the original proof.

for $i \in I$ and $j \in J$. Let $\Gamma = C\mathbb{Z}^d$ be a full-rank lattice in \mathbb{R}^d . The wavelet system associated with the pair $(\{A_i B_j\}_{(i,j) \in I \times J}, \Gamma)$ is a collection of functions of the form

$$\{D_{A_i B_j} T_\gamma \psi_\ell\}_{i \in I, j \in J, \gamma \in \Gamma, \ell \in L}$$

and forms a so-called *wavelet system with composite dilations* in $L^2(\mathbb{R}^d)$ (see e.g. [26]). One usually assumes that one of the two family of matrices, say $\{A_i\}_{i \in I}$, is volume preserving. We will assume that $A_i^T, i \in I$, preserves Γ^* , that is, $A_i^T \Gamma^* = \Gamma^*$; for instance, in the case $\Gamma = \mathbb{Z}^d$, this assumption reads $A_i \in \text{SL}_d(\mathbb{Z})$. Therefore, $\Gamma_{(i,j)}^\perp = B_j^T A_i^T \Gamma^* = B_j^T \Gamma^*$ for $(i, j) \in I \times J$. Thus, for composite wavelet systems satisfying the 1-UCP, an application of Theorem 4.1 yields the frame bound estimates (4.7) and (4.8), where

$$t_0(\omega) = \frac{1}{|\det C|} \sum_{i \in I} \sum_{j \in J} \sum_{\ell \in L} |\hat{\psi}_\ell(A_i^\# B_j^\# \omega)|^2$$

$$R(\omega) = \frac{1}{|\det C|} \sum_{\alpha \in \bigcup_{j \in J} B_j^T \Gamma^* \setminus \{0\}} \left| \sum_{\ell \in L} \sum_{i \in I} \sum_{j \in \kappa(\alpha)} \hat{\psi}_\ell(A_i^\# B_j^\# \omega) \overline{\hat{\psi}_\ell(A_i^\# B_j^\# (\omega + \alpha))} \right|$$

with $\kappa(\alpha) := \{j \in J : \alpha \in B_j^T \Gamma^* \setminus \{0\}\}$.

The classical shearlet system is a special case of wavelets with composite dilations. For simplicity we restrict our attention to $L^2(\mathbb{R}^2)$, but we refer to [26, Section 3.4] for a discussion of shearlet systems in $L^2(\mathbb{R}^d)$. One defines

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and considers the wavelet system associated with the pair $(\{S^k A^j\}_{j,k \in \mathbb{Z}}, \Gamma)$, where $\Gamma = C\mathbb{Z}^2$ for some $C \in \text{GL}_d(\mathbb{R})$. For the *classical shearlet system* of the form $\{D_{S^k A^j} T_\gamma \psi_\ell\}_{j,k \in \mathbb{Z}, \gamma \in \Gamma, \ell \in L}$ we find as above that the corresponding functions $t_0 : \mathbb{R}^2 \rightarrow \mathbb{C}$ and $R : \mathbb{R}^2 \rightarrow [0, \infty]$ are formally given as

$$(4.12) \quad t_0(\omega) = \frac{1}{|\det C|} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{\ell \in L} |\hat{\psi}_\ell((S^\#)^k A^{-j} \omega)|^2$$

and

$$(4.13) \quad R(\omega) = \frac{1}{|\det C|} \sum_{m \in \mathbb{Z}} \sum_{q \in \Gamma^* \setminus A\Gamma^*} \left| \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}} \sum_{\ell \in L} \hat{\psi}_\ell((S^\#)^k A^{n+m} \omega) \overline{\hat{\psi}_\ell((S^\#)^k A^n (A^m \omega + q))} \right|.$$

Since any shearlet system that satisfies the CC-condition satisfies the α -LIC, an application of Theorem 3.5 yields the following result.

THEOREM 4.4. *Let L be a countable index set, let $\{\psi_\ell\}_{\ell \in L} \subset L^2(\mathbb{R}^2)$ and let $\Gamma \subset \mathbb{R}^2$ be a full-rank lattice. Suppose the (classical) shearlet system*

$\{D_{S^k A_j} T_\gamma \psi_\ell\}_{j,k \in \mathbb{Z}, \gamma \in \Gamma, \ell \in L}$ satisfies

$$b_1 := \operatorname{ess\,sup}_{\omega \in \mathbb{R}^2} (t_0(\omega) + R(\omega)) < \infty \quad \text{and} \quad a_1 := \operatorname{ess\,inf}_{\omega \in \mathbb{R}^2} (t_0(\omega) - R(\omega)) > 0,$$

where t_0 and R are given in (4.12) and (4.13), respectively. Then the system is a frame for $L^2(\mathbb{R}^2)$ with bounds a_1 and b_1 .

The estimates in Theorem 4.4 should be compared with previously used sufficient conditions for shearlet systems that are based on the absolute CC-condition and that do not allow for phase cancellations [39].

The rest of this subsection is devoted to cone-adapted shearlet systems. Such shearlets play a more important role in applications than the classical shearlets as they treat directions in an almost uniform manner. A cone-adapted shearlet system is a finite union of shift-invariant systems and wavelet systems with composite dilations. To introduce these systems, we define $A_1 = A$, $S_1 = S$,

$$A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For generators $\phi, \psi_i \in L^2(\mathbb{R}^2)$, $i = 1, 2$, and full-rank lattices $\Gamma_i = C_i \mathbb{Z}^2$, $i = 0, 1, 2$, the *cone-adapted shearlet system* is

$$\{T_\gamma \phi\}_{\gamma \in \Gamma_0} \cup \{D_{S_i^k A_i^j} T_\gamma \psi_i\}_{j \in \mathbb{N}_0, k \in \{-K_j, \dots, K_j\}, \gamma \in \Gamma_i, i \in \{1, 2\}},$$

where $K_j \in \mathbb{N}_0$ for $j \in \mathbb{N}_0$; usually one takes $K_j = 2^j$ or $K_j = 2^j \pm 1$.

For brevity we assume $\Gamma_i = \Gamma = C \mathbb{Z}^2$ for $i = 0, 1, 2$ for some matrix $C \in \operatorname{GL}_d(\mathbb{R})$ so that $C^T A_i C^\sharp$ is integer-valued for $i \in \{1, 2\}$. The auto-correlation functions $t_\alpha : \mathbb{R}^2 \rightarrow \mathbb{C}$, $\alpha \in \Gamma^*$, are then formally given as

$$(4.14) \quad t_0(\omega) = |\hat{\phi}(\omega)|^2 + \sum_{i \in \{1, 2\}} \sum_{j=0}^{\infty} \sum_{k=-K_j}^{K_j} |\hat{\psi}_i((S_i^\sharp)^k A_i^{-j} \omega)|^2,$$

$$(4.15) \quad t_\alpha(\omega) = \hat{\phi}(\omega) \overline{\hat{\phi}(\omega + \alpha)} + \sum_{i \in \{1, 2\}} \sum_{j=0}^{m_i} \sum_{k=-K_j}^{K_j} \hat{\psi}_i((S_i^\sharp)^k A_i^{-j} \omega) \overline{\hat{\psi}_i((S_i^\sharp)^k A_i^{-j} (\omega + \alpha))},$$

where $\alpha \in \Gamma^* \setminus \{0\}$, for each $i \in \{1, 2\}$, is written as $A_i^{m_i} q_i$ for unique $m_i \geq 0$ and $q_i \in \Gamma^* \setminus A_i \Gamma^*$. From the auto-correlation functions (4.15) we see that for the cases $\alpha \in C^\sharp \mathbb{Z}^2 \setminus 2C^\sharp \mathbb{Z}^2$ and $\alpha \in C^\sharp(4\mathbb{Z}^2 + (2, 2))$, the least amount of cancellation is possible. In these cases the auto-correlation function reads

$$t_\alpha(\omega) = \hat{\phi}(\omega) \overline{\hat{\phi}(\omega + \alpha)} + \sum_{i \in \{1, 2\}} \sum_{k=-K_0}^{K_0} \hat{\psi}_i((S_i^\sharp)^k \omega) \overline{\hat{\psi}_i((S_i^\sharp)^k (\omega + \alpha))},$$

hence only cancellation within the 0th scale is possible. On the other hand, when $\alpha \in 4^p C^\# \mathbb{Z}^2$ for some $p \in \mathbb{N}$, then cancellations can happen within all shears and all scales $j = 0, \dots, p$ for *both* shearlet generators ψ_1 and ψ_2 , that is, $m_1 = m_2 = p$ in (4.15).

As local integrability conditions can be ignored for shearlet systems, we arrive at the following Tchamitchian-type estimate for cone-adapted shearlet systems.

THEOREM 4.5. *Let $\phi, \psi_i \in L^2(\mathbb{R}^2)$, $i = 1, 2$, and let Γ be a full-rank lattice in \mathbb{R}^2 . If*

$$(4.16) \quad b_1 := \operatorname{ess\,sup}_{\omega \in \mathbb{R}^2} \sum_{\alpha \in \Gamma^*} |t_\alpha(\omega)| < \infty,$$

$$(4.17) \quad a_1 := \operatorname{ess\,inf}_{\omega \in \mathbb{R}^2} \left(t_0(\omega) - \sum_{\alpha \in \Gamma^* \setminus \{0\}} |t_\alpha(\omega)| \right) > 0,$$

where t_α is given by (4.14) and (4.15), then the cone-adapted shearlet system

$$\{T_\gamma \phi\}_{\gamma \in \Gamma} \cup \{D_{S_i^k A_i^j} T_\gamma \psi_i\}_{j \in \mathbb{N}_0, k \in \{-K_j, \dots, K_j\}, \gamma \in \Gamma, i \in \{1, 2\}}$$

is a frame for $L^2(\mathbb{R}^2)$ with bounds a_1 and b_1 .

The estimates in Theorem 4.5 are improvements of the sufficient conditions for cone-adapted shearlet systems as given in [37], which are based on the absolute CC-condition and do not allow for phase cancellations. Here, it should be noted that the conditions in [37] are currently the only known method for constructing cone-adapted shearlet frames with compactly supported generators. Moreover, the estimates without phase cancellation in [37] are used to “optimize” the choice of shearlet and translation lattice. It would be beneficial to instead use the improved estimates (4.16) and (4.17) in order to optimize the construction of compactly supported shearlets.

4.4. Continuous translation-invariant systems. This section considers “continuous” translation-invariant systems with translation along the whole group, e.g., J being a singleton and $\Gamma = G$. Since $G^\perp = \{0\}$, there is only one correlation function $t_0 : \widehat{G} \rightarrow \mathbb{C}$, and since J is a singleton, the ∞ -UCP trivially holds. Therefore, by combining the necessary condition $A \leq t_0 \leq B$ from [24] and Theorem 3.5, we immediately recover the following characterization of the frame property [30, 31, 50].

COROLLARY 4.6. *Let $0 < A \leq B < \infty$ and let $\{T_\gamma g_p\}_{\gamma \in G, p \in P}$ be a generalized translation-invariant system satisfying the standing hypotheses (I)–(III). Then the system is a frame for $L^2(G)$ with frame bounds A and B if, and only if,*

$$A \leq \int_P |\widehat{g}_p(\omega)|^2 d\mu_P(p) \leq B \quad \text{for } \mu_{\widehat{G}}\text{-a.e. } \omega \in \widehat{G}.$$

For continuous translation-invariant systems, being a frame is equivalent to the transform $C : L^2(G) \rightarrow L^2(P \times G)$, $f \mapsto \{\langle f, T_\gamma g_p \rangle\}_{\gamma \in G, p \in P}$, being an injective, bounded linear operator with closed range. Classical examples of such transforms are the continuous wavelet transform and the windowed Fourier transform. However, the continuous bendlet transform or, more generally, the ℓ th order α -shearlet transform, recently introduced in [43], are also examples of translation-invariant transforms. For these higher-order shearlet transforms the representation-theoretic approach, utilizing orthogonality relations for irreducible, square-integrable representations of an associated locally compact group, is not directly applicable [43, Section 5]. Since no characterizations of the frame property of the higher-order α -shearlet transform are known, in the next example we outline how such a characterization can be obtained from Corollary 4.6.

EXAMPLE 4.7. Let $G = \mathbb{R}^2$. Define the α -scaling operator $A_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $A_a(x_1, x_2) = (ax_1, a^\alpha x_2)$ for $\alpha \in [0, 1]$ and $a > 0$, and define the ℓ th order (non-linear) shearing operator $S_r : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $S_r(x_1, x_2) = (x_1 + \sum_{m=1}^{\ell} r_m x_2^m, x_2)$ for $r = (r_1, \dots, r_\ell) \in \mathbb{R}^\ell$. The Jacobian determinants of A_a and S_r are $a^{1+\alpha}$ and 1, while the inverses are $A_{a^{-1}}$ and S_{-r} , respectively.

Let $P = \mathbb{R}_{>0} \times \mathbb{R}^\ell$ and set $g_p = a^{-(1+\alpha)/2} \psi(A_{a^{-1}} S_{-r} \cdot)$ for some function $\psi \in L^2(\mathbb{R}^2)$ and $p = (a, r) \in P$. The continuous ℓ th order α -shearlet transform is simply the system $\{T_\gamma g_p\}_{\gamma \in G, p \in P}$, which reads

$$\{a^{-(1+\alpha)/2} \psi(A_{a^{-1}} S_{-r}(\cdot - \gamma))\}_{a \in \mathbb{R}_{>0}, r \in \mathbb{R}^\ell, \gamma \in \mathbb{R}^2}.$$

By Corollary 4.6, the system is a frame with bounds A and B if, and only if,

$$A \leq \int_0^\infty \int_{\mathbb{R}^\ell} a^{-(1+\alpha)} |\psi(A_{a^{-1}} S_{-r}(\cdot))^\wedge(\omega)|^2 dr da \leq B \quad \text{for a.e. } \omega \in \mathbb{R}^2.$$

Here, we have not specified the measure $dr da$ on P ; a canonical choice is $a^{-\ell-2+\alpha(\ell-1)}$ times the Lebesgue measure on $\mathbb{R}_{>0} \times \mathbb{R}^\ell$, but the characterization is valid for any measure on P satisfying the standing hypotheses (I)–(III).

The cone-adapted version of the continuous ℓ th order α -shearlet transform is obtained by equipping $\{(a, r) : a \in (0, 1], r \in R\}$ with a measure $dr da$ (satisfying the standing hypotheses), where R is a subset of \mathbb{R}^ℓ ; a canonical choice is $R = [-1 - a^{1-\alpha}, 1 + a^{1-\alpha}] \times \mathbb{R}^{\ell-1}$. Let $Q : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the permutation defined by $Q(x_1, x_2) = (x_2, x_1)$, let $\tilde{A}_a = Q \circ A_a \circ Q$, and $\tilde{S}_r = Q \circ S_r \circ Q$. The cone-adapted continuous ℓ th order α -shearlet system generated by $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$ is given by

$$\begin{aligned} & \{\phi(\cdot - \gamma)\}_{\gamma \in \mathbb{R}^2} \cup \{a^{-(1+\alpha)/2} \psi(A_{a^{-1}} S_{-r}(\cdot - \gamma))\}_{a \in (0,1], r \in R, \gamma \in \mathbb{R}^2} \\ & \cup \{a^{-(1+\alpha)/2} \tilde{\psi}(\tilde{A}_{a^{-1}} \tilde{S}_{-r}(\cdot - \gamma))\}_{a \in (0,1], r \in R, \gamma \in \mathbb{R}^2}, \end{aligned}$$

and forms a frame for $L^2(\mathbb{R}^2)$ with bounds A and B if, and only if,

$$\begin{aligned} A \leq |\hat{\phi}(\omega)|^2 &+ \int_0^1 \int_R a^{-(1+\alpha)} |\psi(A_{a^{-1}} S_{-r}(\cdot))^\wedge(\omega)|^2 dr da \\ &+ \int_0^1 \int_R a^{-(1+\alpha)} |\tilde{\psi}(\tilde{A}_{a^{-1}} \tilde{S}_{-r}(\cdot))^\wedge(\omega)|^2 dr da \leq B \end{aligned}$$

for a.e. $\omega \in \mathbb{R}^2$.

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