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System bandwidth and the existence of generalized shift-invariant frames

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Abstract

We consider the question whether, given a countable family of lattices $(\Gamma_j)_{j \in J}$ in a locally compact abelian group G , there exist functions $(g_j)_{j \in J}$ such that the resulting generalized shift-invariant system $(g_j(\cdot - \gamma))_{j \in J, \gamma \in \Gamma_j}$ is a tight frame of $L^2(G)$. This paper develops a new approach to the study of generalized shift-invariant system via almost periodic functions, based on a novel *unconditional convergence property*. From this theory, we derive characterizing relations for tight and dual frame generators, we introduce the *system bandwidth* as a measure of the total bandwidth a generalized shift-invariant system can carry, and we show that the so-called Calderón sum is uniformly bounded from below for generalized shift-invariant frames. Without the unconditional convergence property, we show, counter intuitively, that even orthonormal bases can have arbitrary small system bandwidth. Our results show that the question of existence of frame generators for a general lattice system is rather subtle and depends on analytical and algebraic properties of the lattice system.

Keywords: Almost periodic, bandwidth, Calderón sum, frame, generalized shift-invariant system

2010 MSC: Primary 42C15; Secondary 43A60

1. Statement of the problem

1.1. Introduction

Generalized shift-invariant (GSI) systems in $L^2(\mathbb{R}^n)$ are structured and flexible function systems of the form $(g_j(\cdot - \gamma))_{j \in J, \gamma \in \Gamma_j}$, where $(\Gamma_j)_{j \in J}$ and $(g_j)_{j \in J}$ are countable families of lattices in \mathbb{R}^n and functions in $L^2(\mathbb{R}^n)$, respectively. GSI systems have since the beginning of the millennium been used as a unifying framework for the study of Gabor, wavelet, curvelet, shearlet,

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etc. systems, but they have only recently been identified as function systems of independent interest, offering adaptive time-frequency and time-scale representations. Yet, even the most fundamental existence questions are still unanswered. One of these questions reads: for which family of lattices $(\Gamma_j)_{j \in J}$ does there exist a family of generators $(g_j)_{j \in J}$ providing stable reproducing formulas for $L^2(\mathbb{R}^n)$ in terms of the associated GSI system $(g_j(\cdot - \gamma))_{j \in J, \gamma \in \Gamma_j}$? Using frame theory terminology, we ask for the existence of generators $(g_j)_{j \in J}$ so that the associated GSI system is a frame for $L^2(\mathbb{R}^n)$. Under a natural regularity assumption we give a complete characterization of lattice systems $(\Gamma_j)_{j \in J}$ admitting GSI frames, i.e., stable GSI reproducing formulas, in dimension one ($n = 1$), and we give partial answers (e.g., sufficient or necessary conditions) in higher dimensions.

The main motivation for this study comes from the Euclidean setting \mathbb{R}^n , however, we will work mainly in the setting of locally compact abelian (LCA) groups since it simplifies the argumentation and notation. As a by-product our results will also be directly applicable for function systems on \mathbb{R}^n , \mathbb{Z}^d , $\mathbb{Z}/(N\mathbb{Z})$, \mathbb{T}^n , and vector-valued function systems. On the other hand, frames in the setting of locally compact groups have recently gained considerable interest among many researchers, e.g., [2, 14, 15, 7, 6] to mention a few of the most recent publications. Moreover, the abstract framework of LCA groups often leads to more transparent arguments and clarifies which results rely on special properties of the group (e.g., discrete/non-discrete groups). We remark that our results are new even in the important special case of \mathbb{R}^n .

In the remainder of this section we will set the stage in all details and give an overview of the main results of this work.

1.2. Some terminology

Let us start by recalling some facts from harmonic analysis on LCA groups; for a thorough introduction, we refer to [28, 13]. Throughout the paper, we let G denote a second countable LCA group. It is endowed with a translation-invariant Radon measure, unique up to normalization, called the Haar measure of G , and denoted by μ_G . We let $L^2(G)$ denote the Hilbert space of square-integrable functions with respect to Haar measure, and $C_b(G)$ the space of bounded, continuous functions. We will typically write LCA groups additively, and let $0 \in G$ denote the neutral element.

A *lattice*, sometimes called a uniform lattice, in G is a discrete subgroup $\Gamma < G$ with the property that the quotient G/Γ is compact. Since we assume G to be second countable, lattices in G are necessarily countable. A GSI system in $L^2(G)$ is constructed by picking a family of lattices $\Gamma_j \subset G$ and a family of vectors $(g_j)_{j \in J} \subset L^2(G)$, and defining the family

$$(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j},$$

where $T_\gamma f = f(\cdot - \gamma)$ denotes the translation operator on $L^2(G)$. This general class of systems of vectors was introduced in [12, 27] for $G = \mathbb{R}^n$, and further studied, e.g., in [16, 18] for the general setting of LCA groups.

GSI systems can be seen as countable filter banks or adaptive time-frequency representations. They are interesting objects in their own right and not only as a framework to unify Gabor and wavelet analysis. We refer to [1] for an implementation and applications of GSI systems in signal processing, to [8] for a construction of dual GSI frames for $L^2(\mathbb{R})$, and to [23, 24] for sparsity properties of GSI frames for $L^2(\mathbb{R})$.

Next, some terminology relating to frames, Bessel systems, and related notions. A family of vectors $(\eta_i)_{i \in I}$ contained in a Hilbert space \mathcal{H} is called a *Bessel system* if there exists a constant B such that, for all $f \in \mathcal{H}$

$$\sum_{i \in I} |\langle f, \eta_i \rangle|^2 \leq B \|f\|^2.$$

The constant B is called a *Bessel bound* of the system. If, in addition, there exists a lower bound $A > 0$ such that, for all $f \in \mathcal{H}$,

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f, \eta_i \rangle|^2, \quad (1.1)$$

the system is called a *frame*. The constants A and B are called *frame bounds*; the optimal frame bounds, denoted A^\dagger and B^\dagger , respectively, are the largest possible value for A and the smallest possible value for B in the above inequalities. If $A = B$, the frame is said to be *tight*. If $A = B = 1$, the frame is called a *Parseval frame*. As a particular case of frames, we mention orthonormal bases, which can be characterized as Parseval frames with normalized vectors.

For a Bessel system $(\eta_i)_{i \in I}$ the frame operator on \mathcal{H} is given as $S_\eta = \sum_i \langle \cdot, \eta_i \rangle \eta_i$; this operator is bounded and, furthermore, invertible if the lower bound (1.1) holds. Two Bessel systems $(\eta_i)_{i \in I}$ and $(\kappa_i)_{i \in I}$ are called *dual frames* if

$$f = \sum_{i \in I} \langle f, \eta_i \rangle \kappa_i \quad \text{for all } f \in \mathcal{H}; \quad (1.2)$$

in this case the two Bessel systems are automatically frames. Finally, a frame $(\eta_i)_{i \in I}$ has at least one dual frame $(\kappa_i)_{i \in I}$ so that (1.2) holds; the canonical choice is $\kappa_i = S_\eta^{-1} \eta_i$ for all $i \in I$.

A *generalized shift-invariant frame* is a generalized shift-invariant system that is a frame at the same time. From the general frame theory outlined above given a GSI frame $(T_\gamma g_j)_{j, \gamma}$, there exists a dual frame $(\tilde{g}_{j, \gamma})_{j, \gamma} \subset L^2(G)$, i.e., for all $f \in L^2(G)$, we have

$$f = \sum_{j \in J} \sum_{\gamma \in \Gamma_j} \langle f, T_\gamma g_j \rangle \tilde{g}_{j, \gamma} = \sum_{j \in J} \sum_{\gamma \in \Gamma_j} \langle f, \tilde{g}_{j, \gamma} \rangle T_\gamma g_j,$$

with unconditional convergence. Parseval frames are characterized by the property that one may take $\tilde{g}_{j, \gamma} = T_\gamma g_j$. However, for a non-tight GSI frame there might not exist any dual frames with GSI structure. Recall that a Riesz basis is a non-redundant frame and that Riesz bases only have one dual frame, namely, the canonical dual.

Proposition 1.1. *The dual basis of a GSI Riesz basis need not have GSI structure.*

Proof. We consider dyadic wavelet systems $(\eta_{j,k})_{j,k \in \mathbb{Z}}$ in $L^2(\mathbb{R})$, where $\eta_{j,k} = D_{2^j} T_k \eta$ and $D_a f = a^{1/2} f(a \cdot)$ for $a > 0$ and $f \in L^2(\mathbb{R})$. We will let ψ be a generator of an orthonormal wavelet basis $(\psi_{j,k})_{j,k \in \mathbb{Z}}$. Furthermore, we assume that ψ is compactly supported. Replacing ψ by a suitably chosen integer shift of ψ , we may then assume that ψ is supported inside the positive half-line $[0, \infty)$.

Daubechies [11, p. 989] and Chui and Shi [9, Section 3] prove that the canonical dual of the dyadic wavelet Riesz basis generated by $\eta = \psi + \epsilon D_2 \psi$ for $0 < \epsilon < 1$, where ψ is any orthonormal wavelet, does not have wavelet structure. Even more is true: it does not have GSI

structure. In fact, the canonical dual basis $(S_\eta^{-1}\eta_{j,k})_{j,k \in \mathbb{Z}}$ of $(\eta_{j,k})_{j,k \in \mathbb{Z}}$ can be computed explicitly as

$$S_\eta^{-1}\eta_{j,k} = \begin{cases} \psi_{j,k} - \varepsilon\psi_{j-1,k/2} + \cdots + (-\varepsilon)^n\psi_{j-n,k/2^n} & j \in \mathbb{Z}, k \in \mathbb{Z} \setminus \{0\}, \\ \sum_{m=0}^{\infty} (-\varepsilon)^m \psi_{j-m,0} & j \in \mathbb{Z}, k = 0, \end{cases} \quad (1.3)$$

where $n = \sup\{m \in \mathbb{N}_0 : 2^m | k\}$. Now, it can be seen by using the properties of $(\psi_{j,k})_{j,k \in \mathbb{Z}}$ that the dual basis is not a GSI system: Indeed, $S_\eta^{-1}\eta_{j,k}$, $k \neq 0$, is compactly supported, whereas it is easy to see that the support of $S_\eta^{-1}\eta_{j,0}$ is contained inside the positive half-line, and not compact, for all $j \in \mathbb{Z}$. But these observations imply that, given any lattice $\Lambda \subset \mathbb{R}$ and any $j \in \mathbb{Z}$, there exists $\lambda \in \Lambda$ negative with $|\lambda|$ sufficiently large, such that the support of $T_\lambda S_\eta^{-1}\eta_{j,0}$ is neither compact nor contained in the positive half-line, and thus $T_\lambda S_\eta^{-1}\eta_{j,0} \notin \{S_\eta^{-1}\eta_{j,k} : j, k \in \mathbb{Z}\}$. \square

Due to Proposition 1.1, we introduce the notion of a *dual generalized shift-invariant system* consisting of a system \mathcal{G} of lattices and two families $(g_j)_{j \in J}$, $(h_j)_{j \in J}$ in $L^2(G)$ such that the generalized shift-invariant systems fulfill the following: $(T_\gamma g_j)$ and $(T_\gamma h_j)$ are Bessel sequences, and

$$f = \sum_{j,\gamma} \langle f, T_\gamma g_j \rangle T_\gamma h_j$$

holds for all $f \in L^2(G)$.

1.3. Aims of this paper

The starting point of this paper is a system of lattices $\mathcal{G} = (\Gamma_j)_{j \in J}$ in G . We want to find sufficient and/or necessary criteria on \mathcal{G} for the existence of an associated system of vectors $(g_j)_{j \in J}$ such that the generalized shift-invariant system arising from these data is a (tight) frame. We then call the system $(g_j)_{j \in J}$ (*tight*) *frame generators* for \mathcal{G} . In the case of existence of dual frames, we call the systems $(g_j)_{j \in J}$ and $(h_j)_{j \in J}$ *dual frame generators*. The associated lower and upper frame bounds shall be denoted by A_g, B_g , etc.

Stated in such general terms, the problem of deciding the existence of frame generators seems somewhat impenetrable at first. We will not be able to fully solve the existence problem for generating systems, but we will derive results and construct examples showing that this question is remarkably subtle, involving both analytic and algebraic aspects.

An analytic condition that we shall investigate has to do with bandwidth. To motivate this notion, it is useful to recall the Shannon Sampling Theorem. Pick an interval $I = [\xi, \xi + L] \subset \mathbb{R}$, and define the closed subspace $PW_I = \{f \in L^2(\mathbb{R}) : \hat{f} \cdot \mathbf{1}_I = \hat{f}\} \subset L^2(\mathbb{R})$, where $\mathbf{1}_I$ denotes the characteristic function on I . Then, letting $g = L^{-1/2}(\mathbf{1}_I)^\vee$, where $(\cdot)^\vee$ denotes the inverse Fourier transform, we find that the system $(T_{k/L}g)_{k \in \mathbb{Z}}$ is an orthonormal basis of PW_I . The length L is commonly called the *bandwidth* of the space PW_I . We now revert this view. Starting from a lattice $\Gamma = c\mathbb{Z}$, we pick an interval I of length $L = 1/c$, and a generator $g \in PW_I$ such that $(T_{ck}g)_{k \in \mathbb{Z}}$ is an orthonormal basis of the Paley–Wiener space PW_I . Now, given a *system* of lattices $(c_j\mathbb{Z})_{j \in J}$, one possible strategy for the construction of compatible tight frame (in fact, orthonormal basis) generators $(g_j)_{j \in J}$ would be to cover the real line (the frequency domain) disjointly by intervals of length $L_j = 1/c_j$, and pick orthonormal basis generators for each PW_{I_j} . The question remains, however, whether the combined intervals suffice to cover

the full real axis, i.e., whether the *system bandwidth*, defined by $\sum_{j \in J} L_j$, is infinite. These considerations motivate the following definition.

Definition 1.2. Let $\mathcal{G} = (\Gamma_j)_{j \in J}$ be a system of lattices in G . Then the quantity, where $\text{covol}(\Gamma_j)$ is the Haar measure of a fundamental domain of Γ_j in G ,

$$BW(\mathcal{G}) = \sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} \in (0, \infty]$$

is called the *bandwidth* of \mathcal{G} ,

Now the above discussion suggests to study the relationship of the existence of tight frame generators to the condition that $BW(\mathcal{G}) \geq \mu_{\widehat{G}}(\widehat{G})$. We shall exhibit situations in which the bandwidth criterion is quite sharp, and other somewhat pathological cases, in which bandwidth is an irrelevant quantity. Hence a general characterization of lattice system admitting dual frame generators will have to involve both analytical, quantitative criteria (such as bandwidth), as well as algebraic ones.

The main results of this paper can be summarized as follows. We first introduce the approach to the analysis of GSI systems via almost periodic functions, established for wavelet systems by Laugesen in works [19, 21], and then further generalized to the GSI setting in [12]. We add a new result to this general approach that allows us to derive characterizing relations for dual frame generators under suitable, rather mild, unconditional convergence conditions (Theorem 3.11), and show, in Example 3.1, that it properly generalizes the known characterizing results for GSI frames [16, 18, 12]. Our mild convergence conditions replace previously used local integrability conditions. Under this convergence property we prove that the so-called Calderón sum for GSI frames is bounded from below by the lower frame bound (Theorem 3.13) which provides a necessary condition on (g_j) for the frame property, but which is also of independent interest. As a corollary, and still under the unconditional convergence property, we obtain that $BW(\mathcal{G}) \geq \mu_{\widehat{G}}(\widehat{G})$ is necessary for the existence of tight frame generators (Theorem 3.14). We then present a general existence result for frames assuming the existence of a dual covering (Theorem 3.16). As a consequence, we prove that lattice systems in $G = \mathbb{R}$ admitting GSI frames satisfying the unconditional convergence property are characterized by infinite bandwidth (Corollary 3.17).

In absence of the unconditional convergence property we construct tight GSI frames with arbitrarily small bandwidth (Theorem 4.1). Using the notion of independent lattices, we then exhibit a rather general class of lattice families for which the unconditional convergence property has an easy characterization and for which the characterizing relations from Theorem 3.11 are rather restrictive (Theorem 5.3). Further illustrations of various interesting features of the problem studied in this paper can be found in Examples 4.1, 5.1, 5.2 and 5.3.

Remark 1. In our considerations, taking some g_j to be the zero function is expressly allowed, unless we want to construct orthonormal bases. Hence, whenever the existence of Bessel, frame, or dual frame generators is shown for a subfamily of a family \mathcal{G} of lattices, it holds for \mathcal{G} itself. Thus one should be aware that the following sufficient conditions only need to be fulfilled by a suitable subfamily of the original lattice family. The LCA group G being second countable implies that $L^2(G)$ is separable, and thus all (discrete) frames in this space are countable. Hence, we will concentrate on countable lattice families Γ .

Remark 2. Shift-invariant (SI) systems are special cases of GTI systems with the property that $\Gamma_j = \Gamma$ for all $j \in J$ some fixed lattice Γ . SI systems are of interest since, e.g., both Gabor and Wilson systems are of this form. However, we remark that the subtlety of the existence problem completely disappears in this special case. Indeed, for any lattice Γ in G , the system $(\Gamma)_{j \in J}$ has orthonormal basis generators of the form $g_j = \sigma(j) \cdot \mathbf{1}_K$, where K is a fundamental domain of Γ and $\sigma : J \rightarrow \Gamma^\perp$ a bijection; we refer to Section 2 for definitions of the here used notation and terminology. Furthermore, the frame property of SI systems can be characterized by fiberization techniques and dual Gramians [6, 3, 26].

2. Notation for LCA groups

As stated above, G will always denote a second countable, locally compact abelian group, its Haar measure will be denoted by μ_G .

A *fundamental domain*, also known as a Borel section, associated to a lattice $\Gamma \subset G$ is a Borel set $K \subset G$ such that the Γ -translates tile G up to sets of measure zero; such sets always exist. A more rigorous formulation of this is as follows: Let $\mathbf{1}_K$ denote the indicator function of K . Then K is a fundamental domain for Γ if

$$\sum_{\gamma \in \Gamma} \mathbf{1}_K(x + \gamma) = 1 \quad (\text{a.e. } x \in G).$$

It is an easy exercise, using translation invariance of Haar measure, to prove that for any two fundamental domains K, K' of the same lattice Γ , one has $\mu_G(K) = \mu_G(K')$. The *covolume* $\text{covol}(\Gamma)$ of Γ in G is then defined as $\mu_G(K)$. Fundamental domains can always be chosen to be pre-compact.

For $G = \mathbb{R}^n$, all lattices are given by $\Gamma = C\mathbb{Z}^n$, where C can be any invertible matrix. Since the cube $[0, 1)^n$ is a fundamental domain for \mathbb{Z}^n , it is immediate that $C[0, 1)^n$ is a fundamental domain for $\Gamma = C\mathbb{Z}^n$, and one obtains $\text{covol}(\Gamma) = |\det(C)|$.

We let \widehat{G} denote the character group of G , i.e., the group of all continuous homomorphisms $G \rightarrow \mathbb{T}$. The duality between G and \widehat{G} is denoted by $\langle \cdot, \cdot \rangle : G \times \widehat{G} \rightarrow \mathbb{T}$; confusion of this notation with inner products in L^2 will be cleared up by the context. The Fourier transform of a function $f \in L^1(G)$ is then given by $\hat{f} : \widehat{G} \rightarrow \mathbb{C}$,

$$\hat{f}(\omega) = \int_G f(x) \overline{\langle x, \omega \rangle} dx.$$

This defines a bounded operator $\mathcal{F} : L^1(G) \rightarrow C_b(\widehat{G})$, $f \mapsto \hat{f}$. The Plancherel theorem states that, after proper normalization of the Haar measure on \widehat{G} , the operator $\mathcal{F}|_{L^1(G) \cap L^2(G)}$ extends uniquely to a unitary operator from $L^2(G)$ onto $L^2(\widehat{G})$ which we will also denote by $\mathcal{F}f = \hat{f}$.

Given a lattice Γ , its dual lattice (or annihilator) is given by

$$\Gamma^\perp = \{\alpha \in \widehat{G} : \langle \alpha, \gamma \rangle = 1 \quad \forall \gamma \in \Gamma\}.$$

By duality theory, $\Gamma^\perp \subset \widehat{G}$ is a lattice as well. In fact, if one normalizes the Haar measure on \widehat{G} in such a way that the Plancherel theorem holds, then the covolumes of Γ and Γ^\perp are related by

$$\text{covol}(\Gamma) \text{covol}(\Gamma^\perp) = 1.$$

In the case $G = \mathbb{R}^n$ and $\Gamma = C\mathbb{Z}^n$ for some invertible matrix C , the dual lattice is computed as $\Gamma^\perp = C^{-T}\mathbb{Z}^n$, with C^{-T} denoting the inverse transpose of C .

To summarize, we let a Haar measure on G be given. We assume dual measures so that Plancherel theorem holds, and we assume the counting measure on discrete subgroups Γ and choose the Haar measure on G/Γ as the quotient measure so that Weil's integral formula holds. Using this quotient measure $\mu_{G/\Gamma}$ on G/Γ , we can express the covolume as $\text{covol}(\Gamma) = \mu_{G/\Gamma}(G/\Gamma)$. The quantity $1/\text{covol}(\Gamma)$ is sometimes called the density of the subgroup, while $\text{covol}(\Gamma)$ is called the lattice size.

3. Almost periodic functions and GSI systems in $L^2(G)$

3.1. Fourier analysis of GSI systems

In order to understand the role of almost periodic functions, let us fix a dual GSI system given by the lattice system $(\Gamma_j)_{j \in J}$ and the associated functions $(g_j)_{j \in J}$ and $(h_j)_{j \in J}$. We fix a closed set $E \subset \widehat{G}$ of measure zero, and define

$$\mathcal{D}_E = \{f \in L^2(G) : \hat{f} \in L^\infty(\widehat{G}) \text{ and } \exists K \subset \widehat{G} \setminus E \text{ compact with } \hat{f}\mathbf{1}_K = \hat{f} \text{ a.e.}\}. \quad (3.1)$$

This is a translation-invariant and dense subspace of $L^2(G)$, and since the frame operator is bounded precisely when the associated system is a Bessel system, the Bessel property and further frame theoretical properties of the system only need to be checked on \mathcal{D}_E . Here $E \subset \widehat{G}$ denotes the *blind spot of the system* [16]; the specific choice of E depends on the application.

For $f \in \mathcal{D}_E$, we define the functions $w_{f;g,h,j} : G \rightarrow \mathbb{C}$ for $j \in J$ by

$$w_{f;g,h,j}(x) = \sum_{\gamma \in \Gamma_j} \langle T_x f, T_\gamma g_j \rangle \langle T_\gamma h_j, T_x f \rangle. \quad (3.2)$$

For each $j \in J$, the series in (3.2) converge pointwise to a continuous limit function as is seen by the following result. The result is a dual version of [18, Lemma 3.4] which is an adaptation of [12, Lemma 2.2].

Lemma 3.1. *Fix $f \in \mathcal{D}_E$ and $j \in J$. Let Γ_j be a lattice in G and $g_j, h_j \in L^2(G)$. Then $w_{f;g,h,j}$ is a trigonometric polynomial. More precisely,*

$$w_{f;g,h,j}(x) = \sum_{\alpha \in \Gamma_j^\perp} d_{j,\alpha} \langle \alpha, x \rangle,$$

where

$$d_{j,\alpha} = \frac{1}{\text{covol}(\Gamma_j)} \int_{\widehat{G}} \hat{f}(\omega) \hat{g}_j(\omega) \overline{\hat{f}(\omega + \alpha)} \hat{h}_j(\omega + \alpha) d\omega.$$

In particular, $d_{j,\alpha} = 0$ for all but finitely many $\alpha \in \Gamma_j^\perp$.

We define the function $w_{f;g,h}$ on G as

$$w_{f;g,h}(x) = \sum_{j \in J} w_{f;g,h,j}(x) = \sum_{j \in J} \sum_{\gamma \in \Gamma_j} \langle T_x f, T_\gamma g_j \rangle \langle T_\gamma h_j, T_x f \rangle \quad (3.3)$$

provided that the series converge. In the case $h_j = g_j$, for all $j \in J$, we write $w_{f;g} = w_{f;g,h}$ and $w_{f;g,j} = w_{f;g,h,j}$. Without any further assumption on the lattice system \mathcal{G} and the generators, we can only say that the series $\sum_{j \in J} w_{f;g,j}(x)$ converges in $[0, \infty]$, hence $w_{f;g} : G \rightarrow [0, \infty]$ is well-defined, while $w_{f;g,h} : G \rightarrow \mathbb{C}$ might not be.

However, under the Bessel property, the partial sums $\sum_{j \in J} w_{f;g,h,j}$ converge pointwise absolutely and uniformly on compact sets, and the limit functions are continuous and bounded. To prove these properties, we need the following well-known result.

Lemma 3.2. *If (Γ_j) , $(g_j)_{j \in J}$ generates a Bessel system in $L^2(G)$, then*

$$\sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\hat{g}_j(\omega)|^2 \leq B_g \quad (\text{a.e. } \omega).$$

The sum $\sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\hat{g}_j(\omega)|^2$ is called the *Calderón sum* of the GSI system in accordance with wavelet analysis. For a proof of Lemma 3.2 in $L^2(\mathbb{R}^n)$ we refer to [12, Proposition 4.1], and for a proof in $L^2(G)$ we refer to [16, 18].

Lemma 3.3. *Fix $f \in \mathcal{D}_E$. Let $(\Gamma_j)_{j \in J}$ denote a system of lattices, and $(g_j)_{j \in J}$, $(h_j)_{j \in J}$ associated function systems of Bessel generators. Then $w_{f;g,h}$ is uniformly continuous. Moreover, the right-hand side of*

$$w_{f;g,h} = \sum_{j \in J} w_{f;g,h,j}$$

converges uniformly and unconditionally on compact sets.

Proof. To begin with, note that the Bessel assumption on the generators guarantees that the sum defining $w_{f;g,h}$ converges pointwise absolutely. We next prove uniform continuity of the limit. Given $x_1, x_2 \in G$, we compute

$$\begin{aligned} & |w_{f;g,h}(x_1) - w_{f;g,h}(x_2)| \\ & \leq \sum_{j \in J} \sum_{\gamma \in \Gamma_j} |\langle T_{x_1} f, T_\gamma g_j \rangle \langle T_\gamma h_j, T_{x_1} f \rangle - \langle T_{x_2} f, T_\gamma g_j \rangle \langle T_\gamma h_j, T_{x_2} f \rangle| \\ & \leq \sum_{j \in J} \sum_{\gamma \in \Gamma_j} (|\langle (T_{x_1} - T_{x_2}) f, T_\gamma g_j \rangle \langle T_\gamma h_j, T_{x_1} f \rangle| + |\langle (T_{x_1} f, T_\gamma g_j) \langle T_\gamma h_j, (T_{x_2} - T_{x_1}) f \rangle|) \\ & \leq B_g \|(T_{x_1} - T_{x_2}) f\| \cdot B_h \|T_{x_1} f\| + B_g \|T_{x_1} f\| \cdot B_h \|(T_{x_1} - T_{x_2}) f\| \\ & \leq 2B_g B_h \|T_{x_1 - x_2} f - f\|^2, \end{aligned}$$

using the Bessel constants B_g, B_h and the fact that the regular representation $x \mapsto T_x$ is a homomorphism. Since this representation is strongly continuous, for any $\epsilon > 0$ there exists a neighborhood U of the identity element such that $\|T_{x_1 - x_2} f - f\|^2 < \epsilon$ whenever $x_1 - x_2 \in U$. This shows uniform continuity of $w_{f;g,h}$.

It remains to show uniform and unconditional convergence on compact sets. We first consider the case $g_j = h_j$. Here the terms on the right-hand side are positive, continuous functions, whose partial sums are bounded by the Bessel constant of the system generated by the $(g_j)_{j \in J}$. We already showed that the limit function is continuous, as well, and since the group

is metrizable [13], we may apply Dini's theorem to conclude that the sum converges uniformly on compact sets.

For the general case, fix $\epsilon > 0$ and a compact set $K \subset G$. Fix a finite set $J' \subset J$ with the property that, for all $J'' \supset J'$, it holds $|w_{f;g}(x) - \sum_{j \in J''} w_{f;g,j}(x)| < \epsilon/B_h$ for all $x \in K$.

The Cauchy–Schwarz inequality yields, for all $j \in J$, that

$$|w_{f;g,h,j}(x)| \leq w_{f;g,j}^{1/2}(x)w_{f;h,j}^{1/2}(x).$$

It follows, by a second application of the Cauchy-Schwarz inequality, that

$$\begin{aligned} \left| w_{f;g,h}(x) - \sum_{j \in J''} w_{f;g,h,j}(x) \right| &= \left| \sum_{j \in J \setminus J''} w_{f;g,h,j}(x) \right| \\ &\leq \underbrace{\left(\sum_{j \in J \setminus J''} w_{f;g,j}(x) \right)^{1/2}}_{< \epsilon/B_h} \underbrace{\left(\sum_{j \in J \setminus J''} w_{f;h,j}(x) \right)^{1/2}}_{\leq B_h} \\ &< \epsilon, \end{aligned}$$

whenever $x \in K$. This proves uniform and unconditional convergence on compact sets. \square

3.2. Almost periodic functions and the unconditional convergence property

The significance of almost periodic functions for GSI systems comes from the fact that the function $w_{f;g,h}$ is a sum of trigonometric polynomials. As soon as this sum converges uniformly, $w_{f;g,h}$ is an almost periodic function. This, in turn, allows us to use facts and results from the Fourier analysis of such functions, which we now recall. Our main sources for this subsection are [13, Chapter 18] and [25].

Definition 3.4. A function $f \in C_b(G)$ is called *almost periodic* if the set $\{T_x f : x \in G\} \subset C_b(G)$ is relatively compact with respect to the uniform norm. The space of all almost periodic functions on G is denoted by $\mathcal{A}(G)$.

As elucidated in [25], almost periodic functions are best understood in connection with the *Bohr compactification* G_B of the group G . This group is constructed by taking the dual group of \widehat{G} , where the latter is endowed with the discrete topology. By construction, G_B is a compact LCA group, and the duality between G and \widehat{G} gives rise to a canonical embedding $i_G : G \rightarrow G_B$, an injective, continuous group homomorphism with dense image. Throughout the following, we will identify G with its image under i_G , i.e., with a subgroup of G_B . If G is noncompact, this image is a proper subset (being noncompact), measurable (being σ -compact), and therefore of measure zero: Any measurable subgroup of positive measure contains a neighborhood of the neutral element, and is therefore open.

We now have the following characterizations of almost periodic functions. Note that by definition, a trigonometric polynomial is a linear combination of characters.

Theorem 3.5. *Let $f \in C_b(G)$. Then the following are equivalent:*

- (a) $f \in \mathcal{A}(G)$.
- (b) f is the uniform limit of trigonometric polynomials.
- (c) f has a (necessarily unique) continuous extension $f_B : G_B \rightarrow \mathbb{C}$.

We will call the function f_B from part (c) the *Bohr extension* of $f \in \mathcal{A}(G)$. Part (c) opens the door to Fourier expansions of almost periodic functions (and therefore for a proof of (b)), by making Fourier expansions of f_B available for the analysis of f . Since G_B is compact, every continuous function on G_B is the continuous limit of trigonometric polynomials. But G and G_B share the same dual \widehat{G} (only the induced topologies are different), hence this approximation result translates to functions in $\mathcal{A}(G)$. In order to compute the Fourier coefficients of f_B , we need to integrate over G_B , or better, devise an integration process on G that allows to compute these integrals without explicitly passing to the extension f_B . This is where the *mean* on $\mathcal{A}(G)$ comes into play, which is described in the following result, which summarizes Theorems 18.8-18.10 from [13].

Theorem 3.6. *Let G denote a second countable LCA group.*

- (a) *There exists a sequence $(H_n)_{n \in \mathbb{N}}$ of open, relative compact subsets $H_n \subset G$ with $G = \bigcup_{n \in \mathbb{N}} H_n$ and such that, for all $x \in G$,*

$$\lim_{n \rightarrow \infty} \frac{\mu_G((x + H_n) \cap (G \setminus H_n))}{\mu_G(H_n)} = 0 .$$

- (b) *Let $(H_n)_{n \in \mathbb{N}}$ be a sequence of subsets as in part (a). For any $f \in \mathcal{A}(G)$, the expression*

$$M(f) = \lim_{n \rightarrow \infty} \frac{1}{\mu_G(H_n)} \int_{H_n} f(x) dx$$

is well-defined and finite. Furthermore, if f_B denotes the Bohr extension of f , then

$$M(f) = \int_{G_B} f_B(y) dy .$$

- (c) *As a consequence of (b), $M(f)$ is independent of the choice of $(H_n)_{n \in \mathbb{N}}$.*

The quantity $M(f)$, as defined in Theorem 3.6, denotes the *mean* of $f \in \mathcal{A}(G)$. Given any $\alpha \in \widehat{G}$, we can then define the *Fourier coefficient* of f as

$$\hat{f}(\alpha) = M(f\bar{\alpha}) .$$

Using the facts that the map $\mathcal{A}(G) \ni f \mapsto f_B \in C(G_B)$ is injective, and that the duals of G and G_B coincide, standard facts of Fourier analysis on G_B give rise to the following important theorem.

Theorem 3.7. *Let $f \in \mathcal{A}(G)$. We then have:*

- (a) *Fourier uniqueness: $f = 0$ if and only if $\hat{f}(\alpha) = 0$, for all $\alpha \in \widehat{G}$.*

(b) Plancherel Theorem: $M(|f|^2) = \sum_{\alpha \in \widehat{G}} |\widehat{f}(\alpha)|^2$. In particular, only countably many Fourier coefficients are nonzero.

We remark that $\mathcal{A}(G)$ under the inner product $\langle f, g \rangle_{AP} = M(f\bar{g})$ for $f, g \in \mathcal{A}(G)$ is a pre-Hilbert space; its completion is called the Besicovitch space $B^2(G)$, for which $(\alpha)_{\alpha \in \widehat{G}}$ is an orthonormal basis. The completion with respect to the norm $M(|\cdot|)$ gives rise to the Besicovitch space $B^1(G)$.

With these definitions in place, we can now introduce a technical condition that will be of central importance to the following.

Definition 3.8. Let $(\Gamma_j)_{j \in J}$ denote a system of lattices, and let $(g_j)_{j \in J}$ and $(h_j)_{j \in J}$ denote generating systems in $L^2(G)$.

(a) The GSI systems $(T_\gamma g_j)_{j, \gamma}$ and $(T_\gamma h_j)_{j, \gamma}$ (or, for short, simply $(g_j)_j$ and $(h_j)_j$) have the *dual 1-unconditional convergence property* (dual 1-UCP) if, for all $f \in \mathcal{D}_E$, $w_{f;g,h} \in \mathcal{A}(G)$ and

$$w_{f;g,h} = \sum_{j \in J} w_{f;g,h,j} \quad (3.4)$$

unconditionally with respect to $M(|\cdot|)$, i.e., for every $\epsilon > 0$ there exists a finite set $J' \subset J$ such that for all finite set $J'' \supset J'$,

$$M \left(\left| w_{f;g,h} - \sum_{j \in J''} w_{f;g,h,j} \right| \right) < \epsilon$$

(b) The systems $(g_j)_j$ and $(h_j)_j$ have the *dual ∞ -UCP* if (3.4) holds with uniform convergence.

(c) If $h_j = g_j$ holds, for all $j \in J$, we say that the system $(g_j)_{j \in J}$ fulfills *p -UCP*, for $p \in \{1, \infty\}$.

Note that the Hölder inequality implies $M(|f|^p)^{1/p} \leq M(|f|^q)^{1/q} \leq \|f\|_\infty$, whenever $1 \leq p \leq q < \infty$. In particular, ∞ -UCP is the stronger condition.

Remark 3. The 1-UCP condition can be rephrased as requiring that

$$(w_{f;g,h})_B = \sum_{j \in J} (w_{f;g,h,j})_B$$

with convergence in $L^1(G_B)$, where we again used the subscript B to denote the (continuous) Bohr extension. This is one of the reasons why the condition $w_{f;g,h} \in \mathcal{A}(G)$ is included in the definition of 1-UCP. Clearly, checking this part can present a nontrivial obstacle. Note however that in the case $p = \infty$, uniform convergence already implies $w_{f;g,h} \in \mathcal{A}(G)$. Also, note that for the derivation of *necessary* conditions for dual systems, one departs from the assumption that $w_{f;g,h} \equiv 1$, which is clearly in $\mathcal{A}(G)$.

Remark 4. As for local integrability conditions, the p -UCP depends on the blind spot set E , which is a closed set $E \subset \widehat{G}$ of measure zero. Since $\mathcal{D}_E \subset \mathcal{D}_\emptyset$ for any blind spot set E , see (3.1), it follows that if p -UCP holds for $E = \emptyset$, it holds for any E . However, we usually only need the p -UCP to hold for *some* blind spot set E .

The Bessel generator assumption and the ∞ -UCP can be seen as regularity assumptions on $w_{f;g,h}$, e.g., both assumptions separately guarantee that $w_{f;g,h}$ is continuous. The two assumptions are in general unrelated. Bessel generators $(g_j)_{j \in J}$ do not imply ∞ -UCP and the ∞ -UCP does not imply Bessel generators. However, the following result shows that the analysis windows $(g_j)_{j \in J}$ and synthesis windows $(h_j)_{j \in J}$ can be separated in the verification of the dual p -UCP condition, when combined with a Bessel assumption.

Lemma 3.9. *Assume that we are given lattices $(\Gamma_j)_{j \in J}$, and generating systems $(g_j)_{j \in J}$ and $(h_j)_{j \in J}$.*

- (a) *Suppose that $(g_j)_{j \in J}$ fulfills ∞ -UCP, and that $(h_j)_{j \in J}$ is a system of Bessel generators. Then $(g_j)_{j \in J}$ and $(h_j)_{j \in J}$ fulfill the dual ∞ -UCP. The same result holds with assumptions on $(h_j)_{j \in J}$ and $(g_j)_{j \in J}$ interchanged.*
- (b) *Suppose that $(g_j)_{j \in J}$ fulfills 1-UCP, and that $(h_j)_{j \in J}$ is a system of Bessel generators such that $w_{f;g,h} \in \mathcal{A}(G)$. Then $(g_j)_{j \in J}$, $(h_j)_{j \in J}$ fulfill the dual 1-UCP. The same result holds with assumptions on $(h_j)_{j \in J}$ and $(g_j)_{j \in J}$ interchanged.*

Proof. Note that, for any finite set $J' \subset J$, and any $x \in G$,

$$\begin{aligned} \left| w_{f;g,h}(x) - \sum_{j \in J'} w_{f;g,h,j}(x) \right| &\leq \sum_{j \in J \setminus J'} \sum_{\gamma \in \Gamma_j} |\langle T_x f, T_\gamma g_j \rangle \langle T_\gamma h_j, T_x f \rangle| \\ &\leq \left| \sum_{j \in J \setminus J'} w_{f;g,j}(x) \right|^{1/2} w_{f;h}(x)^{1/2} \\ &\leq B_h^{1/2} \|f\| \left| w_{f;g}(x) - \sum_{j \in J'} w_{f;g,j}(x) \right|^{1/2} \end{aligned}$$

Now assuming that $(g_j)_{j \in J}$ fulfills ∞ -UCP, yields the desired conclusions about the dual system $(g_j)_{j \in J}$, $(h_j)_{j \in J}$; in particular uniform convergence of the series yields $w_{f;g,h} \in \mathcal{A}(G)$. Since $w_{f;g,h} = \overline{w_{f;h,g}}$, the second statement of (a) follows. In the case of 1-UCP, Hölder's inequality implies

$$M \left(\left| w_{f;g} - \sum_{j \in J'} w_{f;g,j} \right|^{1/2} \right) \leq M \left(\left| w_{f;g} - \sum_{j \in J'} w_{f;g,j} \right| \right)^{1/2},$$

hence part (b) follows in the same way as part (a). \square

Proposition 3.10. *Assume that we are given lattices $(\Gamma_j)_{j \in J}$, and generating systems $(g_j)_{j \in J}$ and $(h_j)_{j \in J}$ that fulfill the dual 1-UCP. Then, for all $f \in \mathcal{D}_E$ and all $\alpha \in \widehat{G}$, we have*

$$\widehat{w_{f;g,h}}(\alpha) = \sum_{j \in J} \widehat{w_{f;g,h,j}}(\alpha) \quad (3.5)$$

with absolute convergence. Hence, the generalized Fourier coefficients of $w_{f;g,h} = \sum_{\alpha \in \widehat{G}} c_\alpha \alpha$ are given by

$$c_\alpha \equiv \widehat{w_{f;g,h}}(\alpha) = \begin{cases} \sum_{j \in J: \alpha \in \Gamma_j^\perp} d_{j,\alpha} & \text{for } \alpha \in \bigcup_{j \in J} \Gamma_j^\perp, \\ 0 & \text{otherwise.} \end{cases} \quad (3.6)$$

Proof. Recall that 1-UCP entails L^1 -convergence of the Bohr extensions, and that the Fourier coefficients of any function in $\mathcal{A}(G)$ coincide with the coefficients of its Bohr extension. Since Fourier coefficients are continuous linear functionals with respect to the 1-norms, equation (3.5) follows. The last statement of the theorem is just a reformulation of (3.5) using Lemma 3.1. \square

Remark 5. Under the dual ∞ -UCP assumption in Proposition 3.10 in place for the dual 1-UCP, the conclusions of Proposition 3.10 still hold true, in particular, that $w_{f;g,h}$ is a continuous and almost periodic function that agrees pointwise with its generalized Fourier series.

The importance of the function $w_{f;g,h}$ and its generalized Fourier series in Proposition 3.10 is that it encodes frame theoretical properties of GSI systems. E.g., under the UCP assumption, a GSI system $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ is a frame with optimal bounds A^\dagger and B^\dagger if and only if

$$B^\dagger = \sup_{f \in \mathcal{D}_E, \|f\|=1} w_{f;g}(0) = \sup_{f \in \mathcal{D}_E, \|f\|=1} \sup_{x \in G} \sum_{j \in J} w_{f;g,j}(x) < \infty \quad (3.7)$$

and

$$A^\dagger = \inf_{f \in \mathcal{D}_E, \|f\|=1} w_{f;g}(0) = \inf_{f \in \mathcal{D}_E, \|f\|=1} \inf_{x \in G} \sum_{j \in J} w_{f;g,j}(x) > 0. \quad (3.8)$$

Estimates of the generalized Fourier series of $w_{f;g}$ were recently employed in (3.7) and (3.8) to obtain new necessary and sufficient conditions for the frame property of GSI systems [22]. Moreover, whenever the mixed frame operator $S_{g,h} = \sum_{j,\gamma} \langle \cdot, T_\gamma g_j \rangle T_\gamma h_j$ on $L^2(G)$ of $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ and $(T_\gamma h_j)_{j \in J, \gamma \in \Gamma_j}$ is well-defined, but not necessarily bounded, the discussions in the next section yield the representation, under appropriate convergence assumptions,

$$\langle \widehat{S_{g,h} f}, \hat{f} \rangle = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} \langle T_\alpha(t_\alpha \cdot \hat{f}), \hat{f} \rangle \quad \text{for each } f \in \mathcal{D}_E,$$

where $t_\alpha = \sum_{j \in J: \alpha \in \Gamma_j^\perp} \frac{1}{\text{covol}(\Gamma_j)} \overline{\hat{g}_j(\cdot)} \hat{h}_j(\cdot + \alpha)$.

3.3. Characterizing equations for dual and tight GSI frames

The following theorem exploits the fact that convergence of the sum

$$w_{f;g,h} = \sum_{j \in J} w_{f;g,h,j} \quad (3.9)$$

in the proper sense results in expressions for the Fourier coefficients

$$d_{j,\alpha} = \frac{1}{\text{covol}(\Gamma_j)} \int_{\widehat{G}} \hat{f}(\omega) \overline{\hat{g}_j(\omega)} \hat{f}(\omega + \alpha) \hat{h}_j(\omega + \alpha) d\omega.$$

Theorem 3.11. *Suppose that $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ and $(T_\gamma h_j)_{j \in J, \gamma \in \Gamma_j}$ are Bessel families fulfilling the dual 1-UCP. Then the following are equivalent:*

- (i) $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ and $(T_\gamma h_j)_{j \in J, \gamma \in \Gamma_j}$ are dual frames for $L^2(G)$,

(ii) for each $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$ we have

$$t_\alpha(\omega) := \sum_{j \in J: \alpha \in \Gamma_j^\perp} \frac{1}{\text{covol}(\Gamma_j)} \overline{\hat{g}_j(\omega)} \hat{h}_j(\omega + \alpha) = \delta_{\alpha,0} \quad a.e. \omega \in \widehat{G} \quad (3.10)$$

with absolute convergence.

Proof. (i) \Rightarrow (ii): Fix $f \in \mathcal{D}_E$. The dual frame assumption yields $w_{f;g,h} \equiv \|f\|^2$, which entails by Fourier uniqueness that, for all $\alpha \in \widehat{G}$,

$$\|f\|^2 \delta_{0,\alpha} = \langle w_{f;g,h}, \alpha \rangle_{AP} = \sum_{j \in J: \alpha \in \Gamma_j^\perp} d_{j,\alpha},$$

with unconditional convergence, where we have used Proposition 3.10.

For the derivation of (3.10) from this fact, we adopt the strategy used in the proof of [16, Theorem 3.4]. Fix $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$, and define

$$t_\alpha(\omega) = \sum_{j \in J: \alpha \in \Gamma_j^\perp} \frac{1}{\text{covol}(\Gamma_j)} \overline{\hat{g}_j(\omega)} \hat{h}_j(\omega + \alpha).$$

Note that the right-hand side actually converges absolutely by the following chain of inequalities:

$$\begin{aligned} \sum_{j \in J: \alpha \in \Gamma_j^\perp} \frac{1}{\text{covol}(\Gamma_j)} |\overline{\hat{g}_j(\omega)} \hat{h}_j(\omega + \alpha)| &\leq \sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\overline{\hat{g}_j(\omega)} \hat{h}_j(\omega + \alpha)| \\ &\leq \left(\sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\hat{g}_j(\omega)|^2 \right)^{1/2} \left(\sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\hat{h}_j(\omega + \alpha)|^2 \right)^{1/2} \\ &\leq B_g^{1/2} B_h^{1/2}, \quad (3.11) \end{aligned}$$

where the last inequality is due to Lemma 3.2, and B_g and B_h are the Bessel constants associated to $(g_j)_{j \in J}$ and $(h_j)_{j \in J}$, respectively. This shows also that

$$\sum_{j \in J: \alpha \in \Gamma_j^\perp} \int_{\widehat{G}} |\hat{f}(\omega) \overline{\hat{g}_j(\omega)} \hat{f}(\omega + \alpha) \hat{h}_j(\omega + \alpha)| d\omega < \infty.$$

Hence the multiplication operator $M_{\overline{t_\alpha}} : L^2(\widehat{G}) \rightarrow L^2(\widehat{G})$, $f \mapsto f \cdot \overline{t_\alpha}$ is well-defined and bounded. For all $f \in \mathcal{D}_E$ we have the relation

$$\langle \hat{f}, M_{\overline{t_\alpha}} T_{-\alpha} \hat{f} \rangle = \sum_{j \in J: \alpha \in \Gamma_j^\perp} d_{j,\alpha} = \delta_{0,\alpha} \langle \hat{f}, \hat{f} \rangle,$$

and since \mathcal{D}_E is dense, this implies

$$M_{\overline{t_\alpha}} T_{-\alpha} = \begin{cases} I & \alpha = 0, \\ 0 & \alpha \notin \bigcup_{j \in J} \Gamma_j^\perp \setminus \{0\}, \end{cases}$$

which in turn yields (3.10).

(ii) \Rightarrow (i): We have

$$\sum_{j \in J: \alpha \in \Gamma_j^\perp} d_{j,\alpha} = \langle \hat{f}, M_{t_\alpha} T_{-\alpha} \hat{f} \rangle = \delta_{0,\alpha} \langle \hat{f}, \hat{f} \rangle,$$

for all $f \in \mathcal{D}_E$, with the last equation provided by the assumption on the t_α . Hence, by Proposition 3.10, the Fourier coefficients of $w_{f;g,h} \in \mathcal{A}(G)$ coincide with the Fourier coefficients of the constant function. Hence the Bohr extension of $w_{f;g,h}$ is constant, and consequently, so is $w_{f;g,h}$. \square

Remark 6. Theorem 3.11 is indeed a generalization of [16, Theorem 3.4]. The cited result is established under the assumption that the so-called *dual α -local integrability condition (dual α -LIC)* holds. This condition involves the coefficients $(c_{j,\alpha})_{j,\alpha}$ associated to each $f \in \mathcal{D}_E$ via

$$c_{j,\alpha} = \frac{1}{\text{covol}(\Gamma_j)} \int_{\widehat{G}} |\hat{f}(\omega) \hat{g}_j(\omega) \hat{f}(\omega + \alpha) \hat{h}_j(\omega + \alpha)| d\omega.$$

A comparison with the definition of $d_{j,\alpha}$ shows that $c_{j,\alpha}$ is obtained by taking the absolute value of the integrand in the definition of $d_{j,\alpha}$, in particular,

$$|d_{j,\alpha}| \leq c_{j,\alpha}.$$

Now the dual α -LIC amounts to the requirement that $\sum_{j,\alpha} |c_{j,\alpha}| < \infty$ for every $f \in \mathcal{D}_E$. Since characters $\alpha \in \widehat{G}$ are bounded, the α -LIC implies in fact that

$$w_{f;g,h} = \sum_{j \in J} \sum_{\alpha \in \Gamma_j^\perp} d_{j,\alpha} \alpha,$$

with the right hand side converging *unconditionally and uniformly*. Thus dual ∞ -UCP is guaranteed, and we may apply Theorem 3.11 to recover [16, Theorem 3.4].

Note also that this argument goes through under the assumption $\sum_{j,\alpha} |d_{j,\alpha}| < \infty$. This condition could be strictly weaker than the dual α -LIC because of possible cancellations inside the integrals defining the $d_{j,\alpha}$ that could entail $|d_{j,\alpha}| \ll c_{j,\alpha}$. However, we are not aware of examples where this observation pays off. Finally, we remark that the classical LIC [12, 18] amounts to the requirement that $\sum_{j,\alpha} |\tilde{c}_{j,\alpha}| < \infty$ for every $f \in \mathcal{D}_E$, where

$$\tilde{c}_{j,\alpha} = \frac{1}{\text{covol}(\Gamma_j)} \int_{\widehat{G}} |\hat{f}(\omega) \hat{f}(\omega + \alpha)| |\hat{g}_j(\omega)|^2 d\omega,$$

which implies the α -LIC and therefore also the ∞ -UCP.

In the case of tight frames, the Bessel assumption in Theorem 3.11 is not necessary.

Theorem 3.12. *Suppose that $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ fulfills the 1-UCP. Then the following are equivalent:*

- (i) $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ is a Parseval frame for $L^2(G)$,

(ii) for each $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$ we have

$$\sum_{j \in J: \alpha \in \Gamma_j^\perp} \frac{1}{\text{covol}(\Gamma_j)} \overline{\hat{g}_j(\omega)} \hat{g}_j(\omega + \alpha) = \delta_{\alpha, 0} \quad a.e. \omega \in \widehat{G}. \quad (3.12)$$

Proof. We claim that if either (i) or (ii) holds, the convergence in (3.12) is absolute. The proof of this claim for (i) is clear from the chain of inequalities (3.11) with $h_j = g_j$, $j \in J$, since $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ is a Bessel system by assumption. On the other hand, if (ii) holds, then for $\alpha = 0$, we have $\sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\hat{g}_j(\omega)|^2 = 1$. By computations as in (3.11), this proves the claim.

The rest of the proof follows the proof of Theorem 3.11. \square

The following example was discovered by Bownik and Rzeszotnik [5] to show that the Calderón sum for Parseval GSI frames is not necessarily equal to one. The example was also the first construction to show that the t_α -equations (3.12) do not characterize Parseval GSI frames without some regularity assumption on Γ_j and g_j , e.g., the LIC. In the context of this paper, the example serves as an illustration that 1-UCP allows finer distinctions than LIC. It is constructed in $\ell^2(\mathbb{Z})$, but can be easily transferred to $L^2(\mathbb{R})$.

Example 3.1. Let $G = \mathbb{Z}$ and, for each $N = 2, 3, \dots$, write \mathbb{Z} as a disjoint union:

$$\mathbb{Z} = \bigsqcup_{j \in \mathbb{N}} \tau_j + N^j \mathbb{Z},$$

where $\tau_1 = 0$ and τ_j , $j \geq 2$, are chosen inductively as the smallest $t \in \mathbb{Z}$ in absolute value satisfying

$$\left(\bigsqcup_{i=1}^{j-1} (\tau_i + N^i \mathbb{Z}) \right) \cap (t + N^j \mathbb{Z}) = \emptyset.$$

It case t and $-t$ both are minimizers, we pick τ_j to be positive.

For $j \in J = \mathbb{N}$, let $\Gamma_j = N^j \mathbb{Z}$ and $g_j = \delta_{\tau_j}$, where δ_k denotes the sequence with $\delta_k(k) = 1$ and $\delta_k(\ell) = 0$ for $\ell \neq k \in \mathbb{Z}$. The GSI system $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$ since it is a reordering of the canonical orthonormal basis $(\delta_k)_{k \in \mathbb{Z}}$. Bownik and Rzeszotnik [5] show that $t_0(\omega) = \frac{1}{N-1}$ and that the GSI systems do not satisfy the LIC. For $N \geq 3$ this shows that the local integrability condition of [12, Theorem 2.1] cannot be removed. Kutyniok and Labate [18] used the example with $N = 2$ to show that Parseval frames need not satisfy the LIC.

In [16] it was noted that the characterizing equations (3.12) are satisfied for $N = 2$; to be precise, the example in [16] is slightly different from the present one, but the verification of the characterizing equations for our example is very similar, hence we leave out the details. Since GSI systems can satisfy the characterizing equations, but not the local integrability conditions nor the weaker α -LIC, it leaves room for an improvement of the results in both [12] and [16]. Hence, none of the known results on characterizing t_α -equations can be applied for the case $N = 2$. However, we will now show that Theorem 3.11 indeed can capture this phenomenon:

For $N = 2$ the 1-UCP holds, while it fails for $N \geq 3$. This is the desired conclusion as only the case $N = 2$ satisfies the characterizing equations.

A first indication of the striking difference between $N = 2$ and $N \geq 3$ comes from the growth rate of τ_j . For $N = 2$ we find by induction that $\tau_j = -\frac{1}{3}(-2)^j + \frac{1}{3}$, while $|\tau_j|$ grows linearly as $\frac{j}{2} \leq |\tau_j| \leq j$ for $N \geq 3$.

Let $N \geq 2, N \in \mathbb{N}$ be given. Since $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ is an orthonormal basis, we have that $w_{f;g}(x) = \sum_j w_{f;g,j}(x) = \|f\|^2$ for every $x \in \mathbb{Z}$. For simplicity, assume $f \in \mathcal{D}_\emptyset$ is normalized, that is, $\|f\|^2 = 1$ and $E = \emptyset$. For each $j \in \mathbb{N}$,

$$w_{f;g,j}(x) = \sum_{\gamma \in \Gamma_j} \left| \langle T_\gamma f, T_\gamma g_j \rangle \right|^2 = \sum_{\ell \in H_{x,j}} |f(\ell)|^2,$$

where $H_{x,j} = x + \tau_j + N^j \mathbb{Z}$. Hence, $w_{f;g,j}(x)$ is the value of the squared norm of the orthogonal projection of $T_x f$ onto $\text{span} \{ \delta_k : k \in \Gamma_j \}$.

We first prove that ∞ -UCP is not satisfied for any choice of N . For this purpose, fix $m \in \mathbb{Z}$ with $f(m) \neq 0$. Let $J \subset \mathbb{N}$ be any finite subset. Let $x \in \mathbb{Z}$ be such that $m - x \notin \bigcup_{j \in J} \tau_j + N^j \mathbb{Z}$; note that x exists by construction of the τ_j . It then follows that $\sum_{j \in J} w_{f;g,j}(x) \leq 1 - |f(m)|^2$, or

$$w_{f;g}(x) - \sum_{j \in J} w_{f;g,j}(x) \geq |f(m)|^2,$$

and thus $\|w_{f;g} - \sum_{j \in J} w_{f;g,j}\|_\infty \geq |f(m)|^2$. Hence, ∞ -UCP does not hold for $E = \emptyset$. Note that allowing a general blind spot set E , i.e., a closed subset of $\widehat{G} = \mathbb{T}$ of measure zero, does not change this conclusion as \mathcal{D}_E is a non-trivial, translation-invariant subspace of $\ell^2(\mathbb{Z})$.

Let us next consider 1-UCP. Note that $w_{f;g} = 1$ which is indeed almost periodic. We first compute the mean of $w_{f;g,j}$. Note that $w_{f;g,j}$ is N^j -periodic, and that the mean of a periodic function is just the average over one period. Hence we get

$$M(w_{f;g,j}) = \frac{1}{N^j} \sum_{x=0}^{N^j-1} w_{f;g,j}(x) = \frac{1}{N^j} \underbrace{\sum_{x=0}^{N^j-1} \sum_{m \in x + \tau_j + N^j \mathbb{Z}} |f(m)|^2}_{=\|f\|^2} = \frac{1}{N^j}.$$

Using linearity of the mean, we get for any finite set $J' \subset \mathbb{N}$

$$M(1 - \sum_{j \in J'} w_{f;g,j}) = 1 - \sum_{j \in J'} N^{-j}. \quad (3.13)$$

In particular, for $N = 2$ convergence of the geometric series yields

$$M(1 - \sum_{j \in J} w_{f;g,j}) \rightarrow 0$$

as J runs through any increasing family of finite subsets covering \mathbb{N} , and thus 1-UCP holds. For $N > 2$, however,

$$M(1 - \sum_{j \in J} w_{f;g,j}) \geq 1 - \sum_{j \in \mathbb{N}} N^{-j} = \frac{N-2}{N-1}$$

shows that 1-UCP is violated.

3.4. Necessary conditions for the frame property

From the characterizing equations in Theorem 3.11 we can derive a necessary condition for the frame property of a GSI system in terms of the Calderón sum. The condition can be seen as a quantitative version of the fact that the Fourier supports of the generators need to cover \widehat{G} .

Theorem 3.13. *Suppose that $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ is a frame for $L^2(G)$ satisfying the 1-UCP. Then*

$$A_g \leq \sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\hat{g}_j(\omega)|^2 \quad \text{for a.e. } \omega \in \widehat{G}. \quad (3.14)$$

Proof. Let $f \in \mathcal{D}_E$. By assumption, $w_{f;g} \in \mathcal{A}(G)$, and its mean is equal to the constant term of its Fourier series by Definition 3.8, that is,

$$M(w_{f;g}) = \langle w_{f;g}, \mathbf{1}_G \rangle_{AP} = \sum_{j \in J} d_{j,0} = \int_{\widehat{G}} |\hat{f}(\omega)|^2 t_0(\omega) d\omega,$$

where we have used (3.5).

The frame inequality implies that $A_g \|f\|^2 \leq w_{f;g}(x)$ for all $x \in G$. Since $w_{f;g} - A_g \|f\|^2 \geq 0$ a.e., it follows that

$$M(w_{f;g}) \geq M(A_g \|f\|^2) = M(\mathbf{1}_G) A_g \|f\|^2 = A_g \|f\|^2.$$

Hence, we arrive at

$$A_g \|f\|^2 \leq \int_{\widehat{G}} |\hat{f}(\omega)|^2 t_0(\omega) d\omega. \quad (3.15)$$

This, in turn, implies that $A_g \leq t_0(\omega)$ for a.e. $\omega \in \widehat{G}$. To see this, assume towards a contradiction that $t_0(\omega) < A_g$ for $\omega \in F$, where F is of positive measure. Let $\hat{f}(\omega) = \mathbf{1}_F$. Then $\int_{\widehat{G}} |\hat{f}(\omega)|^2 t_0(\omega) d\omega = \int_F t_0(\omega) d\omega < A_g \|f\|^2$, which contradicts (3.15). \square

Remark 7. Note that Theorem 3.13 is a generalization of results in [8, 9]. The argument in the proof of Theorem 3.13 can also be used to prove an upper bound, but this bound holds without any LIC/UCP assumptions by Lemma 3.2. We refer to Section 6 for applications to wavelet systems and generalizations of Theorem 3.13.

The following theorem substantiates the intuition on the role of bandwidth for the existence of generators. It proves that infinite bandwidth is necessary for the existence of frame generators $(g_j)_{j \in J}$ in non-discrete groups.

Theorem 3.14. *Suppose that $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ is a frame for $L^2(G)$ which satisfies the 1-UCP. Then*

$$BW(\mathcal{G}) \equiv \sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} \geq \frac{A_g}{B_g} \mu_{\widehat{G}}(\widehat{G}).$$

In particular, if G is non-discrete, then $BW(\mathcal{G}) = \infty$.

Proof. By integrating the lower bound in (3.14) over \widehat{G} , we obtain

$$\int_{\widehat{G}} A_g d\omega \leq \sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} \|\hat{g}_j\|^2. \quad (3.16)$$

From frame theory we know that a countable Bessel family $(\eta_i)_{i \in I}$ with Bessel bound B in a Hilbert space satisfies the norm bound $\|\eta_i\|^2 \leq B$ for all $i \in I$. In our settings, using isometry of translations and the Plancherel theorem, this fact yields $\|\hat{g}_j\|^2 \leq B_g$ which, combined with (3.16), proves the sought inequality. Finally, if G is non-discrete, the dual group \widehat{G} is non-compact, hence $\int_{\widehat{G}} A_g d\omega$ is infinite. \square

The following result notes a further basic fact: Lattice systems that generate a frame must be infinite, if G is non-discrete.

Corollary 3.15. *Assume that G is non-discrete. Let $(\Gamma_j)_{j \in J}$ denote a finite system of lattices. Then, for every system of generators $(g_j)_{j \in J}$, the associated GSI system does not possess a lower frame bound.*

Proof. Since G is non-discrete, \widehat{G} is non-compact, and therefore has infinite Haar measure. A GSI system $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ with a finite system of lattices, i.e., $\#|J| < \infty$, obviously satisfies ∞ -UCP and therefore 1-UCP. Let $C = \max_{j \in J} \frac{1}{\text{covol}(\Gamma_j)}$. Then $BW(\mathcal{G}) \leq C(\#|J|) < \infty$. The $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ system cannot be a frame by Theorem 3.14. \square

We end this subsection by remarking that, for discrete LCA groups G , the above discussions yield the following additional necessary condition for the Bessel property:

$$\sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} \|\hat{g}_j\|^2 \leq B_g \mu_{\widehat{G}}(\widehat{G}).$$

3.5. Sufficient conditions for the frame property

The next theorem builds on the intuition that motivated the introduction of our notion of bandwidth. Note in particular that the conditions of the following theorem can only be fulfilled if $BW(\mathcal{G}) \geq \mu_{\widehat{G}}(\widehat{G})$: Condition (i) implies that K_j must be contained in a fundamental domain modulo Γ_j^\perp , and then (ii) implies that \widehat{G} can be covered by fundamental domains mod Γ_j^\perp , as j runs through J . The latter can only hold if the measures of these domains at least sum up $\mu_{\widehat{G}}(\widehat{G})$.

Theorem 3.16. *Let $\mathcal{G} = (\Gamma_j)_{j \in J}$ denote a family of lattices. Assume that there exist Borel sets $K_j \subset \widehat{G}$, for $j \in J$, fulfilling the following two properties:*

- (i) $\mu_{\widehat{G}}(K_j \cap \gamma + K_j) = 0$ for all $\gamma \in \Gamma_j^\perp \setminus \{0\}$ and for all $j \in J$,
- (ii) $\mu_{\widehat{G}}(\widehat{G} \setminus \bigcup_{j \in J} K_j) = 0$.

Then there exists a family $(g_j)_{j \in J}$ such that the associated GSI system is a Parseval frame. In addition, the system can be chosen to fulfill the relations

$$\forall j \in J \forall \alpha \in \Gamma_j^\perp : \hat{g}_j(\omega) \hat{g}_j(\omega + \alpha) = 0 \quad (\text{a.e. } \omega), \quad (3.17)$$

and

$$\forall j_1, j_2 \in J \text{ with } j_1 \neq j_2 : \hat{g}_{j_1}(\omega) \hat{g}_{j_2}(\omega) = 0 \quad (\text{a.e. } \omega). \quad (3.18)$$

If, in addition to (i) and (ii), the sets $\{K_j\}_{j \in J}$ fulfill $\mu_{\widehat{G}}(K_j \cap K_{j'}) = 0$, for $j \neq j'$, as well as $\mu_{\widehat{G}}(K_j) = \frac{1}{\text{covol}(\Gamma_j)}$, there exist orthonormal basis generators with these properties.

Proof. Without loss of generality, we can assume either $J = \mathbb{N}$ or $J = \{1, 2, \dots, N\}$. For each j , pick a fundamental domain K_j^1 of Γ_j^\perp which satisfies $K_j \subset K_j^1 \bmod \Gamma_j^\perp$, and define the function h_j by $\hat{h}_j = \text{covol}(\Gamma_j)^{1/2} \mathbf{1}_{K_j^1}$. Then $(T_\gamma h_j)_{\gamma \in \Gamma_j}$ is an orthonormal basis of the closed subspace

$$\mathcal{H}_j^1 = \{f \in L^2(G) : \hat{f} \cdot \mathbf{1}_{K_j^1} = \hat{f}\}$$

by Kluvanek's Theorem [17]. Next define, for $j \geq 1$,

$$K_j^2 = K_j \setminus \bigcup_{l < j} K_l.$$

Then $(K_j^2)_{j \in J}$ is a disjoint covering of \widehat{G} , and if we define

$$\mathcal{H}_j^2 = \{f \in L^2(G) : \hat{f} \cdot \mathbf{1}_{K_j^2} = \hat{f}\},$$

we obtain $L^2(G) = \bigoplus_j \mathcal{H}_j^2$. Furthermore, for any given j , the function g_j , defined by $\hat{g}_j = \text{covol}(\Gamma_j)^{1/2} \mathbf{1}_{K_j^2}$ is the projection of h_j into \mathcal{H}_j^2 . Since this projection commutes with translations, one gets that the associated shift-invariant system $(T_\gamma g_j)_{\gamma \in \Gamma_j} \subset \mathcal{H}_j^2$ is the image of an orthonormal basis under the projection onto the subspace \mathcal{H}_j^2 , and thus a Parseval frame of that subspace. Finally, taking the union over Parseval frames of an orthogonal sequence of subspaces spanning the whole space yields a Parseval frame of the latter.

In the case where the K_j fulfill $\mu_{\widehat{G}}(K_j \cap K_{j'}) = 0$, for $j \neq j'$ and $\mu_{\widehat{G}}(K_j) = 1/\text{covol}(\Gamma_j)$, it follows that K_j and K_j^2 only differ by a set of measure zero, and the system $(T_\gamma g_j)_{\gamma \in \Gamma_j}$ is an orthonormal basis of \mathcal{H}_j^2 . Hence the full system is an orthonormal basis of $L^2(G)$. \square

If the underlying group is $G = \mathbb{R}$, we can now formulate the following characterization of existence of frame generators.

Corollary 3.17. *Suppose that $\mathcal{G} = (\Gamma_j)_{j \in J}$ is a family of lattices in \mathbb{R} . Then the following are equivalent:*

- (i) *There exists a system $(g_j)_{j \in J}$ of frame generators satisfying the LIC-condition.*
- (ii) *There exists a system $(g_j)_{j \in J}$ of frame generators satisfying the 1-UCP condition.*

(iii) $BW(\mathcal{G}) = \infty$.

Proof. The implication (i) \Rightarrow (ii) is clear by Remark 6. Implication (ii) \Rightarrow (iii) is provided by Theorem 3.14. Finally, if $BW(\mathcal{G}) = \infty$, we use Theorem 3.16 to construct generators for \mathcal{G} . Given any $f \in \mathcal{D}_E$, we use (3.17) and the construction of the \hat{g}_j to verify LIC via

$$\begin{aligned}
\sum_{j,\alpha} |c_{j,\alpha}| &= \sum_{j,\alpha} \frac{1}{\text{covol}(\Gamma_j)} \int_{\mathbb{R}^1} |\hat{f}(\omega) \hat{g}_j(\omega) \hat{g}_j(\omega + \alpha) \hat{f}(\omega + \alpha)| d\omega \\
&= \sum_j \frac{1}{\text{covol}(\Gamma_j)} \int_{\mathbb{R}^1} |\hat{f}(\omega)|^2 |g_j(\omega)|^2 d\omega \quad (\text{by (3.17)}) \\
&= \int_{\mathbb{R}^1} |\hat{f}(\omega)|^2 \underbrace{\sum_j \frac{1}{\text{covol}(\Gamma_j)} |g_j(\omega)|^2}_{\equiv 1} d\omega \\
&= \|f\|^2.
\end{aligned}$$

□

We remark that the equivalences in Corollary 3.17 are false without the LIC/UCP assumption. This follows from Theorem 4.1, proved in Section 4.

The next result describes classes of lattice systems in \mathbb{R}^n for which the intuition from the one-dimensional case remains valid. Example 5.1 in Section 5 shows that the assumption on the singular values cannot be dropped.

Proposition 3.18. *Assume that the system $\mathcal{G} = (C_j \mathbb{Z}^n)_{j \in J}$ of lattices in \mathbb{R}^n has the property that for all $j \in J$, the quotient of maximal singular value of C_j , divided by the minimal singular value, is bounded by a constant. Then $BW(\mathcal{G}) = \infty$ implies the existence of a family of tight frame generators.*

Proof. Fix $j \in J$, and let $\sigma_{\min}(j)$ and $\sigma_{\max}(j)$ denote the minimal and maximal singular value of C_j^{-T} , respectively. By the assumption on the family, we have

$$\frac{\sigma_{\max}(j)}{\sigma_{\min}(j)} \leq K, \quad (3.19)$$

for $K > 0$ fixed. We let $C_j^{-T} = UDV$ denote the singular value decomposition, where U and V are orthogonal, and D diagonal with diagonal entries ranging between $\sigma_{\min}(j)$ and $\sigma_{\max}(j)$. To simplify notation, we suppress the dependence of j in the singular values. Denoting by $B_{1/2}(0)$ the open ball around zero with respect to the euclidean norm, we have the inclusions

$$(-1/2, 1/2)^n \subset 2\sqrt{n}B_{1/2}(0) \subset 2n(-1/2, 1/2)^n.$$

This gives the following chain of inclusions

$$\begin{aligned}
\sigma_{\min} U^{-1}(-1/2, 1/2)^n &\subset 2\sqrt{n}\sigma_{\min} U^{-1} B_{1/2}(0) \\
&\subset 2n\sigma_{\min}(-1/2, 1/2)^n.
\end{aligned}$$

On the other hand, we have $\sigma_{\min}(B_{1/2}(0)) \subset D(B_{1/2}(0))$, and thus, since $VB_{1/2}(0) = B_{1/2}(0)$, we get

$$\begin{aligned}\sigma_{\min}(-1/2, 1/2)^n &\subset 2\sqrt{n}\sigma_{\min}B_{1/2}(0) \subset 2\sqrt{n}\sigma_{\min}VB_{1/2}(0) \\ &\subset 2\sqrt{n}DVB_{1/2}(0) \subset 2nDV(-1/2, 1/2)^n.\end{aligned}$$

Combining these inclusions yields

$$\begin{aligned}\frac{\sigma_{\min}}{4n^2}(-1/2, 1/2)^n &\subset \frac{\sigma_{\min}}{2n}U(-1/2, 1/2)^n \\ &\subset UDV(-1/2, 1/2)^n = C_j^{-T}(-1/2, 1/2)^n.\end{aligned}$$

Furthermore, recalling the dependence of j , we have

$$|\det(C_j)^{-T}| \leq \sigma_{\max}(j)^n \leq K^n \sigma_{\min}(j)^n$$

via (3.19), and thus the infinite bandwidth assumption yields

$$\sum_{j \in J} \sigma_{\min}(j)^n = \infty.$$

To summarize, we have shown that the fundamental domains $C_j[-1/2, 1/2]^n$ modulo Γ_j^\perp contain cubes of infinite combined volume. Now the elementary, but somewhat technical following Lemma 3.19 shows the desired covering property. \square

Lemma 3.19. *Let $K_j = k_j[0, 1]^n$, $j \in \mathbb{N}$, denote a sequence of cubes in \mathbb{R}^n , with $\sum_{j \in \mathbb{N}} k_j^n = \infty$. Then there exist vectors $\tau_j \in \mathbb{R}^n$ such that $\mathbb{R}^n = \bigcup_{j \in \mathbb{N}} \tau_j + K_j$.*

Proof. For the following argument, it is helpful to recall the notion of a dyadic cube. By this we mean a subset $2^k(m + [0, 1]^n) \subset \mathbb{R}^n$, with $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. What we need in the following is that each dyadic cube decomposes into disjoint dyadic cubes of smaller size.

We first show the simpler statement that there exist τ_j ($j \in \mathbb{N}$) such that

$$[0, 1]^n \subset \bigcup_{j \in \mathbb{N}} \tau_j + K_j.$$

To see this, we first observe that we may assume that $\{k_j : j \in \mathbb{N}\} \subset \{2^m : m \in \mathbb{Z}\}$: If \tilde{k}_j denotes the largest power of 2 that is less than or equal to k_j , then we have $\sum_j \tilde{k}_j^n = \infty$ as well, and any covering by the smaller cubes solves the problem, as well.

Secondly, note that the problem is easy to solve if the k_j do not converge to zero. In that case, there is either an unbounded subsequence (in which case a single cube from the system can cover the unit cube), or there exist infinitely many cubes of the same size, which then can be used to cover the unit cube.

Hence, possibly after reindexing the sequence, we are left with the case where $(k_j)_{j \in \mathbb{N}}$ is a decreasing sequence of powers of 2, converging to zero. Here we can inductively pick τ_j , $j = 1, \dots$, with the property that for all $\ell \in \mathbb{N}$ satisfying

$$[0, 1]^n \setminus \bigcup_{j \leq \ell} \tau_j + k_j[0, 1]^n \neq \emptyset \tag{3.20}$$

we have

$$\bigcup_{j=1}^{\ell+1} \tau_j + k_j [0, 1)^n \subset [0, 1)^n.$$

To see this, we pick $\tau_1 = 0$. Assuming that τ_1, \dots, τ_ℓ are determined, and (3.20) holds for ℓ , we note that by the inductive assumption, $[0, 1)^n \setminus \bigcup_{j \leq \ell} \tau_j + k_j [0, 1)^n$ is the complement of a union of dyadic cubes in $[0, 1)$, with side-lengths greater than or equal to k_ℓ , which in turn is greater than or equal to $k_{\ell+1}$. In particular, if this complement is nonempty, it is the union of dyadic cubes of side-length $k_{\ell+1}$. Hence there exists $\tau_{\ell+1} = 2^{k_{\ell+1}} m$, with $m \in \mathbb{Z}^n$, with the desired property.

Since the volumes of the cubes add up to infinity, the condition (3.20) can only hold for finitely many ℓ . Hence we have $[0, 1)^n \subset \bigcup_{j=1}^N \tau_j + k_j [0, 1)^n$, for sufficiently large N .

In order to cover all of \mathbb{R}^n by shifted cubes, we reindex the sequence $(k_j)_{j \in \mathbb{N}}$ into a double sequence $(r_{j,\ell})_{(j,\ell) \in \mathbb{N}^2}$ with the property that, for all $j \in \mathbb{N}$, $\sum_{\ell \in \mathbb{N}} r_{j,\ell}^n = \infty$. Numbering the cubes of the type $m + [0, 1)^n$, with $m \in \mathbb{Z}^n$, as $(M_j)_{j \in \mathbb{N}}$, the first step of the proof shows that we can cover M_j using the cubes with side-lengths $(t_{j,\ell})_{\ell \in \mathbb{N}}$. Hence we have achieved the desired covering of \mathbb{R}^n using the full family of cubes. \square

4. Finite system bandwidth

By the intuition outlined in the introduction, and substantiated in Theorem 3.14, large bandwidth $BW(\mathcal{G}) \geq \mu_{\widehat{G}}(\widehat{G})$ is necessary for the existence of tight frame generators. On non-discrete groups even *infinite* bandwidth is necessary. Note however that these conclusions required additional assumptions, in the form of LIC or UCP. We will see that without these assumptions, the bandwidth intuition fails. Surprisingly, we will even see that $BW(\mathcal{G})$ can be arbitrarily small, while still preserving the frame property; actually, even orthonormal bases can have arbitrarily small system bandwidth.

Theorem 4.1. *Assume that there exists a sequence $(\Gamma_n)_{n \in \mathbb{N}_0}$ of strictly decreasing lattices $\Gamma_0 \supsetneq \Gamma_1 \supsetneq \dots$ in G . Then, given any $\epsilon > 0$, there exists a system $\mathcal{L} = (\Lambda_i)_{i \in I}$ of lattices in G with $BW(\mathcal{L}) < \epsilon$, and a system of functions $(h_i)_{i \in I}$ such that the associated GSI system is an orthonormal basis.*

The remainder of this section will prove the theorem. The following results exploit an idea introduced by Bownik and Rzesotnik in [5], namely that it is possible to index the same family of vectors as GSI system over different lattice families. While this relabeling does not affect any pertinent property of the associated frame operator(s), it may influence other properties of the system, most notably its bandwidth.

Definition 4.2. Let $\mathcal{G} = (\Gamma_j)_{j \in J}$ and $\mathcal{L} = (\Lambda_i)_{i \in I}$ denote lattice systems. We say that \mathcal{L} is a *refinement* of \mathcal{G} if there exists a partition $(I_j)_{j \in J}$ of I and group elements $(\gamma_i)_{i \in I}$ with the property

$$\forall j \in J : \Gamma_j = \bigcup_{i \in I_j} \gamma_i + \Lambda_i.$$

We then have the following obvious fact.

Lemma 4.3. Let $\mathcal{G} = (\Gamma_j)_{j \in J}$ be a system of lattices, and $\mathcal{L} = (\Lambda_i)_{i \in I}$ a refinement of \mathcal{G} . Given a system of functions $(g_j)_{j \in J}$, and define

$$h_i = T_{\gamma_i} g_j, i \in I_j.$$

Then the GSI system $(T_{\lambda} h_i)_{i \in I, \lambda \in \Lambda_i}$ is obtained by reindexing the GSI system $(T_{\gamma} g_j)_{j \in J, \gamma \in \Gamma_j}$. In particular, it is a Bessel system, a tight/Parseval frame, or an orthonormal basis if and only if the original system has the same properties. This observation extends to dual systems.

Remark 8. Note that whenever one has

$$\Gamma \supset \bigcup_{i \in I} \gamma_i + \Lambda_i$$

with finitely many lattices $\Lambda_i, i \in I$, then

$$\frac{1}{\text{covol}(\Gamma)} \geq \sum_{i \in I} \frac{1}{\text{covol}(\Lambda_i)}.$$

Hence, if \mathcal{L} is a refinement of \mathcal{G} , then one has that

$$BW(\mathcal{L}) \leq BW(\mathcal{G}).$$

The whole point of introducing refinements to our discussion is the fact that this inequality can be proper. It is also worth noting that whenever the index sets I_j occurring in a refinement are all finite, the bandwidth does not change.

We next show that refinements can be constructed from chains of subgroups. Note that Lemma 4.4 is valid also for non-abelian groups, but we only formulate it for the setting we need.

Lemma 4.4. Let H denote a countable abelian group, and let $(H_j)_{j \in \mathbb{N}}$ denote a sequence of proper subgroups with finite index, and $H_j \supsetneq H_{j+1}$ for all $j \in \mathbb{N}$. Then there exists a sequence $(\gamma_j)_{j \in \mathbb{N}} \subset H$ such that

$$H = \bigcup_{j \in \mathbb{N}} \gamma_j + H_j.$$

Proof. Let $(h_k)_{k \in \mathbb{N}}$ denote an enumeration of H . We choose the γ_j inductively, with $\gamma_1 = h_1$. Then $H \setminus \gamma_1 + H_1$ is nonempty.

Assume that after j steps, we have found $\gamma_1, \dots, \gamma_j$ such that

$$K_j = H \setminus \bigcup_{\ell \leq j} \gamma_\ell + H_\ell$$

is nonempty. Since $H_j \subset H_\ell$ for all $\ell < j$, K_j is a union of H_j -cosets. Now pick $k \in \mathbb{N}$ minimal with $h_k \notin K_j$, and let $\gamma_{j+1} = h_k$. Then $\gamma_{j+1} + H_{j+1} \subset K_j$, because $H_j \supset H_{j+1}$, and thus $(\gamma_{j+1} + H_{j+1} \cap \gamma_\ell + H_\ell) \subset (K_j \cap \gamma_\ell + H_\ell) = \emptyset$, for all $\ell \leq j$. Finally, the fact that $H_j \supsetneq H_{j+1}$ implies that

$$K_{j+1} = H \setminus \bigcup_{\ell \leq j+1} \gamma_\ell + H_\ell$$

is a nonempty union of H_{j+1} -cosets.

Thus the inductive procedure can be continued to yield a sequence $(\gamma_j)_{j \in \mathbb{N}}$ with $\gamma_j + H_j \cap \gamma_\ell + H_\ell = \emptyset$ for $j \neq \ell$. In addition, the choice of γ_{j+1} in the induction step allows to prove inductively that $h_k \in \bigcup_{j \leq k} \gamma_j + H_j$, for all $k \in \mathbb{N}$. Hence the sequence has all the desired properties. \square

Proof of Theorem 4.1. First consider a constant lattice system $\mathcal{G} = (\Gamma_0)_{\alpha \in \Gamma_0^\perp}$. If $K \subset \widehat{G}$ is any fundamental domain modulo Λ_0^\perp , then its translates $K_\alpha = \alpha + K$, with $\alpha \in \Lambda_0^\perp$ are a disjoint covering of \widehat{G} , and Theorem 3.16 provides the existence of a family of orthonormal basis generators for \mathcal{G} .

We will now construct a refinement \mathcal{L} of \mathcal{G} with finite bandwidth, as follows: Let $I = \Gamma_0^\perp \times \mathbb{N}$, and fix a bijection $\varphi : I \rightarrow \mathbb{N}$ with the property that, for every $\alpha \in \Gamma_0^\perp$, the sequence $\varphi(\alpha, \cdot)$ is strictly increasing. Given $i = (\alpha, k) \in I$, let $\Lambda_i = \Gamma_{\varphi(i)}$. By choice of φ , we have for all $\alpha \in \Gamma_0^\perp$, that

$$\Gamma_0 \supseteq \Lambda_{\alpha,1} \supseteq \Lambda_{\alpha,2} \supseteq \dots$$

Hence Lemma 4.4 implies that \mathcal{L} is a refinement of \mathcal{G} , and since φ is bijective, the refined system has bandwidth

$$BW(\mathcal{L}) = \sum_{i \in I} \frac{1}{\text{covol}(\Lambda_i)} = \sum_{n \in \mathbb{N}} \frac{1}{\text{covol}(\Gamma_n)}.$$

Now for every $n \in \mathbb{N}$, the inclusion $\Gamma_n \subset \Gamma_{n-1}$ yields $\text{covol}(\Gamma_n) = [\Gamma_n : \Gamma_{n-1}] \text{covol}(\Gamma_{n-1})$. Since the inclusions are strict, all subgroup indices are at least 2, which leads to $\text{covol}(\Gamma_n) \geq 2^n \text{covol}(\Gamma_0)$, and we finally arrive at

$$BW(\mathcal{L}) \leq \sum_{n \in \mathbb{N}} \frac{1}{2^n \text{covol}(\Gamma_0)} = \frac{1}{\text{covol}(\Gamma_0)}$$

Hence, there exist GSI orthonormal bases with bandwidth $1/\text{covol}(\Gamma_0)$.

Now, starting from the constant lattice system $\mathcal{G} = (\Gamma_m)_{\alpha \in \Gamma_m^\perp}$ in the above construction, we obtain orthonormal bases with bandwidth less than or equal to $1/\text{covol}(\Gamma_m)$. \square

Lemma 4.3 showed that a system of frame, Bessel, or dual frame generators for a system \mathcal{G} can be used to provide a system of generators for \mathcal{L} , whenever \mathcal{L} is a refinement of \mathcal{G} . The converse is generally not true as the following example shows.

Example 4.1. The system $\mathcal{L} = (2^j \mathbb{Z})_{j \in \mathbb{N}}$ is a refinement of the single lattice system $\mathcal{G} = (\mathbb{Z})_{i=1}$. By Theorem 4.1, there exist orthonormal basis generators in $L^2(\mathbb{R})$ for \mathcal{L} , but by Corollary 3.15, \mathcal{G} has no frame generators.

5. Independent lattices and UCP

The aim of this section is to exhibit a general setup for which ∞ -UCP holds as soon as $w_{f,g,h}$ is bounded. Furthermore, we argue that this setup is the generic case of GSI systems and that it leads to rather stringent condition on the frame generators.

Definition 5.1. A lattice system $(\Gamma_j)_{j \in J}$ is called *independent* if for all families $(x_j)_{j \in J'}$ with finite $J' \subset J$ and $x_j \in \Gamma_j$, we have the implication

$$\sum_{j \in J'} x_j = 0 \Rightarrow \forall j \in J' : x_j = 0.$$

We call the system *pairwise independent* if $\Gamma_j \cap \Gamma_k = \{0\}$, whenever $j \neq k$.

We will be interested in lattice families whose dual lattices are independent. The following lemma characterizes this condition in terms of a density property.

Lemma 5.2. *Let $(\Gamma_j)_{j \in J}$ denote a family of lattices in G . Then the following are equivalent:*

(i) *The dual lattices $(\Gamma_j^\perp)_{j \in J}$ are independent.*

(ii) *For all finite subsets J' , the subgroup*

$$\{(x + \Gamma_j)_{j \in J'} : x \in G\} \subset \prod_{j \in J'} G/\Gamma_j$$

is dense with respect to the product topology.

(iii) *The subgroup*

$$\{(x + \Gamma_j)_{j \in J} : x \in G\} \subset \prod_{j \in J} G/\Gamma_j$$

is dense with respect to the product topology.

Proof. For the proof of (i) \Leftrightarrow (ii), consider the continuous group homomorphism $\varphi : G \rightarrow \prod_{j \in J'} G/\Gamma_j$, defined by $\varphi(x) = (x + \Gamma_j)_{j \in J'}$. Let

$$\hat{\varphi} : \left(\prod_{j \in J'} G/\Gamma_j \right)^\wedge \rightarrow \widehat{G}$$

denote the dual homomorphism, defined by

$$\langle x, \hat{\varphi}(\alpha) \rangle = \langle \varphi(x), \alpha \rangle.$$

We may identify $\left(\prod_{j \in J'} G/\Gamma_j \right)^\wedge$ with $\prod_{j \in J'} \Gamma_j^\perp$, using the duality

$$\langle (x_j + \Gamma_j)_{j \in J'}, (\alpha_j)_{j \in J'} \rangle = \prod_{j \in J'} \langle x_j, \alpha_j \rangle.$$

With this identification, we obtain

$$\begin{aligned} \langle x, \hat{\varphi}((\alpha_j)_{j \in J'}) \rangle &= \langle (x + \Gamma_j)_{j \in J'}, (\alpha_j)_{j \in J'} \rangle \\ &= \prod_{j \in J'} \langle x, \alpha_j \rangle \end{aligned}$$

$$= \langle x, \sum_{j \in J'} \alpha_j \rangle ,$$

leading to

$$\hat{\varphi}((\alpha_j)_{j \in J'}) = \sum_{j \in J'} \alpha_j .$$

Now (i) is equivalent to injectivity of $\hat{\varphi}$, for all choices of finite $J' \subset J$, whereas (ii) is equivalent to the fact that φ has dense image. But the statements about the homomorphisms are equivalent by duality theory: If φ has dense image, then two continuous functions (for example, characters) coinciding on $\varphi(G)$ must coincide everywhere, which shows (ii) \Rightarrow (i). And if the image of φ is not dense, there exists a nontrivial character on the quotient $(\prod_{j \in J'} G/\Gamma_j) / \overline{\varphi(G)}$, which gives rise to a character on $(\prod_{j \in J'} G/\Gamma_j)$ that coincides on $\varphi(G)$ with the trivial character, showing that $\hat{\varphi}$ is not injective, and thus (ii) \Rightarrow (i).

Finally, the equivalence (ii) \Leftrightarrow (iii) is a standard fact about product topologies. \square

Remark 9. For $G = \mathbb{R}$ and $\Gamma_i = c_i \mathbb{Z} \subset G$, independence of the dual lattices is equivalent to linear independence of $(1/c_j)_{j \in J}$ over the rationals. We note that this is not the same as linear independence of $(c_j)_{j \in J}$ over the rationals. E.g., if $c \in \mathbb{R}$ is transcendental, then the family $(c_n)_{n \in \mathbb{N}}$ given by $c_n = n + c$, $n \in \mathbb{N}$, is linearly dependent over the rationals, but $(1/c_n)_{n \in \mathbb{N}}$ is not.

While the condition of rational independence may seem strong, one can argue that in a sense, it is the generic case: If one chooses the lattice generators c_j randomly, with independent Lebesgue-absolute continuous probability densities for each $j \in J$, then the system $(1/c_j)_{j \in J}$ will be rationally independent with probability one.

If $G = \mathbb{Z}$ and $\Gamma_j = c_j \mathbb{Z} \subset G$, then the dual lattices are independent if and only if the c_j are pairwise prime. This can be seen by Lemma 5.2 and the Chinese Remainder Theorem, stating that

$$\mathbb{Z} \ni n \mapsto (n + c_j \mathbb{Z})_{j \in J'} \in \prod_j \mathbb{Z}/c_j \mathbb{Z}$$

is onto if and only if the c_j are pairwise prime.

The following theorem shows the scope of the ∞ -UCP condition.

Theorem 5.3. *Let $(\Gamma_j)_{j \in J}$ denote a system of lattices, and let $(g_j)_{j \in J}, (h_j)_{j \in J} \subset L^2(G)$.*

- (i) *Suppose that the dual lattices are independent. Let $f \in \mathcal{D}_E$. Then $w_{f;g} \in L^\infty(G)$ if and only if $w_{f;g} = \sum_{j \in J} w_{f;g,j}$ converges uniformly. In particular, every Bessel family $(g_j)_{j \in J}$ fulfills the ∞ -UCP with respect to any closed set $E \subset \widehat{G}$ of measure zero.*
- (ii) *Suppose that the dual lattices are pairwise independent. Then two families $(g_j)_{j \in J}$ and $(h_j)_{j \in J}$ of Bessel generators satisfying the 1-UCP are dual frame generators if and only if*

$$\sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} \overline{\hat{g}_j(\omega)} \hat{h}_j(\omega) = 1 ,$$

and

$$\forall j \in J \forall \alpha \in \Gamma_j^\perp \setminus \{0\} : \overline{\hat{g}_j(\omega)} \hat{h}_j(\omega + \alpha) = 0 ,$$

for almost all $\omega \in \widehat{G}$.

Proof. Fix $f \in \mathcal{D}_E$. If ∞ -UCP holds, then $w_{f;g} \in \mathcal{A}(G) \subset L^\infty(G)$. To finish the proof of (i), we need to show that

$$w_{f;g} = \sum_{j \in J} w_{f;g,j} \quad (5.1)$$

converges uniformly. In fact, we will show that

$$\sum_{j \in J} \|w_{f;g,j}\|_\infty < \infty. \quad (5.2)$$

For this purpose, fix $0 < \epsilon < 1$ and a finite, nonempty set $J' \subset J$. Each $w_{f;g,j}$ induces a continuous function on the compact group G/Γ_j , hence there exists an open set $M_j \subset G$ such that

$$\forall j \in J', \forall y \in M_j + \Gamma_j : w_{f;g,j}(y) \geq (1 - \epsilon)\|w_{f;g,j}\|_\infty.$$

By the independence assumption on the dual lattices and Lemma 5.2, there exists $x \in \bigcap_{j \in J'} M_j + \Gamma_j$. Since all $w_{f;g,j}$ are positive, and their sum is pointwise bounded by the $\|w_{f;g}\|_\infty$, we get

$$\|w_{f;g}\|_\infty \geq \sum_{j \in J'} w_{f;g,j}(x) \geq \sum_{j \in J'} (1 - \epsilon)\|w_{f;g,j}\|_\infty.$$

Since $0 < \epsilon < 1$ and $J' \subset J$ were chosen arbitrary, (5.2) is shown, and thus part (i).

For the remainder of the proof, it is enough to observe that the characterizing equations from Theorem 3.11, which are applicable by the 1-UCP assumption, simplify to the form given in (ii), when the dual lattices are pairwise independent. \square

The point of the following result is that if a family of lattices has pairwise independent dual lattices, and there exist dual frame generators $(g_j)_{j \in J}, (h_j)_{j \in J}$, then the somewhat simple-minded procedure from Theorem 3.16 will also provide such generators.

Corollary 5.4. *Assume that $(\Gamma_j)_{j \in J}$ is a family of lattices with pairwise independent dual lattices. If there exist dual frame generators $(g_j)_{j \in J}, (h_j)_{j \in J}$ satisfying the 1-UCP, then there exist Borel sets $(K_j)_{j \in J}$ satisfying*

$$(i) \mu_{\widehat{G}}(K_j \cap \gamma + K_j) = 0, \text{ for all } \gamma \in \Gamma_j^\perp \setminus \{0\} \text{ and for all } j \in J,$$

$$(ii) \mu_{\widehat{G}}(\widehat{G} \setminus \bigcup_{j \in J} K_j) = 0.$$

Proof. This follows from the characterizing equations in Theorem 5.3(ii), if we let

$$K_j = \{\omega \in \widehat{G} : \hat{g}_j(\omega)\hat{h}_j(\omega) \neq 0\}$$

for each $j \in J$ \square

If one restricts further to orthonormal basis generators, the characterizing equations become even more stringent.

Corollary 5.5. Let $\mathcal{G} = (\Gamma_j)_{j \in J}$ denote a system of lattices whose dual lattices are pairwise independent. Let $(g_j)_{j \in J}$ an associated system of orthonormal basis generators fulfilling 1-UCP. Then

$$|\hat{g}_j| = c_j^{1/2} \mathbf{1}_{K_j}$$

where K_j is a measurable fundamental domain mod Γ_j^\perp , and, up to sets of measure zero,

$$\widehat{G} = \bigcup_{j \in J} K_j.$$

Proof. Let $K_j = \hat{g}_j^{-1}(\mathbb{C} \setminus \{0\})$. Then Theorem 5.3(ii) implies that, up to a set of measure zero, the set K_j is contained in a fundamental domain modulo Γ_j^\perp . Now the fact that the Γ_j -shifts of g_j are an orthonormal system forces K_j to have measure $1/\text{covol}(\Gamma_j)$, and that $|\hat{g}_j| = c_j^{1/2} \mathbf{1}_{K_j}$, with $c_j = \text{covol}(\Gamma_j)$. Thus the Γ_j -shifts of g_j are an orthonormal basis of

$$\mathcal{H}_j = \{f \in L^2(G) : \hat{f} \cdot \mathbf{1}_{K_j} = \hat{f}\}.$$

The assumption that the full system $(T_\gamma g_j)_{j, \gamma}$ is orthonormal therefore forces the \mathcal{H}_j to be pairwise orthogonal, and thus the K_j to be essentially disjoint. Finally, it is clear that completeness of the system forces $\widehat{G} = \bigcup_{j \in J} K_j$ up to sets of measure zero. \square

As a further application of Theorem 5.3, we now construct an example of a lattice family in dimension two with infinite bandwidth, but without dual frame generators.

Example 5.1. Fix a transcendental number $c > 1$, and let $\Gamma_j = C_j \mathbb{Z}^2$, where

$$C_j = \begin{pmatrix} c^{-j} & 0 \\ 0 & c^j \end{pmatrix} \quad \text{for } j \in \mathbb{N}.$$

Hence $\sum_{j \in \mathbb{N}} \frac{1}{\text{covol}(\Gamma_j)} = \infty$, but there do not exist families of dual generators for this system. To see this, assume otherwise. Note that by choice of c , the dual lattices are independent, hence Theorem 5.3(i) implies that the dual generators fulfill ∞ -UCP. Hence Corollary 5.4 applies and yields Borel sets $(K_j)_{j \in \mathbb{N}}$ with $\lambda(\mathbb{R}^2 \setminus \bigcup_{j \in \mathbb{N}} K_j) = 0$ and $\lambda(K_j \cap (\gamma + K_j)) = 0$ for all $j \in \mathbb{N}$ and all $\gamma \in \Gamma_j^\perp = c^j \mathbb{Z} \times c^{-j} \mathbb{Z}$, where λ denotes the Lebesgue measure on \mathbb{R}^2 .

Without loss of generality, we may assume that $K_j \cap \gamma + K_j = \emptyset$, for all $j \in \mathbb{N}$ and $\gamma \in \Gamma_j^\perp$. Define, for $j \in \mathbb{N}$ and $x \in \mathbb{R}$, the Borel set

$$G_{j,x} = \{y \in \mathbb{R} : (x, y) \in K_j\}.$$

Assume that there exists $k \in \mathbb{Z} \setminus \{0\}$ such that $G_{j,x} \cap (G_{j,x} + c^{-j}k) \neq \emptyset$. Then there exists $y \in \mathbb{R}^n$ such that $(x, y) \in K_j$ and $(x, y - c^{-j}k) \in K_j$. Hence $(x, y) \in K_j \cap ((0, c^{-j}k) + K_j)$, and $(0, c^{-j}k) \in \Gamma_j^\perp$, which contradicts our assumption on the K_j .

Hence $G_{j,x}$ is contained in a fundamental domain mod $c^{-j} \mathbb{Z}$, which entails $\lambda(G_{j,x}) \leq c^{-j}$. Now, let us assume that $\mathbb{R}^2 \subset \bigcup_{j \in \mathbb{N}} K_j$, up to a null set. We then get

$$\infty = \int_0^1 \int_{\mathbb{R}} 1 dy dx$$

$$\begin{aligned}
&= \int_0^1 \int_{\mathbb{R}} \sum_{j \in \mathbb{N}} \mathbf{1}_{K_j}(x, y) dy dx \\
&= \sum_{j=1}^{\infty} \int_0^1 \underbrace{\lambda(G_{j,x})}_{\leq c^{-j}} dx \\
&\leq \sum_{j=1}^{\infty} c^{-j} < \infty,
\end{aligned}$$

which is the desired contradiction.

Remark 10. The results in this section align nicely with results from wavelet analysis. For example, Corollary 5.5 is related to so-called MSF wavelets. Such wavelets ψ are characterized by the property that $|\hat{\psi}|$ is, up to scalar multiplication, given by the characteristic function of a Borel set. It was shown by Chui and Shi in [10], that whenever the dilation $a > 1$ is such that all integer powers of a are irrational, every orthonormal wavelet associated to a must be an MSF wavelet. Corollary 5.5, applied to the family $\Gamma_j = a^j \mathbb{Z}$, for $j \in \mathbb{Z}$, provides this answer under the strictly stronger assumption that a is transcendental (which is equivalent to independence of the dual lattices). Note however that here our corollary also provides a stronger conclusion, since the generators $(g_j)_{j \in \mathbb{Z}}$ are not assumed to be dilates of a single mother wavelet.

But also Theorem 5.3(i) and its proof have a precedent in wavelet analysis. Note that the proof of the Theorem yields

$$\|w_{f;g,h}\|_{\infty} = \sum_{j \in J} \|w_{f;g,h,j}\|_{\infty}. \quad (5.3)$$

This phenomenon is related to the question how to estimate frame bounds of the full system $(T_{\gamma}g_j)_{\gamma \in \Gamma_j, j \in J}$ from the bounds of the individual layers $(T_{\gamma}g_j)_{\gamma \in \Gamma_j}$ indexed by $j \in J$, which was investigated for wavelet systems with transcendental dilations in [20]. Indeed, $(T_{\gamma}g_j)_{j \in J, \gamma \in \Gamma_j}$ is a Bessel system with optimal bound B^{\dagger} precisely when

$$B^{\dagger} = \sup_{f \in \mathcal{D}_E, \|f\|=1} \sum_{j \in J} \max_{x \in G} w_{f;g,j}(x) < \infty;$$

which furthermore is a frame with optimal lower bound A^{\dagger} if and only if

$$A^{\dagger} = \inf_{f \in \mathcal{D}_E, \|f\|=1} \sum_{j \in J} \min_{x \in G} w_{f;g,j}(x) > 0.$$

These estimates should be compared to (3.7) and (3.8); they can be viewed as a generalization of [20, Theorem 2.1].

As a further application of our criteria for the existence of frame generators for lattice systems, we present an example showing the instability of this property with respect to perturbations. More precisely, the existence of normalized tight frame generators is *not* robust with respect to arbitrarily small perturbations of the lattice generators.

Example 5.2. Consider $G = \mathbb{R}$ with the Lebesgue measure. Consider the system $\mathcal{G} = (2^j \mathbb{Z})_{j \in \mathbb{N}}$, and let $(\epsilon_j)_{j \in \mathbb{N}}$ be an arbitrary sequence of strictly positive numbers. Pick a sequence $(c_j)_{j \in \mathbb{N}} \subset \mathbb{R}$ with $|2^j - c_j| < \min(\epsilon_j, 1)$, and the additional property that $(1/c_j)_{j \in \mathbb{N}}$ is \mathbb{Q} -linearly independent (this is easily done inductively). Then Theorem 4.1 yields a system of tight frame (even orthonormal basis) generators associated with \mathcal{G} . However, for the perturbed lattice system $\mathcal{G}' = (c_j \mathbb{Z})_{j \in \mathbb{N}}$, we can estimate $BW(\mathcal{G}') \leq 2$, hence Theorem 5.3 shows that no generators can exist for \mathcal{G}' .

A question that is somewhat similar to the notion of refinements of lattice families is whether the existence of frame generators is robust with respect to enlarging each lattice in the family individually. At first glance, this may seem like a reasonable conjecture; after all, enlarging the lattices leads to systems with more redundancy (and larger bandwidth), which should make frame construction easier. However, this intuition is generally misleading, as the following example shows.

Example 5.3. Consider $G = \mathbb{Z}$ with the counting measure. Fix a family $(c_j)_{j \in \mathbb{N}}$ of pairwise prime integers such that $\sum_{j \in \mathbb{N}} c_j^{-1} < 1 = \mu_{\widehat{G}}(\widehat{G})$. Then the lattices are independent. Hence, there does not exist a family of dual frame generators in $\ell^2(\mathbb{Z})$ for the lattices $\Gamma_j = c_j \mathbb{Z}$ by Theorem 5.3(i) and Theorem 3.14. On the other hand, if we let

$$\Lambda_j = \bigcap_{i \leq j} \Gamma_i$$

we obtain a strictly decreasing family of lattices. By the proof of Theorem 4.1, there is a system of dual frame generators for the Λ_j , and $\Lambda_j \subset \Gamma_j$ holds for all $j \in \mathbb{N}$. Thus increasing the lattices can have a *negative* impact on the availability of tight frame generators.

6. Applications and extensions

We end this paper with further discussions of the necessary conditions for the frame property in Section 3.4.

Intuitively, the Calderón sum $\sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\hat{g}_j(\cdot)|^2$ measures the total energy concentration of the generators g_j in the frequency domain. If the Calderón sum is zero on some domain in frequency, then clearly none of the frequencies in this domain can be represented by the corresponding GSI system. In other words, the corresponding GSI system is not complete/total. Furthermore, whenever a GSI system has the frame property, which is a stronger assumption than the spanning property, one would even expect the Calderón sum to be bounded uniformly from below since the GSI frame can reproduce all frequencies in a *stable* way.

However, as we saw in Theorem 3.13 and Example 3.1, this is again a situation where our intuition only holds true if we assume the 1-UCP. Under the 1-UCP, the Calderón sum of a GSI frame $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$ with bounds A_g and B_g takes values in $[A_g, B_g]$, that is,

$$A_g \leq \sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\hat{g}_j(\omega)|^2 \leq B_g \quad \text{for a.e. } \omega \in \widehat{G}.$$

Without the 1-UCP, the best one can say is that

$$0 < \sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} |\hat{g}_j(\omega)|^2 \leq B_g \quad \text{for a.e. } \omega \in \widehat{G}.$$

As mentioned, the terminology ‘‘Calderón sum’’ comes from wavelet analysis. Let us show that our results on GSI systems extends known results in wavelet analysis. Fix an $n \times n$ matrix $A \in \text{GL}_n(\mathbb{R})$ and a full-rank lattice $\Gamma \subset \mathbb{R}^n$. The wavelet system $(D_{A^j} T_\gamma \psi)_{j \in \mathbb{Z}, \gamma \in \Gamma}$, where $D_{A^j} T_\gamma \psi = |\det A|^{j/2} \psi(A \cdot -\gamma)$, can be written as a GSI system in the following *standard form*:

$$J = \mathbb{Z}, \quad \Gamma_j = A^{-j}\Gamma, \quad g_j = D_{A^j} \psi, \quad \text{for all } j \in \mathbb{Z}.$$

The Calderón sum then reads $\sum_{j \in \mathbb{Z}} |\hat{\psi}(A^j \cdot)|^2$. It is a classical result by Chui and Shi [9] that for univariate frame wavelets ($n = 1, A = a$) with bounds C_1 and C_2 , it holds

$$C_1 \leq \sum_{j \in \mathbb{Z}} |\hat{\psi}(a^j \omega)|^2 \leq C_2 \quad \text{for a.e. } \omega \in \mathbb{R}.$$

In wavelet analysis the case $n = 1$ is special: it is the only dimension where the LIC/UCP automatically holds once we assume local integrability in $\mathbb{R}^n \setminus \{0\}$ of the Calderón sum. Hence, for univariate wavelets the issue of LIC/UCP is, in most cases, completely absent.

Theorem 6.1. *Let $A \in \text{GL}_n(\mathbb{R})$, $|\det A| > 1$, let $\Gamma \subset \mathbb{R}^n$ be a full-rank lattice, and let L be an at most countable index set. Suppose that (A^T, Γ^\perp) satisfies the lattice counting estimate, that is,*

$$\# |\Gamma^\perp \cap (A^T)^j(B_r(0))| \leq C \max(1, |\det A|^j) \quad \text{for all } j \in \mathbb{Z}.$$

If the wavelet system $(D_{A^j} T_\gamma \psi_\ell)_{\ell \in L, j \in \mathbb{Z}, \gamma \in \Gamma}$ is a frame with bounds C_1 and C_2 , then

$$C_1 \leq \sum_{\ell \in L} \sum_{j \in \mathbb{Z}} |\hat{\psi}_\ell(A^j \omega)|^2 \leq C_2 \quad \text{for a.e. } \omega \in \mathbb{R}^n.$$

Proof. We consider the wavelet system as a GSI system in the standard form. By Lemma 3.2, it holds that $\sum_{\ell \in L} \sum_{j \in \mathbb{Z}} |\hat{\psi}_\ell(A^j \omega)|^2 \leq C_2$ for a.e. $\omega \in \mathbb{R}^n$. Since (A^T, Γ^\perp) satisfies the lattice counting estimate, this implies, by a result in [4], that the wavelet system satisfies that LIC. Since the LIC implies 1-UCP, the result follows from Theorem 3.13. \square

The lattice counting estimate was introduced in [4], where Bownik and the second named author show that almost all wavelet systems satisfy the lattice counting estimate. In particular, a dilation matrix A that is *expanding on a subspace* (i.e., matrices with eigenvalues bigger than one in modulus, at least one strictly bigger, and eigenvalues of modulus one have Jordan blocks of order one) and any translation lattice $\Gamma \subset \mathbb{R}^n$ will satisfy the lattice counting estimate.

The two proofs of the lower bound of the Calderón sum of wavelet frames and GSI frames for $L^2(\mathbb{R})$ in [9] and [8], respectively, are of similar nature, and they rely on the fact that lattices $(c_j \mathbb{Z})_{j \in \mathbb{Z}}$ in \mathbb{R} have a natural ordering. Indeed, one can assume $c_j \leq c_{j+1}$. This is not the case in higher dimensions nor for general LCA groups, and the mentioned proofs break down.

Moreover, the proof of Theorem 3.13 is conceptually much simpler than the proofs in [9, 8] once the theory of almost periodic functions of GSI systems is in place. In fact, our proof extends to a larger class of systems, called *generalized translation-invariant* (GTI) systems, introduced in [16] as families of the form

$$(T_\gamma g_{j,p})_{j \in J, p \in P_j, \gamma \in \Gamma_j}, \quad g_{j,p} \in L^2(G),$$

where J is countable, and $\Gamma_j \subset G$ is a co-compact subgroup (with some given Haar measure) and P_j is a σ -finite measure space (satisfying the three “standing assumptions” in [16]) for each $j \in J$. GTI systems are continuous or semi-continuous variants of GSI systems, and therefore encompass, e.g., the continuous (and semi-continuous) wavelet, Gabor and shearlet transforms. The following version of Theorem 3.13 provides generalized admissibility conditions for GTI frames.

Theorem 6.2. *Suppose that the GTI system $(T_\gamma g_{j,p})_{j \in J, p \in P_j, \gamma \in \Gamma_j}$ is a (continuous) frame with bounds A_g and B_g that satisfies 1-UCP (with the straightforward modifications). Then*

$$A_g \leq \sum_{j \in J} \frac{1}{\text{covol}(\Gamma_j)} \int_{P_j} |\hat{g}_{j,p}(\omega)|^2 dp \leq B_g \quad \text{for a.e. } \omega \in \widehat{G},$$

where $\text{covol}(\Gamma_j) := \mu_{G/\Gamma_j}(G/\Gamma_j)$ for each $j \in J$.

Theorems 3.11, 3.12 and 3.14 also extend to GTI frames as Theorem 6.2, albeit Theorem 3.14 needs the additional assumption that $\int_{P_j} \|g_{j,p}\|^2 dp$ is uniformly bounded over $j \in J$. We remark that there exist LCA groups that have no lattices, while any LCA group has a co-compact subgroup. We leave the existence question of GTI frames for a family of co-compact subgroups $(\Gamma_j)_{j \in J}$ for future research.

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