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A simple characterization of H-convergence for a class of nonlocal problems

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Abstract

This is a follow-up of the paper J. Fernández-Bonder, A. Ritorto and A. Salort, *H-convergence result for nonlocal elliptic-type problems via Tartar's method*, SIAM J. Math. Anal., 49 (2017), pp. 2387-2408, where the classical concept of H-convergence was extended to fractional p-Laplace type operators. In this short paper we provide an explicit characterization of this notion by demonstrating that the weak-$*$ convergence of the coefficients is an equivalent condition for H-convergence of the sequence of nonlocal operators. This result takes advantage of nonlocality and is in stark contrast to the local p-Laplacian case.

1 Introduction

For $p \in (1,\infty)$, $s \in (0,1)$ and a *nonlocal* conductivity $a(x,y)$ belonging to the class

$$\mathcal{A}_{\lambda,\Lambda} = \{a \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \,:\, a(x,y) = a(y,x),\, \lambda \le a(x,y) \le \Lambda, \text{ a.e. in } \mathbb{R}^n \times \mathbb{R}^n\},$$

where $0 < \lambda \le \Lambda < \infty$ are given constants, let us consider the following nonlocal operator related to the fractional p-Laplacian:

$$\mathcal{L}_a\, u(x) = p.v. \int_{\mathbb{R}^n} a(x,y) \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy.$$

For a fixed bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and $f \in L^{p'}(\Omega)$, with $p' = p/(p-1)$ being the conjugate exponent to p, we consider the nonlocal problem

$$(1) \qquad \begin{cases} \mathcal{L}_a\, u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

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This problem is well-posed, with a unique solution found in the space

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^n) \ : \ u = 0 \text{ a.e. in } \mathbb{R}^n \backslash \Omega\},$$

where $W^{s,p}(\mathbb{R}^n)$ is the classical fractional Sobolev space over $\mathbb{R}^n$, see [5, 1]:

$$W^{s,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) \ : \ D_{s,p}u(x,y) \in L^p(\mathbb{R}^n \times \mathbb{R}^n)\},$$

and

$$D_{s,p}u(x,y) = \frac{u(x) - u(y)}{|x-y|^{\frac{n}{p}+s}} \tag{2}$$

is the (s,p)-nonlocal gradient of u. In view of existence and uniqueness of solutions we can employ the shorthand notation $(u,q) = \mathcal{S}_a f$ to denote the solution $u \in W_0^{s,p}(\Omega)$ to (1) corresponding to coefficients a and the right hand side f, and the *non-local flux* $q = a|D_{s,p}u|^{p-2} D_{s,p}u \in L^{p'}(\Omega \times \Omega)$.

In this paper we are concerned with H-convergence of the nonlocal operators $\mathcal{L}_{a_k}$ for a given sequence of coefficients a_k.

Definition 1.1 *Given the sequence of coefficients $\{a_k\}_{k=1}^{\infty} \subset \mathcal{A}_{\lambda,\Lambda}$, we say that $\mathcal{L}_{a_k}$ H-converges to $\mathcal{L}_a$ if for any $f \in W^{-s,p}(\Omega)$ (the dual space of $W_0^{s,p}(\Omega)$) the following conditions are satisfied:*

1. *convergence of states: $u_k \rightharpoonup u$, weakly in $W_0^{s,p}(\Omega)$;*
2. *convergence of non-local fluxes: $q_k \rightharpoonup q$, weakly in $L^{p'}(\Omega \times \Omega)$,*

where $(u_k, q_k) = \mathcal{S}_{a_k} f$, and $(u,q) = \mathcal{S}_a f$.

In [8] is was shown that for any sequence $\{a_k\}_{k=1}^{\infty} \subset \mathcal{A}_{\lambda,\Lambda}$ there exists a subsequence $\{a_{k'}\}$ and a function $a \in \mathcal{A}_{\lambda, \frac{\Lambda p'}{\lambda}}$ such that $\mathcal{L}_{a_{k'}}$ H-converges to $\mathcal{L}_a$. We show that in fact a belongs to the same class $\mathcal{A}_{\lambda,\Lambda}$. Furthermore, our main result establishes that weak-$*$ convergence of the sequence of coefficients is a necessary and sufficient condition for H-convergence in the considered case.

Theorem 1.1 *If $a_k, a \in \mathcal{A}_{\lambda,\Lambda}$, then $a_k \rightharpoonup a$ weakly-$*$ in $L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ if and only if $\mathcal{L}_{a_k}$ H-converges to $\mathcal{L}_a$.*

This result generalizes, giving a simpler proof, previous results for a related nonlocal situation in the linear case [3, Th. 6]. We also refer to [16], where an abstract, general setting for nonlocal H-convergence is analyzed. The reference [9] addresses the application of the more general framework of Γ-convergence to several physical problems, including macroscopic behavior of conducting materials mixtures. In particular, in [9, Ex. 2.23] the Γ-limit of nonlocal quadratic energies with varying coefficients is characterized.

The outline of the paper is as follows. Section 2 is devoted to setting the functional analysis framework of this work, and to stating the main results from [8], which are the starting point of the investigation presented here. Section 3 deals with the relation of the nonlocal H-convergence notion introduced in Definition 1.1 with the weaker notion of G-convergence. We show that the two notions are equivalent. Additionally, we establish the uniqueness of H-limit. Finally, Section 4 contains the proof of Theorem 1.1.

2 Preliminaries

In this section we set the functional analysis framework in which problems are set and recall the main results from [8].

We start by recalling some fundamental facts about fractional Sobolev spaces. The space $W^{s,p}(\mathbb{R}^n)$, previously defined, is equipped with the norm

$$\|u\|_{s,p} = \left(\|u\|_p^p + |u|_{s,p}^p\right)^{\frac{1}{p}},$$

where $\|u\|_p$ is the usual norm of u in $L^p(\mathbb{R}^n)$ and

$$|u|_{s,p} = \left(\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{\frac{1}{p}} = \left(\int\int_{\mathbb{R}^n\times\mathbb{R}^n} |D_{s,p}u(x,y)|^p\,dx\,dy\right)^{\frac{1}{p}}$$

is the Gagliardo seminorm [5, 1]. With this definition $W^{s,p}(\mathbb{R}^n)$ is a separable and reflexive Banach space for $1 < p < \infty$. $W_0^{s,p}(\Omega)$ is usually defined as

$$W_0^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{s,p}},$$

and in the case Ω has a Lipschitz boundary the following identification holds

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^n) \;:\; u = 0 \text{ a.e. in } \mathbb{R}^n\backslash\Omega\}.$$

An important mathematical fact is that, for Ω bounded, $W_0^{s,p}(\Omega)$ embeds continuously into $L^p(\Omega)$, thanks to the Poincaré's inquality in this fractional situation: there exists $C = C(n, s, |\Omega|) > 0$ such that

$$\|u\|_p \leq C|u|_{s,p},$$

for all $u \in W_0^{s,p}(\Omega)$. Furthermore, the Rellich-Kondrachov theorem can be extended to fractional Sobolev spaces, and the embedding of $W_0^{s,p}(\Omega)$ into $L^p(\Omega)$ is compact. Proofs of these results can be found, for instance, in [5]. The dual space of $W_0^{s,p}(\Omega)$ is denoted by $W^{-s,p'}(\Omega)$, and its norm is given by

$$\|f\|_{-s,p'} = \sup\{\langle f, u\rangle \;:\; u \in W_0^{s,p}(\Omega),\, |u|_{s,p} = 1\}.$$

We now focus on the precise statement of the problem (1), which should be understood in the weak sense. Thus, we require that $\mathcal{L}_a\, u = f$ holds in the sense of distributions, and we say that $u \in W_0^{s,p}(\Omega)$ is a solution to (1) if

(3)
$$\frac{1}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y)\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}\,dx\,dy = \langle f, v\rangle,$$

for all test functions $v \in C_0^\infty(\Omega)$. The first result documents the well-posedness of the problem (1) [8, Prop. 2.2, Cor. 2.4].

Proposition 2.1 *For each $a \in \mathcal{A}_{\lambda,\Lambda}$ and $f \in W^{-p,s}(\Omega)$, the problem* (1) *admits a unique solution $u \in W_0^{s,p}(\Omega)$. Furthermore, this solution is also the unique minimizer in $W_0^{s,p}(\Omega)$ of functional*

$$I_a(v) = \frac{1}{2p}\int\int_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y)|D_{s,p}v(x,y)|^p\,dx\,dy - \langle f, v\rangle.$$

In addition to the non-local gradient (2), for each $\phi \in L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)$ it will be convenient to define the corresponding (s,p)-divergence operator by

$$d_{s,p}\phi(x) = p.v. \int_{\mathbb{R}^n} \frac{\phi(x,y) - \phi(y,x)}{|x-y|^{\frac{n}{p}+s}}\, dy.$$

The following result summarizes several properties of these operators [8, Th. 3.1., Lem. 3.3].

Theorem 2.2 *The following assertions hold:*

1. ***Integration by parts:*** *For each $\phi \in L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)$ and $u \in W^{s,p}(\mathbb{R}^n)$ we have the inclusion $d_{sp}\phi \in W^{-s,p'}(\mathbb{R}^n)$, the dual of $W^{s,p}(\mathbb{R}^n)$, and the integration by parts formula*

$$\int\int_{\mathbb{R}^n \times \mathbb{R}^n} \phi D_{s,p} u \, dx\, dy = \langle d_{s,p}\phi, u\rangle;$$

2. ***Nonlocal div-curl lemma:*** *given $\phi_k, \phi \in L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)$ and v_k, $v \in W^{s,p}(\mathbb{R}^n)$, $k = 1, 2, \dots$ such that $v_k \rightharpoonup v$, weakly in $W^{s,p}(\mathbb{R}^n)$, $\phi_k \rightharpoonup \phi$, weakly in $L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)$ and $d_{s,p}\phi_k \to d_{s,p}\phi$, strongly in $W^{-s,p'}_{loc}(\mathbb{R}^n)$ then*

$$\phi_k D_{s,p} v_k \to \phi D_{s,p} v$$

in the sense of distributions.

Note that owing to the integration by parts identity and the assumed symmetry of the conductivity $a(x,y) = a(y,x)$, equation (3) can be equivalently understood as $d_{s,p}q = 2f$, where $(u,q) = \mathcal{S}_a f$.

There are several other results in the literature, which are related to the previous one. In other works dealing with nonlocal or fractional problems a nonlocal vector calculus has been developed in order to deal with the involved operators. References including integration by parts formulas are [6, 10, 11, 4]. Regarding the div-curl lemma, this is a very interesting compensated compactness-type result in the nonlocal context. In [15] a general analytic perspective for div-curl lemma that includes the nonlocal situation is considered. It is interesting to refer to [4], where in a very related situation to the one analyzed here, the weak convergence of any minor of the Riesz fractional gradient of vector fields has been shown by means of a nonlocal Piola identity.

Now we are prepared to state our point of departure, [8, Theorem 4.6], which establishes that $\mathcal{A}_{\lambda,\Lambda}$ is sequentially relatively compact with respect to H-convergence.

Theorem 2.3 $0 < \lambda \leq \Lambda$. *For any sequence $\{a_k\} \subset \mathcal{A}_{\lambda,\Lambda}$, there exists a subsequence $\{a_{k'}\}$ and $a \in \mathcal{A}_{\lambda,\Lambda}$ such that $\mathcal{L}_{a_k}$ H-converges to $\mathcal{L}_a$.*

To be precise this is not the exact statement of [8, Theorem 4.6], as it differs in the upper bound on the coefficients of the H-limiting problem. In [8, Theorem 4.6] it is claimed that the H-limit $a \in \mathcal{A}_{\lambda, \frac{\Lambda^{p'}}{\lambda}}$. However, we will show that the more natural upper bound

$$a(x,y) \le \Lambda, \quad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}^n, \tag{4}$$

holds in this case. Indeed, let us assume that $\mathcal{L}_{a_k}$ H-converges to a. Let us further fix an arbitrary $f \in L^{p'} \setminus \{0\}$, and put $(u_k, q_k) = \mathcal{S}_{a_k} f$, and $(u, q) = \mathcal{S}_a f$. Since $a_k \in \mathcal{A}_{\lambda,\Lambda}$, for any $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\varphi \ge 0$ we have

$$\begin{aligned}\int\int_{\mathbb{R}^n \times \mathbb{R}^n} |q_k|^{p'} \varphi \, dx\, dy &= \int\int_{\mathbb{R}^n \times \mathbb{R}^n} a_k^{p'} |D_{s,p} u_k|^p \varphi \, dx\, dy \\ &\le \Lambda^{p'-1} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi q_k D_{s,p} u_k \, dx\, dy.\end{aligned}$$

The term on the left is weakly lower semicontinuous with respect to the fluxes, which converge weakly owing to the H-convergence assumption. The term on the right converges owing to the non-local div-curl lemma. Passing to the limit we therefore arrive at the inequality

$$\int\int_{\mathbb{R}^n \times \mathbb{R}^n} a^{p'} |D_{s,p} u|^p \varphi \, dx\, dy \le \Lambda^{p'-1} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} a |D_{s,p} u|^p \varphi \, dx\, dy,$$

and as φ is nonnegative but otherwise arbitrary,

$$a^{p'} |D_{s,p} u|^p \le \Lambda^{p'-1} a |D_{s,p} u|^p, \quad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}^n. \tag{5}$$

Additionally, since f is arbitrary, u is also arbitrary, and (5) holds for any $u \in W_0^{s,p}(\Omega)$, and hence the inequality (4) holds.

3 G-convergence

In the local case, H-convergence was proposed by Murat and Tartar, see for example [12], as an extension of the previously proposed G-convergence concept [14]. G-convergence was formulated for linear elliptic equations with symmetric coefficients in divergence form and only requires weak convergence of the states. H-convergence, on the other hand, requires convergence of both states and fluxes, and has been formulated for problems with non-symmetric coefficients. In the case of elliptic PDEs with symmetric coefficients both notion are known to coincide. In the more general case of non-symmetric coefficients G-convergence is less useful in the sense that the G-limit is not guaranteed to be unique [2, Section 1.3.2].

The previous definition of nonlocal H-convergence requires both convergence of the states and convergence of the fluxes. As a consequence of this, owing to the div-curl lemma, the associated energy

$$E(a) = \int\int_{\mathbb{R}^n \times \mathbb{R}^n} a |D_{s,p} u|^p \, dx\, dy = \int\int_{\mathbb{R}^n \times \mathbb{R}^n} q D_{s,p} u \, dx\, dy,$$

with $(u,q) = \mathcal{S}_a f$ is continuous with respect to H-convergence. This is a remarkable and a desirable property, especially when dealing with optimal design problems. In this section, we show that in the considered nonlocal situation of scalar and symmetric coefficients in $\mathcal{A}_{\lambda,\Lambda}$, the requirement on flux convergence in the definition of H-convergence is unnecessary. In other words, nonlocal G-convergence implies H-convergence, precisely as in the local case. First of all we make rigorous the definition of nonlocal G-convergence.

Definition 3.1 *Given the sequence of coefficients $\{a_k\}_{k=1}^{\infty} \subset \mathcal{A}_{\lambda,\Lambda}$, we say that $\mathcal{L}_{a_k}$ G-converges to $\mathcal{L}_a$ if for any $f \in W^{-s,p}(\Omega)$ we have*

$$u_k \rightharpoonup u, \quad \textit{weakly in } W^{s,p}(\Omega),$$

where $(u_k, q_k) = \mathcal{S}_{a_k} f$ and $(u,q) = \mathcal{S}_a f$.

Proposition 3.1 *Consider a sequence $\{a_k\}_{k=0}^{\infty} \subset \mathcal{A}_{\lambda,\Lambda}$. Then $\mathcal{L}_{a_k}$ H-converges to $\mathcal{L}_a$ if and only if $\mathcal{L}_{a_k}$ G-converges to $\mathcal{L}_a$, with $a \in \mathcal{A}_{\lambda,\Lambda}$.*

Proof: Obviously H-convergence implies G-convergence. For the sake of contradiction, let us now assume that we have G-convergence, but not H-convergence. For an arbitrary $f \in W^{-s,p}(\Omega)$ let $(u_k^f, q_k^f) = \mathcal{S}_{a_k} f$ and $(u^f, q^f) = \mathcal{S}_a f$. Owing to the assumed G-convergence we have $u_k^f \rightharpoonup u^f$ in $W_0^{s,p}(\Omega)$. The assumed lack of H-convergence is equivalent to saying that for some $\hat{f} \in W^{-s,p}(\Omega)$ there is a subsequence $a_{k'}$ and a $L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)$-weakly open neighbourhood $\hat{N}$ of $q^{\hat{f}}$, such that $q_{k'}^{\hat{f}} \notin \hat{N}$, for all k'. Owing to Theorem 2.3, the set $\{\mathcal{L}_{a_{k'}}\}$ is relatively sequentially compact with respect to H-convergence. Therefore there is a further subsequence $a_{k''}$ and $\tilde{a} \in \mathcal{A}_{\lambda,\Lambda}$ such that $\mathcal{L}_{a_{k''}}$ H-converges to $\mathcal{L}_{\tilde{a}}$. Consequently, if we put $(\tilde{u}^f, \tilde{q}^f) = \mathcal{S}_{\tilde{a}} f$, then $u^f = \tilde{u}^f$ owing to the assumed G-convergence and the uniqueness of the weak limit, and the uniqueness of solutions to (3). Therefore we have the equality

$$\int\int_{\mathbb{R}^n \times \mathbb{R}^n} (a - \tilde{a}) |D_{s,p} u^f|^p \, dx \, dy = 0,$$

which holds for an arbitrary f. Since f is arbitrary, $u^f \in W_0^{s,p}(\Omega)$ is also arbitrary. Applying [7, Proposition 17], which characterizes null nonlocal functionals, we obtain that necessarily $a = \tilde{a}$, a.e. in $\mathbb{R}^n \times \mathbb{R}^n$. But then $\tilde{q}^f = q^f$, $q_{k''}^f \rightharpoonup q^f$ in $L^{p'}(\mathbb{R}^n \times \mathbb{R}^n)$, and consequently $q_{k''}^{\hat{f}} \in \hat{N}$ for all large k'', which is a contradiction. □

Another important point is the uniqueness of the G-limit. This is established in the next result.

Proposition 3.2 *The G-limit of a sequence in $\mathcal{A}_{\lambda,\Lambda}$ is unique.*

Proof: The proof follows the lines of the second part of proof of Proposition 3.1. Let us assume that the sequence $\{a_n\} \subset \mathcal{A}_{\lambda,\Lambda}$ G-converges to both a and $\tilde{a}$. Then, arguing as in the proof of Proposition 3.1, we have that

$$\int\int_{\mathbb{R}^n \times \mathbb{R}^n} (a - \tilde{a}) |D_{s,p} u^f|^p \, dx \, dy = 0,$$

for any f, where u^f is the solution of the Dirichlet problem (1) for both coefficients a and $\tilde{a}$. The same argument as above yields the conclusion

$$a = \tilde{a}, \text{ a.e. in } \mathbb{R}^n \times \mathbb{R}^n.$$

□

4 Proof of Theorem 1.1

This section is devoted to the proof Theorem 1.1.

We claim that weak-* convergence is sufficient for H-convergence Indeed, let us assume that $\{a_k\}_{k=1}^{\infty} \in \mathcal{A}_{\lambda,\Lambda}$ be such that $a_k \rightharpoonup a \in \mathcal{A}_{\lambda,\Lambda}$, weak-* in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Owing to Theorem 2.3 there exists a subsequence $\{a_{k'}\}$ and $\tilde{a} \in \mathcal{A}_{\lambda,\Lambda}$ such that $\mathcal{L}_{a_{k'}}$ H-converges to $\mathcal{L}_{\tilde{a}}$. Let $f \in W^{-s,p}(\Omega)$ be fixed but arbitrary, and let $(u_k, q_k) = \mathcal{S}_{a_k} f$, $(u, q) = \mathcal{S}_a f$, and $(\tilde{u}, \tilde{q}) = \mathcal{S}_{\tilde{a}} f$. Owing to H-convergence, $u_{k'} \rightharpoonup \tilde{u}$, weakly in $W_0^{s,p}(\Omega)$. Recalling that $u_{k'} = 0$ in $\mathbb{R}^n \backslash \Omega$ and the compact embedding of $W_0^{s,p}(\Omega)$ into $L^p(\Omega)$, $\|u_{k'} - \tilde{u}\|_{L^p(\mathbb{R}^n)} \to 0$. In particular, there is a further subsequence $u_{k''}$ of $u_{k'}$, such that $u_{k''}(x) \to \tilde{u}(x)$, for almost all $x \in \mathbb{R}^n$.

Owing to the variational characterization of solutions to (3) given in Proposition 2.1, we have the inequality $I_{a_k}(u_k) \leq I_{a_k}(u)$, $\forall k = 1, 2, \ldots$ Taking into account the facts that $|D_{s,p} u|^p \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and $a_k \rightharpoonup a$ weak-$*$ in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, we obtain the inequality

$$\limsup_{k'' \to \infty} I_{a_{k''}}(u_{k''}) \leq \lim_{k'' \to \infty} I_{a_{k''}}(u) = I_a(u) \leq I_a(\tilde{u}). \tag{6}$$

On the other hand, let us define the measures

$$\nu_k(E) = \int\int_E a_k(x, y)\, dx\, dy = \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_E(x, y) a_k(x, y)\, dx\, dy, \quad k \geq 1,$$

and

$$\nu(E) = \int\int_E a(x, y)\, dx\, dy = \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_E(x, y) a(x, y)\, dx\, dy,$$

where $E \subset \mathbb{R}^n \times \mathbb{R}^n$ is an arbitrary Lebesgue measurable set. Weak-$*$ convergence of a_k to a implies the strong convergence of these measures, that is, $\lim_{k \to \infty} \nu_k(E) = \nu(E)$ for any measurable set $E \subset \mathbb{R}^n \times \mathbb{R}^n$. Since $u_{k''}(x) \to \tilde{u}(x)$ for almost all $x \in \mathbb{R}^n$, it follows that $|D_{s,p} u_{k''}(x, y)|^p \to |D_{s,p} \tilde{u}(x, y)|^p$, for almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. These facts together with the upper bound (6) allow us to apply the generalized Fatou's lemma [13, Proposition 17, p. 269] to get the inequality

$$\begin{aligned} I_a(\tilde{u}) &= \frac{1}{2p} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} |D_{s,p} \tilde{u}(x, y)|^p\, d\nu(x, y) - \langle f, u \rangle \\ &\leq \liminf_{k'' \to \infty} \frac{1}{2p} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} |D_{s,p} u_{k''}(x, y)|^p\, d\nu_{k''}(x, y) - \langle f, u_{k''} \rangle = \liminf_{k'' \to \infty} I_{a_{k''}}(u_{k''}). \end{aligned}$$

Therefore $I_a(u) = I_a(\tilde{u})$, whence $u = \tilde{u}$, owing to the uniqueness of solutions to (3) and their variational characterization. Arguing further as in the proof of Proposition 3.1, we conclude that $a = \tilde{a}$, almost everywhere in $\mathbb{R}^n \times \mathbb{R}^n$. Finally, since from every subsequence of $\mathcal{L}_{a_k}$ we can extract a further subsequence, which H-converges to $\mathcal{L}_a$, the whole sequence must converge to $\mathcal{L}_a$.

We now claim that weak-* convergence is also necessary for H-convergence Assume that $\mathcal{L}_{a_k}$ H-converges to $\mathcal{L}_a$, but for some weak-* open neighbourhood N of $a \in \mathcal{A}_{\lambda,\Lambda}$ and a subsequence k' we have $a_{k'} \notin N$. Since $\{a_{k'}\}_{k'=1}^{\infty} \subset \mathcal{A}_{\lambda,\Lambda}$ and is thus bounded in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, it has a non-empty set of weak-* limit points. Suppose that $a_{k''} \rightharpoonup \tilde{a} \in \mathcal{A}_{\lambda,\Lambda}$ for some further subsequence $k'' = 1, 2, \ldots$ By the already established implication, $\mathcal{L}_{a_{k''}}$ H-converges to $\mathcal{L}_{\tilde{a}}$. Owing to Propositions 3.1, 3.2, and [7, Proposition 17] we necessarily have $\tilde{a} = a$. But then $a_{k''} \in N$, for all large enough k'', which is a contradiction. This completes the proof.

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