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Gaps in the cycle spectrum of 3-connected cubic planar graphs

Martin Merker*

Abstract

We prove that, for every natural number k , every sufficiently large 3-connected cubic planar graph has a cycle whose length is in $[k, 2k + 9]$. We also show that this bound is close to being optimal by constructing, for every even $k \geq 4$, an infinite family of 3-connected cubic planar graphs that contain no cycle whose length is in $[k, 2k + 1]$.

Keywords: cycles, cycle spectrum, 3-connected, cubic, planar graphs

1 Introduction

The *cycle spectrum* of a graph G , denoted $\mathcal{C}(G)$, is the set of lengths of cycles in G . The *circumference* of a graph is the length of its longest cycle. The size of the cycle spectrum has been studied for many different graph classes, in particular for graphs of large minimum degree and Hamiltonian graphs. For example, Sudakov and Verstraëte [8] showed that if G has average degree d and girth g , then $|\mathcal{C}(G)| \geq \Omega(d^{\lfloor \frac{g-1}{2} \rfloor})$, proving a conjecture by Erdős [2]. Milans et al. [7] showed that if G is Hamiltonian, then $|\mathcal{C}(G)| \geq \sqrt{p} - \frac{1}{2} \ln p - 1$ where $p = |E(G)| - |V(G)|$.

There are various problems in structural graph theory which can be rephrased in terms of the cycle spectrum of graphs with certain properties. A typical question is to decide for a class of graphs \mathcal{G} and $A \subset \mathbb{N}$ if every graph in \mathcal{G} contains a cycle whose length is in A . For example, Grötzsch's theorem [5] states that $3 \in \mathcal{C}(G)$ for every planar graph G of chromatic number 4, while Steinberg's Conjecture states that $\mathcal{C}(G) \cap \{4, 5\} \neq \emptyset$. Steinberg's Conjecture has been disproved [1] but it is still open if $\mathcal{C}(G) \cap \{4, 5, 6\} \neq \emptyset$ for every planar graph G of chromatic number 4. There are also many theorems and open problems where A is an infinite set. For example, the Erdős-Gyárfás Conjecture [3] states that if G is a graph of minimum degree at least 3, then $\mathcal{C}(G) \cap \{2^k : k \in \mathbb{N}\} \neq \emptyset$. For $m, k \in \mathbb{N}$, let $A(m, k)$ denote the set of natural numbers congruent to m modulo k .

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Very recently, Gao et al. [4] showed that $\mathcal{C}(G) \cap A(2m, k) \neq \emptyset$ for every graph G of minimum degree at least $k + 1$, proving a conjecture by Thomassen [9]. Lyngsie and Merker [6] showed that $\mathcal{C}(G) \cap A(m, 2k + 1) \neq \emptyset$ if G is a sufficiently large 3-connected cubic graph.

In this article we focus on the case where A is an interval and \mathcal{G} is a family of 2-connected simple graphs. Let $a, b \in \mathbb{N}$ with $3 \leq a \leq b$. We say $[a, b]$ is a *gap* of G if $\mathcal{C}(G) \cap [a, b] = \emptyset$ and G has circumference greater than b . Note that if $[a, b]$ is a gap of G then also every interval contained in $[a, b]$ is a gap of G . We investigate which intervals occur as gaps of graphs in \mathcal{G} . If \mathcal{G} contains graphs of arbitrarily large girth, then every interval is a gap of some graph in \mathcal{G} . To avoid this, we focus on 2-connected cubic planar graphs.

Proposition 1. *Let $a, b \in \mathbb{N}$ with $3 \leq a \leq b$. The interval $[a, b]$ is a gap of some 2-connected cubic planar graph if and only if $a = 3, b \leq 4$ or $a = 4, b \leq 9$ or $a \geq 5$.*

Proof. Every 2-connected cubic plane graph contains a face of length 3, 4, or 5. Thus, if $[3, b]$ is a gap of some 2-connected cubic planar graph, then $b \in \{3, 4\}$. The interval $[3, 4]$ is a gap of every cubic planar graph of girth 5. If G is a cubic planar graph of girth 5, then replacing every vertex of G with a triangle yields a graph G' with $\mathcal{C}(G') \cap [4, 9] = \emptyset$. Thus, $[4, 9]$ is a gap of some 2-connected cubic planar graph. A simple discharging argument shows that $\mathcal{C}(G) \cap [4, 10] \neq \emptyset$ for every 2-connected cubic planar graph G .

Now assume $a \geq 5$ and let $C = v_1 v_2 \dots v_{3k} v_1$ be a cycle on $3k$ vertices where $3k > b$. Let G be the graph consisting of C and k vertices u_1, \dots, u_k such that u_i is joined to v_{3i-2}, v_{3i-1} , and v_{3i} for $i \in \{1, \dots, k\}$. It is easy to see that G is a Hamiltonian 2-connected cubic planar graph and the only cycles of length less than $3k$ have length 3 or 4. Thus, $\mathcal{C}(G) \cap [a, b] = \emptyset$ and $[a, b]$ is a gap of G . \square

Proposition 1 completely characterizes the intervals which are gaps of 2-connected cubic planar graphs. For 3-connected cubic planar graphs the characterization of gaps appears to be more difficult. In Section 3 we construct 3-connected cubic planar graphs for which $[k, 2k + 1]$ is a gap (with k even). The main theorem of this paper shows that gaps in 3-connected cubic planar graphs cannot be much larger than that.

Theorem 2. *If $k \in \mathbb{N}$ and G is a 3-connected cubic planar graph of circumference at least k , then $\mathcal{C}(G) \cap [k, 2k + 9] \neq \emptyset$.*

We make the following conjecture which states that the graphs constructed in Section 3 have the largest possible gaps among all 3-connected cubic planar graphs.

Conjecture 3. *If $k \in \mathbb{N}$ with $k \geq 2$ and G is a 3-connected cubic planar graph of circumference at least k , then $\mathcal{C}(G) \cap [k, 2k + 2] \neq \emptyset$.*

It is not clear whether 3-connected planar graphs can have larger gaps in their cycle spectrum compared to 3-connected cubic planar graphs. We conjecture that a similar result to Theorem 2 holds for this more general class of graphs.

Conjecture 4. *There exists $c \in \mathbb{N}$ such that $\mathcal{C}(G) \cap [k, 2k + c] \neq \emptyset$ for every $k \in \mathbb{N}$ and 3-connected planar graph G of circumference at least k .*

2 Proof of Theorem 2

We prove Theorem 2 by contradiction. Suppose there exists a 3-connected cubic plane graph G of circumference at least k such that $\mathcal{C}(G) \cap [k, 2k + 9] = \emptyset$. Now every cycle in G is either *short* (shorter than k) or *long* (longer than $2k + 9$). Since the circumference of G is at least k , there is at least one long cycle in G . The following lemma shows that G also contains a long facial cycle.

Lemma 5. *Let G be a 3-connected cubic plane graph and $k, c \in \mathbb{N}$. If the circumference of G is at least k , then $\mathcal{C}(G) \cap [k, 2k + c] \neq \emptyset$ or G has a facial cycle of length greater than $2k + c$.*

Proof. Suppose $\mathcal{C}(G) \cap [k, 2k + c] = \emptyset$ and every facial cycle of G has length less than k . Let C be a cycle of length greater than $2k + c$ for which the number of faces in its interior is minimal. Since every face has length less than k , there are at least three faces in the interior of C . For every edge e of C , let C_e denote the facial cycle in the interior of C which is incident with e . Let D_e denote the symmetric difference of C and C_e . Note that D_e is a union of cycles for every $e \in E(C)$. It is easy to see that there exists an edge $f \in E(C)$ such that D_f is a cycle. Since every facial cycle has length less than k , we have $|E(D_f)| \geq |E(C)| - k$. Since $|E(C)| > 2k + c$ and $\mathcal{C}(G) \cap [k, 2k + c] = \emptyset$, we have $|E(D_f)| > 2k + c$. However, D_f contains fewer faces than C in its interior, contradicting our choice of C . \square

Next we show that we can remove edges between short faces of G to obtain a 2-connected subgraph G' where no two short faces are adjacent. Moreover, the subgraph G' also contains a long facial cycle and each edge of a 2-edge-cut in G' is incident with both a short face and a long face. Note that in subcubic graphs any two cycles intersecting in a vertex also share an edge.

Lemma 6. *Let G be a 3-connected cubic plane graph and $k \in \mathbb{N}$. If $\mathcal{C}(G) \cap [k, 2k] = \emptyset$ and G contains a facial cycle C of length at least $2k + 1$, then G contains a 2-connected subgraph G' such that*

- (A) *no two facial cycles of G' of length less than k intersect,*
- (B) *$E(C) \subseteq E(G')$,*
- (C) *and every edge of G' which is part of a 2-edge-cut is incident with a face longer than $2k$ and a face shorter than k .*

Proof. We call a cycle *short* if it has length less than k and *long* if it has length greater than $2k$. We call a face *short* (*long*) if its boundary is a short (*long*) cycle. Since

$\mathcal{C}(G) \cap [k, 2k] = \emptyset$, every cycle of G is short or long.

We construct a sequence of 2-connected graphs G_0, G_1, \dots by successively gluing adjacent short faces together. Let $G_0 = G$ and suppose we have constructed G_i . If G_i does not contain two adjacent short faces, then we set $G' = G_i$ and stop. Otherwise, let F_1 and F_2 be two adjacent short faces in G_i . We form a graph H_i by deleting the edges of $F_1 \cap F_2$ and we also delete any resulting isolated vertices. Now H_i has a face F whose boundary is the symmetric difference of F_1 and F_2 . This boundary might consist of several cycles, but each cycle in the boundary is short. This operation cannot create cut-edges, but it can disconnect the graph, in which case F is incident with each component. If H_i is disconnected, then we choose the component containing C as G_{i+1} , otherwise we set $G_{i+1} = H_i$. Note that every long face of G_{i+1} is also a long face of G_i . Clearly G_{i+1} is a proper subgraph of G_i , so the sequence terminates.

In G' there are no two adjacent short faces, so G' satisfies (A). We have $E(C) \subset E(G_i)$ for each i , so G' satisfies (B). Every G_i is 2-connected, so G' is 2-connected. If e_1, e_2 is a 2-edge-cut in G' , then there are two faces F_1 and F_2 which are incident with both e_1 and e_2 . Since G' satisfies (A), at least one of these two faces is long. If both F_1 and F_2 are long, then F_1, F_2 are also faces of G and e_1, e_2 would be a 2-edge-cut in G , contradicting 3-connectivity. Thus, every edge of a 2-edge-cut is incident with a short face and a long face, so G' satisfies (C). \square

Let C_1 and C_2 be two facial cycles in G' . Any two distinct edges in $E(C_1 \cap C_2)$ form a 2-edge-cut. Thus, if $|E(C_1 \cap C_2)| \geq 2$, then by (A) and (C) one of the two facial cycles C_1 and C_2 is long while the other one is short. In particular, (C) implies that if C_1 and C_2 are two facial cycles in G' with length greater than k , then $|E(C_1 \cap C_2)| \leq 1$. We can now prove Theorem 2 by a counting argument.

Proof of Theorem 2. We fix an embedding of G in the plane. Suppose the circumference of G is at least k and $\mathcal{C}(G) \cap [k, 2k + 9] = \emptyset$. By Lemma 5 we can assume that G has a facial cycle C with $|E(C)| \geq 2k + 10$. We apply Lemma 6 to obtain a 2-connected subgraph G' of G satisfying the conditions (A), (B), and (C). Let H be the graph we obtain from G' by suppressing all vertices of degree 2. Note that H might have multiple edges. The embedding of G in the plane also gives us an embedding of G' and H such that there is a canonical bijection between the faces of G' and H . Let $F(G')$ denote the set of faces of G' . For $F \in F(G')$ we write $\ell(F)$ for the length of F and $\ell_H(F)$ for the length of the corresponding face in H . We define $X = \{F \in F(G') : \ell(F) < k\}$ and $Y = \{F \in F(G') : \ell(F) > 2k + 9\}$. Note that $F(G') = X \cup Y$. Let $x = |X|$, $y = |Y|$, and $n = |V(H)|$. The graph H is a 2-connected cubic planar graph, so

$$x + y = \frac{n}{2} + 2 \tag{1}$$

by Euler's formula.

Suppose H is a triple edge. In this case G' has exactly three facial cycles, say C_1, C_2 , and C_3 , and they pairwise intersect. By (A), at least two of the cycles are long, so we may assume that C_1 and C_2 are long. By (C), we have $|E(C_1 \cap C_2)| \leq 1$ and thus

$|E(C_1 \cap C_2)| = 1$. If all three cycles were long, then by (C) their pairwise intersections would have size one and G' would be a triple edge, contradicting (B). Thus, $|E(C_3)| < k$ and we get the contradiction

$$2k + 9 < |E(C_1)| = |E(C_1 \cap C_2)| + |E(C_1 \cap C_3)| < 1 + k.$$

Suppose H contains a facial cycle of length 2. Let C' be the corresponding facial cycle in G' , and let C_1, C_2 denote the two facial cycles in G' that intersect C' . Let u, v denote the two vertices of degree 3 in G' that are incident with C' . Let e_u, e_v denote the edges incident with u, v that are not incident with C' . Since H is not a triple edge, the edges e_u and e_v are distinct. It is easy to see that $e_u, e_v \in E(C_1 \cap C_2)$ and e_u, e_v form a 2-edge-cut in G' . By (C), we may assume $|E(C_1)| < k$ and $|E(C_2)| > 2k + 9$. By (A), we have $|E(C')| > 2k + 9$. Now

$$2k + 9 < |E(C')| = |E(C' \cap C_1)| + |E(C' \cap C_2)| < k + |E(C' \cap C_2)|$$

implies $|E(C' \cap C_2)| > k + 9$ which contradicts (C). Thus, every facial cycle of H has length at least 3. By (A) the faces in X are pairwise non-adjacent, thus

$$n = |V(H)| \geq \sum_{F \in X} \ell_H(F) \geq 3x \quad (2)$$

which implies

$$\sum_{F \in Y} \ell_H(F) = \sum_{F \in F(G')} \ell_H(F) - \sum_{F \in X} \ell_H(F) = 2|E(H)| - \sum_{F \in X} \ell_H(F) \leq 3n - 3x. \quad (3)$$

In G' , each edge incident with a face in Y is also an edge in H or it is incident with a face in X , so by (3) we have

$$\sum_{F \in Y} \ell(F) \leq \sum_{F \in Y} \ell_H(F) + \sum_{F \in X} \ell(F) \leq 3(n - x) + (k - 1)x = 3n + (k - 4)x. \quad (4)$$

Each face in Y has length at least $2k + 10$ in G' , so by (1) we have

$$\sum_{F \in Y} \ell(F) \geq (2k + 10)y = (2k + 10) \left(\frac{n}{2} + 2 - x \right) > (k + 5)n - (2k + 10)x. \quad (5)$$

Combining (4) and (5) yields

$$(k + 5)n - (2k + 10)x < 3n + (k - 4)x$$

which is equivalent to

$$(k + 2)n < 3(k + 2)x.$$

This implies $n < 3x$ which contradicts (2). \square

3 Graphs with large cycle gaps

In this section we show that for every $k \geq 2$ the interval $[2k, 4k + 1]$ is a gap of a 3-connected cubic planar graph and thus the interval $[k, 2k + 9]$ in Theorem 2 is close to the optimal bound.

Theorem 7. *For $k \in \mathbb{N}$ with $k \geq 2$ there exists a 3-connected cubic planar graph G of circumference at least $2k$ such that $\mathcal{C}(G) \cap [2k, 4k + 1] = \emptyset$.*

Proof. For $k \in \mathbb{N}$, let H_k be the graph consisting of two disjoint cycles $C_1 = x_0x_1 \dots x_{2k}x_0$ and $C_2 = y_0y_1 \dots y_{2k}y_0$ together with chords $x_i x_{2k-i}$ and $y_i y_{2k-i}$ for $i \in \{1, \dots, k-1\}$ as well as one edge $x_k y_k$ joining C_1 and C_2 , see Figure 1.

Let D be a cubic graph and $e \in E(D)$. We define $D(e, H_k)$ as the graph obtained from D by replacing e with a copy of H_k . To be more precise, if $e = uv$ where u is adjacent to u_1, u_2, v and v is adjacent to v_1, v_2, u in D , then $D(e, H_k)$ is obtained from the disjoint union of $D - u - v$ and H_k by adding the edges $u_1x_0, u_2x_{2k}, v_1y_0$, and v_2y_{2k} . If M is a matching in D , then we define $D(M, H_k)$ as the graph we obtain by successively replacing the edges in M by H_k . Note that if D is a 3-connected cubic planar graph, then so is $D(M, H_k)$. If M is a perfect matching, then $D(M, H_1)$ is the graph where every vertex of D is replaced with a triangle.

For $n \in \mathbb{N}$ with $n \geq 4k + 2$, let D_n be a graph consisting of two cycles $C_1 = u_1 \dots u_n u_1$, $C_2 = v_1 \dots v_n v_1$ of length n and one cycle $C_3 = w_1 w_2 \dots w_{2n} w_1$ of length $2n$ together with edges $v_i w_{2i-1}$ and $u_i w_{2i}$ for every $i \in \{1, \dots, n\}$. It is easy to see that D_n is a 3-connected cubic planar graph. Let M denote the perfect matching in D consisting of the edges of the form $v_i w_{2i-1}$ and $u_i w_{2i}$ for $i \in \{1, \dots, n\}$. We define $G(n, k) = D_n(M, H_{k-1})$, see Figure 2 for an example. Clearly $G(n, k)$ is a 3-connected cubic planar graph. Since $n \geq 4k + 2$, the circumference of $G(n, k)$ is greater than $2k$. It remains to show that $\mathcal{C}(G(n, k)) \cap [2k, 4k + 1] = \emptyset$.

Let C be a cycle of $G(n, k)$ and suppose $2k \leq |E(C)| \leq 4k + 1$. The circumference of H_{k-1} is $2k - 1$, so C is not contained in one of the copies of H_{k-1} . Thus, C corresponds to a cycle C' in D (by contracting each $(2k - 1)$ -cycle in each copy of H_{k-1} into a vertex) and $|E(C')| \leq |E(C)|$. The shortest cycle in D not containing an edge of M has length n , so C' contains at least one edge of M . There are no cycles in D containing precisely one edge of M , so $|E(C') \cap M| \geq 2$. Note that in C the edges used by C' in M correspond to paths between vertices x_0 and y_0 in H_{k-1} and these have length at least $2k - 1$. In particular $|E(C') \cap M| = 2$. Moreover, C' contains two consecutive edges not belonging to M and these two edges correspond to a path of length at least 3 in C . Since the girth of D is 5, we have $|E(C)| \geq |E(C')| + 2(2k - 2) + 1 \geq 5 + 4k - 3 = 4k + 2$, contradicting our choice of C . \square

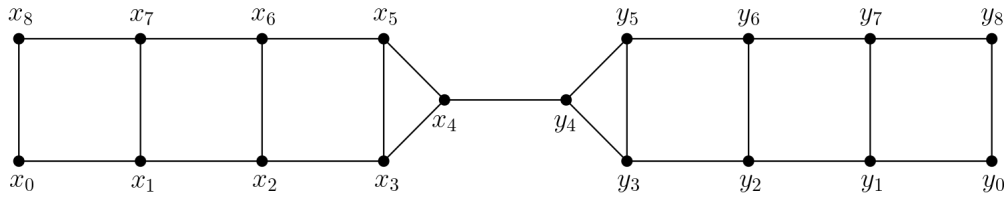


Figure 1: The graph H_4

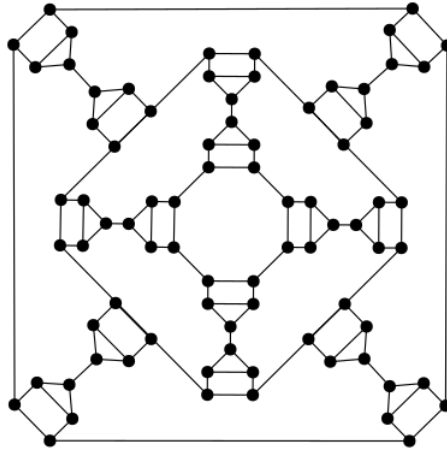


Figure 2: The graph $G(4,3)$

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