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Light meson masses from baryon interaction states?

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Abstract – We examine interaction states between baryons in $U(3)$ configurations. Such interaction states may represent the meson mass spectrum above the pion triplet. Our configuration space is the Lie group $U(3)$ with a Hamiltonian structure for baryons as stationary states. Mesonic states come about via an interaction potential. The Hamiltonian can be diagonalized by a Rayleigh-Ritz method resulting in matrix element integrals that can be solved analytically for the toroidal degrees of freedom by expanding on a suitable set of base functions. We compare calculated eigenvalues for indefinite parity states to observed unflavoured meson masses.



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Introduction. – Most work on meson mass spectra select specific groups of states trying to understand them from a quark model perspective [1,2]. More fundamental are lattice gauge theory considerations based on quantum chromodynamics [3] where masses are derived from time decay of correlation functions [4]. Here a select choice of meson and baryon masses, *e.g.*, from π, K and N, Δ are used to settle quark masses for hadron spectroscopy.

The present work wants to avoid quark masses as fitting parameters. The goal is to calculate meson masses without specific reference to quark models. Instead, we want to treat light mesons as interaction states between two baryons. We first introduce the baryons for whom the mesons act as interaction quanta under the strong interactions.

Baryons. – We consider baryons as stationary states on an intrinsic $U(3)$ Lie group configuration space. One may consider the configuration space as a generalization of an intrinsic spin space, now with nine degrees of freedom. The intrinsic dynamics to produce the baryon spectrum is a Hamiltonian structure [5],

$$\frac{\hbar c}{a} \left[-\frac{1}{2}\Delta + \frac{1}{2}\text{Tr} \chi^2 \right] \Psi(u) = \mathcal{E}\Psi(u), \quad (1)$$

with configuration variable $u = e^{ix} \in U(3)$, Laplacian Δ and a Manton-like trace potential [6] radically reinterpreted from lattice gauge theory for non-perturbative quantum chromodynamics [7]. The trace potential in (1) folds

out in periodic potentials of nine dynamical variables [8], see fig. 1.

The scale $\Lambda \equiv \hbar c/a$ is set by a projection [5] of the intrinsic baryon dynamics to space. The length scale a for the projection is related to the classical electron radius $r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2}$ [9,10] by the expression $r_e = \pi a$ [5]. The factor π manifests the toroidal shape of the intrinsic configuration space, $U(3)$. Our baryonic energy scale $\Lambda = \hbar c/a = \frac{\pi}{\alpha} m_e c^2$ is close to the scale of quantum chromodynamics $\Lambda_{\text{MS}}^{(5)} \approx 210(14)$ MeV [3].

The projection scaled by a led to a compact relation for the neutron to electron mass ratio [5]

$$\frac{m_n}{m_e} = \frac{\pi}{\alpha} E_n = 1839(1), \quad (2)$$

in agreement with the experimental value [3]. Here $E_n \equiv \mathcal{E}_n/\Lambda = 4.382(2)$ from a Rayleigh-Ritz solution [11] of (1) with 3078 base functions —at the limit of our computer programme— and the fine structure coupling $\alpha^{-1}(m_n) = 133.61$ is sliding from $\alpha^{-1} = \alpha^{-1}(m_\tau) = 133.476(7)$ [3,12]. The Hamiltonian structure in (1) also gives good agreement with observed four star resonances for unflavoured baryons [13,14] and with the proton spin structure function [15,16].

Unfolding the theory for baryons. – To solve (1), we first express the configuration variable in nine dynamical variables $\theta_j, \alpha_j, \beta_j \in \mathbb{R}$, $j = 1, 2, 3$ spanning the

$$\frac{\hbar c}{a} \left[-\frac{1}{2} \sum_{j=1}^3 \frac{\partial^2}{\partial \theta_j^2} - 1 + \sum_{\substack{1 \leq i < j \leq 3 \\ k \neq i, j}}^3 \frac{(S_k^2 + M_k^2)/\hbar^2}{16 \sin^2 \frac{1}{2}(\theta_i - \theta_j)} + \sum_{j=1}^3 w(\theta_j) \right] \Phi(u) = \mathcal{E} \Phi(u). \quad (12)$$

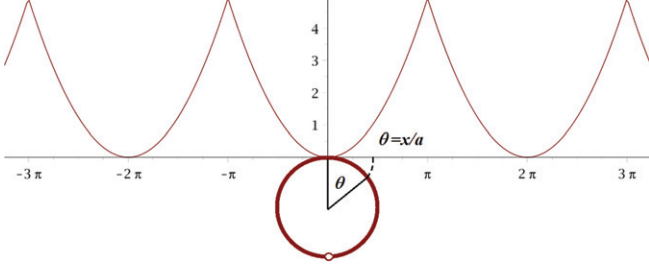


Fig. 1: Trace potential (10) in dynamical variables and projection of these to laboratory space. Figure taken from [16].

nine degrees of freedom laid out by the nine generators T_j, S_j, M_j of the algebra $u(3)$, thus

$$u = e^{i(\theta_j T_j + \alpha_j S_j + \beta_j M_j/\hbar)}, \quad T_j, S_j/\hbar, M_j/\hbar \in u(3). \quad (3)$$

The three T_j 's generate the maximal, Abelian torus of $U(3)$, the S_j 's are off-diagonal generators equivalent to three of the off-diagonal Gell-Mann matrices [17] and take care of spin degrees of freedom. The three M_j 's take care of flavour degrees of freedom. The nine generators are equivalent to kinematic generators in laboratory space, thus one may imagine the intrinsic degrees of freedom to be excited kinematically from laboratory space. In a coordinate representation [17] with $p_j = -i\hbar \frac{\partial}{\partial \theta_j}$, we have, *e.g.*,

$$\begin{aligned} S_1 &= a\theta_2 p_3 - a\theta_3 p_2 = \hbar \lambda_7, \\ M_1/\hbar &= \theta_2 \theta_3 + \frac{a^2}{\hbar^2} p_2 p_3 = \lambda_6. \end{aligned} \quad (4)$$

We use the Laplacian in a polar decomposition [18]

$$\Delta = \sum_{j=1}^3 \frac{1}{J} \frac{\partial^2}{\partial \theta_j^2} J + 2 - \sum_{\substack{1 \leq i < j \leq 3 \\ k \neq i, j}}^3 \frac{(S_k^2 + M_k^2)/\hbar^2}{8 \sin^2 \frac{1}{2}(\theta_i - \theta_j)}, \quad (5)$$

where $e^{i\theta_j}, j = 1, 2, 3, \theta_j \in \mathbb{R}$ are the three eigenvalues of u in a matrix representation. The derivatives are equivalent to the toroidal generators with generalizations to left invariant coordinate fields ∂_j and forms $d\theta_j$ [5]

$$\frac{\partial}{\partial \theta_j} = iT_j, \quad [iT_j, \theta_i] \sim d\theta_i(\partial_j) = \delta_{ij}, \quad \partial_j = u iT_j, \quad (6)$$

and the van de Monde determinant is [19]

$$J = \prod_{1 \leq i < j \leq 3} 2 \sin \left(\frac{1}{2}(\theta_i - \theta_j) \right). \quad (7)$$

The van de Monde determinant only depends on the three eigenangles θ_j . The trace in the potential is invariant under conjugation $u \rightarrow v^{-1} u v, v \in U(3)$. In particular we can use a conjugation to diagonalize u to get

$$u \sim v^{-1} u v = \begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{i\theta_3} \end{pmatrix}. \quad (8)$$

Thus, the potential likewise only depends on the eigenangles and is periodic in these [8], see fig. 1,

$$\frac{1}{2} \text{Tr} \chi^2 = \sum_{j=1}^3 w(\theta_j), \quad (9)$$

where

$$w(\theta) = \frac{1}{2}(\theta - n \cdot 2\pi)^2, \quad \theta \in [(2n-1)\pi, (2n+1)\pi], \quad n \in \mathbb{Z}. \quad (10)$$

The polar decomposition (5) suits the potential (10) such that we can factorize the wave function

$$\Psi(u) = \tau(\theta_1, \theta_2, \theta_3) \Upsilon(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3). \quad (11)$$

We namely multiply through by J with (5) and (9) inserted in (1) and introduce $\Phi \equiv J\Psi \equiv R\Upsilon$ to get

see eq. (12) above

The nominator in the third term in (12) acts only on the off-toroidal degrees of freedom and we want to integrate these out like when solving the hydrogen atom [17]. This is possible by exploiting the fact that the indexing $j = 1, 2, 3$ of the three sets of dynamical variables $\theta_j, \alpha_j, \beta_j$ is arbitrary. Therefore, we can average over the off-toroidal degrees of freedom if only we know the spectrum of $\mathbf{S} = (S_1, S_2, S_3)$ and $\mathbf{M} = (M_1, M_2, M_3)$. The spectrum of the spin operators S_j is well known,

$$S_j/\hbar = -s, -s+1, \dots, s, \quad \mathbf{S}^2/\hbar^2 = s(s+1), \quad (13)$$

from the commutation relations (cf., *e.g.*, p. 145 in [20])

$$[S_i, S_j] = -i\hbar \varepsilon_{ijk} S_k. \quad (14)$$

Note the minus sign on the right-hand side as in intrinsic body fixed coordinate systems in nuclear physics [21]. The three operators M_j connect the algebra by commuting into the spin subalgebra generated by the three spin generators S_j

$$[M_i, M_j] = [S_i, S_j] = -i\hbar \varepsilon_{ijk} S_k. \quad (15)$$

The spectrum for \mathbf{M}^2 follows from a lengthy calculation [13,16]. Here we give the result

$$\mathbf{M}^2 = \frac{4}{3} \left(n + \frac{3}{2} \right)^2 - s(s+1) - 3 - \frac{1}{3}y^3 - 4i_3^2, \quad (16)$$

where $n \geq 0$ is a non-negative integer, y is the hypercharge and i_3 is the isospin three-component. For $n = 1, y = 1, i_3 = \pm \frac{1}{2}$ and spin $s = \frac{1}{2}$ we get the minimum value

$$(\mathbf{S}^2 + \mathbf{M}^2)/\hbar^2 = 4. \quad (17)$$

Thus, integrating out the off-toroidal degrees of freedom our problem in (12) greatly simplifies to yield for the toroidal wave function $R = J\tau$

$$\left[-\frac{1}{2}\Delta_e + W(\theta_1, \theta_2, \theta_3) \right] R = ER, \quad (18)$$

where the dimensionless eigenvalue $E \equiv \mathcal{E}/\Lambda$ with energy scale $\Lambda \equiv \hbar c/a$, total potential

$$W(\theta_1, \theta_2, \theta_3) \equiv -1 + C + \sum_{j=1}^3 w(\theta_j), \quad (19)$$

where the potential C and the Laplacian Δ_e are

$$C = \sum_{1 \leq i < j \leq 3} \frac{4/3}{16 \sin \frac{1}{2}(\theta_i - \theta_j)}, \quad \Delta_e = \sum_{j=1}^3 \frac{\partial^2}{\partial \theta_j^2}. \quad (20)$$

As mentioned, the labelling of the eigenangles θ_j is arbitrary so the toroidal wave function $\tau(\theta_1, \theta_2, \theta_3)$ should be symmetric in these. The Jacobian J in (7), on the other hand is antisymmetric, so the *measure-scaled* toroidal wave function $R \equiv J\tau$ must be antisymmetric. Thus, R can be expanded on Slater determinants

$$f_{pqr} = \begin{vmatrix} e^{ip\theta_1} & e^{ip\theta_2} & e^{ip\theta_3} \\ e^{iq\theta_1} & e^{iq\theta_2} & e^{iq\theta_3} \\ e^{ir\theta_1} & e^{ir\theta_2} & e^{ir\theta_3} \end{vmatrix}, \quad p < q < r \in \mathbb{Z}. \quad (21)$$

Colour quark fields are generated by the momentum form as $c_j(u) = dR(uiT_j)$ and used, *e.g.*, for parton distribution functions, *e.g.*, for u and d quarks with $T_u = \frac{2}{3}T_1 - T_3$ and $T_d = -\frac{1}{3}T_1 - T_3$ [5].

Rayleigh-Ritz method. – We want to find eigenvalues E in (18) equivalent to

$$[-\Delta_e + 2W]R = 2ER. \quad (22)$$

In the Rayleigh-Ritz method [11,16] one expands the eigenfunctions on an orthogonal set of base functions with a set of expansion coefficients, multiply the equation by the complex conjugate expansion, integrates over the entire variable volume and ends up with a matrix problem in the expansion coefficients from which a set of eigenvalues can be obtained. In standard quantum mechanics lingo

this is called *diagonalization* of the Hamiltonian. Thus with the approximation

$$R_N = \sum_{l=1}^N a_l f_l, \quad (23)$$

we have the integral equation

$$\begin{aligned} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} R_N^* \cdot (-\Delta_e + 2W) R_N d\theta_1 d\theta_2 d\theta_3 = \\ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} R_N^* \cdot 2ER_N d\theta_1 d\theta_2 d\theta_3. \end{aligned} \quad (24)$$

The counting variable l in (23) is a suitable ordering of the set of triples p, q, r in (21) such that we expand on an orthogonal set. Equation (24) can be interpreted as a vector eigenvalue problem, where \mathbf{a} is a vector, whose elements are the expansion coefficients a_l . Thus, (24) is equivalent to the eigenvalue problem

$$\mathbf{a}^\dagger \mathbf{H} \mathbf{a} = 2E \mathbf{a}^\dagger \mathbf{F} \mathbf{a}, \quad (25)$$

where the matrix elements of \mathbf{H} and \mathbf{F} are given by

$$H_{lm} \equiv \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_l^* \cdot (-\Delta_e + 2W) f_m d\theta_1 d\theta_2 d\theta_3 \quad (26)$$

and

$$F_{lm} \equiv \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_l^* f_m d\theta_1 d\theta_2 d\theta_3. \quad (27)$$

When the set of expansion functions is orthogonal, (25) implies

$$\mathbf{H} \mathbf{a} = 2E \mathbf{F} \mathbf{a}, \quad (28)$$

from which we get a spectrum of N eigenvalues determined as the set of components of a vector \mathbf{E} generated from the eigenvalues of the matrix $\mathbf{F}^{-1} \mathbf{H}$, *i.e.*,

$$\mathbf{E} = \frac{1}{2} \text{eig}(\mathbf{F}^{-1} \mathbf{H}). \quad (29)$$

The lowest-lying eigenvalues will be better and better determined for increasing values of N in (23). For the base (21) the integrals (26) and (27) can be solved analytically.

The exact expressions to be used in constructing \mathbf{H} and \mathbf{F} are given below for the base (21). For $p < q < r$ and $s < t < u$ we have the following orthogonality relations:

$$\begin{aligned} \langle f_{pqr} | f_{stu} \rangle \equiv \\ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_{pqr}^*(\theta_1, \theta_2, \theta_3) \cdot f_{stu}(\theta_1, \theta_2, \theta_3) d\theta_1 d\theta_2 d\theta_3 = \\ (2\pi)^3 \cdot 6 \delta_{ps} \delta_{qt} \delta_{ru}. \end{aligned} \quad (30)$$

Here we have used the Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \quad (31)$$

The Laplacian yields

$$\left\langle f_{pqr} \left| \frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2} + \frac{\partial^2}{\partial \theta_3^2} \right| f_{stu} \right\rangle = (-p^2 - q^2 - r^2) \cdot (2\pi)^3 \cdot 6\delta_{ps}\delta_{qt}\delta_{ru}. \quad (32)$$

The matrix elements for the intrinsic potential (9) and the centrifugal potential C in (19) are more involved [16]. We give explicit expressions for these when considering interaction states.

Interaction states. – The trace potential in (1) is half the square of the shortest geodesic $d(e, u)$ from the *origo* e , the neutral element, to the configuration point $u \in U(3)$. Thus we could equally well write (1) as

$$\frac{\hbar c}{a} \left[-\frac{1}{2}\Delta + \frac{1}{2}d^2(e, u) \right] \Psi(u) = \mathcal{E}\Psi(u). \quad (33)$$

This form opens for the introduction of interaction states between two baryons with intrinsic configuration variables u, u' , respectively. We namely take the interaction quanta to be eigenstates of

$$\frac{\hbar c}{a} \left[-\frac{1}{2}(\Delta_u + \Delta_{u'}) + \frac{1}{2}d^2(u, u') \right] \Psi(u, u') = \mathcal{E}\Psi(u, u'), \quad (34)$$

for proper choices of spins, hypercharges and isospins, hidden in the Laplacians for the respective configuration variables. Note that we have deliberately omitted the intrinsic potentials $d^2(e, u)/2$ and $d^2(e, u')/2$, respectively, as these relate to the intrinsic structure of the individual baryons whereas here we want to consider their interaction keeping the kinetic terms at the point of interaction. It is our task in the present work to solve (34).

The shortest geodesic is left (and right) invariant, thus

$$d(u, u') = d(e, u^{-1}u') \quad (35)$$

by left translation with u^{-1} . This left invariance is what secures local gauge invariance of strong interactions in laboratory space [14,22].

We shall interpret the interaction potential as a representation of unflavoured antiquark-quark states, *e.g.*, \bar{u} -quark $\sim u^{-1}$ -configuration and d -quark $\sim u'$ -configuration leading to mesonic interaction states. In table 1, we are therefore going to take the following set of quark quantum numbers,

$$s = s' = \frac{1}{2}, \quad n = n' = 1, \quad y = y' = \frac{1}{3}, \quad i_3 = -i'_3 = \pm \frac{1}{2}, \quad (36)$$

to get the centrifugal potentials in (12) for the configuration variables u and u' , respectively.

The new thing about making (35) operational in the interaction potential $\frac{1}{2}d^2(e, u^{-1}u')$ is that we *cannot* in general diagonalize both u and u' by conjugation with the same $v \in U(3)$ and thus the shortest geodesic cannot be expressed only by the eigenangles θ_j and θ'_j . Therefore

Table 1: Light unflavoured mesons above the pion triplet compared to interaction states from (34). Isospin and spin assignments are for the observed states only. The masses for interaction states are tentative. They are sensitive to averaging over off-toroidal degrees of freedom (60). The values shown are for base sets combined from $-2 \leq p < q < r \leq 4$ and $-2 \leq h < k < l \leq 4$ (*i.e.*, 1225 base functions) and for $\Lambda = 214$ MeV. They are given for indefinite parity Slater determinants (21).

Meson [3] experiment	Quantum numbers I, J	Observed (MeV)	Interaction state (MeV)
η	0,0	547.862	452
f_0	0,0	400–550	705
$\rho(770)$	1,1	775.26	714
ω	0,1	782.65	809
η'	0,0	957.78	816
$f_0(980)$	0,0	990	886
a_0	1,0	980	911
–	–	–	–

our wave function cannot be factorized in toroidal and off-toroidal factors like in (11) for our single baryon case.

We have to find a way of averaging over the off-toroidal degrees of freedom in u and u' . We will do this by what we could call a *one shot Monte Carlo* integration. As an exemplar situation we take $u = e^{i\lambda_1}, u' = e^{i\lambda_2}$ and we want to calculate the value of the interaction potential for these two configuration variables which share eigenvalues but are at finite distance. The result is (57)

$$d^2(e^{i\lambda_1}, e^{i\lambda_2}) = 3.248984884\dots \approx \frac{\pi^2}{3}. \quad (37)$$

Calculation of off-diagonal distance. – To calculate the squared distance in (37) requires quite a bit of algebra, which we here go through. To simplify we use the two-dimensional analogues of λ_1 and λ_2

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (38)$$

and define

$$u_1 = e^{i\tau_1}, \quad u_2 = e^{i\tau_2}. \quad (39)$$

With $u_1 u_2^{-1} \equiv e^{iX}$, we consider

$$d^2(u_1, u_2) = d^2(u_1 u_2^{-1}, e) = \text{Tr } X^2. \quad (40)$$

Now

$$u_1 u_2^{-1} = u_1 u_2^\dagger = e^{i\tau_1} e^{-i\tau_2^\dagger} = e^{i\tau_1} e^{-i\tau_2} \quad (41)$$

and we have the nice relation [23]

$$e^{i\theta \cdot \tau} = \cos \theta I + i\hat{\theta} \cdot \tau \sin \theta, \quad \hat{\theta} \equiv \theta/\theta, \quad (42)$$

where

$$\theta = |\boldsymbol{\theta}|, \quad \boldsymbol{\theta}_1 = (1, 0, 0), \quad \boldsymbol{\theta}_2 = (0, -1, 0) \quad (43)$$

and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3). \quad (44)$$

Thus,

$$\begin{aligned} u_1 &= e^{i\boldsymbol{\theta}_1 \cdot \boldsymbol{\tau}} = \cos 1I + i\tau_1 \sin 1 \\ u_2^{-1} &= e^{i\boldsymbol{\theta}_2 \cdot \boldsymbol{\tau}} = \cos 1I - i\tau_2 \sin 1. \end{aligned} \quad (45)$$

Omitting the unit matrix I we get

$$\begin{aligned} u_1 u_2^{-1} &= (\cos 1 + i\tau_1 \sin 1) \cdot (\cos 1 - i\tau_2 \sin 1) = \\ &= \cos^2 1 - i\tau_2 \cos 1 \sin 1 + i\tau_1 \sin 1 \cos 1 + \tau_1 \tau_2 \sin^2 1. \end{aligned} \quad (46)$$

This yields

$$\begin{aligned} u_1 u_2^{-1} &= \begin{pmatrix} \cos^2 1 + i \sin^2 1 & (i-1) \cos 1 \sin 1 \\ (i+1) \cos 1 \sin 1 & \cos^2 1 - i \sin^2 1 \end{pmatrix} \\ &= \cos^2 1 \begin{pmatrix} 1 + i\alpha^2 & \alpha(i-1) \\ \alpha(i+1) & 1 - i\alpha^2 \end{pmatrix}, \quad \alpha \equiv \tan 1. \end{aligned} \quad (47)$$

We want to exploit the conjugation relation [24]

$$B e^A B^{-1} = e^{B A B^{-1}}. \quad (48)$$

Using (48) we can calculate the trace needed in (40) from just finding the eigenvalues of $u_1 u_2^{-1}$ in (47). To do this we set up the eigenvalue equation

$$\begin{pmatrix} 1 + i\alpha^2 & \alpha(i-1) \\ \alpha(i+1) & 1 - i\alpha^2 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = e \begin{pmatrix} 1 \\ x \end{pmatrix} \quad (49)$$

and find two complex conjugate eigenvalues,

$$e_a = 1 - i\alpha\sqrt{2 + \alpha^2} \quad \text{and} \quad e_b = 1 + i\alpha\sqrt{2 + \alpha^2}, \quad (50)$$

from which the eigenvalues λ for $u_1 u_2^{-1}$ are

$$\begin{aligned} \lambda &= e_{a,b} \cos^2 1 = \cos^2 1 \pm i \sin 1 \sqrt{2 \cos^2 1 + \sin^2 1} \\ &= 1 - \beta^2 \pm i\beta\sqrt{2 - \beta^2}, \quad \beta \equiv \sin 1. \end{aligned} \quad (51)$$

To get the eigenvalues for the generator X , we need the complex logarithm [25]

$$z = r e^{i\theta} \rightarrow \log z \equiv \log r + i\theta. \quad (52)$$

For our case we have

$$r^2 = \lambda \lambda^* = 1 \rightarrow \log \lambda = \pm i\Theta, \quad (53)$$

with the angular argument determined from (51) by

$$\tan \Theta = \frac{\beta\sqrt{2 - \beta^2}}{1 - \beta^2}. \quad (54)$$

This yields by diagonalization with some $B \in U(3)$

$$B u_1 u_2^{-1} B^{-1} = e^{iX} = \exp \begin{pmatrix} i\Theta & \\ & -i\Theta \end{pmatrix} = e^{i\Theta\boldsymbol{\tau}_3}, \quad (55)$$

from which we get

$$X^2 = X X^\dagger = \Theta\boldsymbol{\tau}_3 \cdot (\Theta\boldsymbol{\tau}_3^\dagger) = \Theta^2 I, \quad (56)$$

and finally as disclosed in (37)

$$\text{Tr } X^2 = 2\Theta^2 = 2 \left(\arctan \frac{\beta\sqrt{2 - \beta^2}}{1 - \beta^2} \right)^2 \approx \frac{\pi^2}{3}. \quad (57)$$

Integrating the trace potential in the off-diagonal degrees of freedom. – We cannot factorize the wave function $\Psi(u, u')$ in (34) like we did in (11) because the interaction trace potential cannot be reduced to an expression containing only the eigenangles θ_j and θ'_j for u and u' , respectively. But we can *expand* the measure-scaled function

$$\Phi(u, u') = J J' \Psi(u, u') \quad (58)$$

on factorized functions of the form

$$\Phi(u, u') = R(\boldsymbol{\theta}) R'(\boldsymbol{\theta}') \Upsilon(\boldsymbol{\alpha}, \boldsymbol{\beta}) \Upsilon'(\boldsymbol{\alpha}', \boldsymbol{\beta}'). \quad (59)$$

We then have

$$\begin{aligned} &\int d\boldsymbol{\alpha}^3 d\boldsymbol{\beta}^3 d\boldsymbol{\alpha}'^3 d\boldsymbol{\beta}'^3 \Upsilon^\dagger(\boldsymbol{\alpha}, \boldsymbol{\beta}) \Upsilon'(\boldsymbol{\alpha}', \boldsymbol{\beta}') \\ &\cdot \frac{1}{2} d^2(u, u') \Upsilon(\boldsymbol{\alpha}, \boldsymbol{\beta}) \Upsilon'(\boldsymbol{\alpha}', \boldsymbol{\beta}') \approx \frac{\pi^2}{3} \sum_{j=1}^3 w(\theta_j - \theta'_j). \end{aligned} \quad (60)$$

Here we used the exemplar off-toroidal average from (57) and independent diagonalizations of u and u' like in (8).

Rayleigh-Ritz matrix elements for interaction states. – We here explain how to obtain matrix elements for the centrifugal terms and the interaction term from (34). In (59) we expand R and R' on Slater determinants (21)

$$R_N = \sum_{l=1}^N a_l f_l, \quad R'_N = \sum_{l=1}^N a'_l f'_l. \quad (61)$$

For clarity, below, we use the following notation:

$$f_l = f_{pqr} \equiv R_{pqr}(\theta_1, \theta_2, \theta_3) \quad f'_l = f_{hkl} \equiv R'_{hkl}(\theta'_1, \theta'_2, \theta'_3). \quad (62)$$

The *norm integrals* follow from (30):

$$\begin{aligned} &\langle R_{stu} R'_{mno} | R_{pqr} R'_{hkl} \rangle_{(2\pi)^6} = \\ &\langle R_{stu} | R_{pqr} \rangle_{(2\pi)^3} \cdot \langle R'_{mno} | R'_{hkl} \rangle_{(2\pi)^3} = \\ &(2\pi)^3 \cdot 6\delta_{ps}\delta_{qt}\delta_{ru} \cdot (2\pi)^3 \cdot 6\delta_{hm}\delta_{kn}\delta_{lo}. \end{aligned} \quad (63)$$

Note that l and o are letters, not the numbers one and zero! The subscripts $(2\pi)^3$ indicate that we integrate from $-\pi$ to π in all three eigenangles θ_j and θ'_j , respectively. Also the *Laplacian integrals* for (20) factorize

$$\begin{aligned} & \langle R_{stu}R'_{mno} | \Delta_e + \Delta'_e | R_{pqr}R'_{hkl} \rangle_{(2\pi)^6} = \\ & \langle R_{stu} | \Delta_e | R_{pqr} \rangle_{(2\pi)^3} \langle R'_{mno} | R'_{hkl} \rangle_{(2\pi)^3} \\ & + \langle R_{stu} | R_{pqr} \rangle_{(2\pi)^3} \langle R'_{mno} | \Delta'_e | R'_{hkl} \rangle_{(2\pi)^3} = \\ & -(p^2 + q^2 + r^2 + h^2 + k^2 + l^2) \\ & \cdot (2\pi)^3 \cdot 6\delta_{ps}\delta_{qt}\delta_{ru} \cdot (2\pi)^3 \cdot 6\delta_{hm}\delta_{kn}\delta_{lo}. \end{aligned} \quad (64)$$

The *centrifugal potential* likewise comes in factors depending on the three θ_j s and the three θ'_j s respectively

$$\begin{aligned} & \left\langle R_{stu}R'_{mno} \left| \sum_{1 \leq i < j \leq 3} \frac{1}{\sin^2 \frac{1}{2}(\theta_i - \theta_j)} \right. \right. \\ & + \left. \sum_{1 \leq i < j \leq 3} \frac{1}{\sin^2 \frac{1}{2}(\theta'_i - \theta'_j)} \left. R_{pqr}R'_{hkl} \right\rangle_{(2\pi)^6} = \\ & 3 \left\langle R_{stu} \left| \frac{1}{\sin^2 \frac{1}{2}(\theta_1 - \theta_2)} \right. R_{pqr} \right\rangle_{(2\pi)^3} \langle R'_{mno} | R'_{hkl} \rangle_{(2\pi)^3} \\ & + \langle R_{stu} | R_{pqr} \rangle \cdot 3 \left\langle R'_{mno} \left| \frac{1}{\sin^2 \frac{1}{2}(\theta'_i - \theta'_j)} \right. R'_{hkl} \right\rangle_{(2\pi)^3}. \end{aligned} \quad (65)$$

Here we exploited the arbitrary labelling of the eigenangles to make do with 3 times the terms for $\theta_1 - \theta_2$ and $\theta'_1 - \theta'_2$, respectively. However, to do the integrals over these factors we change variables to $\frac{\theta_1 - \theta_2}{2}$ and $\frac{\theta'_1 - \theta'_2}{2}$ in order to manage the seemingly singular expression for $\theta_1 = \theta_2$. It namely turns out that the Slater determinants can be separated in terms that suit these new variables in such a way that the integrals can be calculated analytically to give finite results.

To do the integrals in (65) let

$$b_{pqr} \equiv e^{i(p\theta_1 + q\theta_2 + r\theta_3)}, \quad B_{pqr} = b_{pqr} - b_{qpr}. \quad (66)$$

Then we find

$$\begin{aligned} B_{pqr} &= e^{ir\theta_3} (e^{i(p\theta_1 + q\theta_2)} - e^{i(q\theta_1 + p\theta_2)}) \\ &= e^{ir\theta_3} e^{i(p+q)\frac{\theta_1 + \theta_2}{2}} 2i \sin \left((p-q) \frac{\theta_1 - \theta_2}{2} \right). \end{aligned} \quad (67)$$

Applying this we have

$$\begin{aligned} & \left\langle B_{stu} \left| \frac{1}{\sin^2 \frac{1}{2}(\theta_1 - \theta_2)} \right. B_{pqr} \right\rangle_{(2\pi)^3} = \\ & (2\pi)^3 \delta_{ru} \delta_{p+q, s+t} \cdot 4 \cdot \pi nn(p-q, s-t), \end{aligned} \quad (68)$$

$p < q < r \wedge s < t < u,$

where the factor 4 comes from the shift of variables and nn is a shorthand notation for

$$nn(i, j) = \begin{cases} |i+j| - |i-j|, & \text{for } i+j \equiv 0 \pmod{2}, \\ 0, & \text{otherwise,} \end{cases} \quad (69)$$

from the sine integrals (cf., e.g., pp. 172–180 in [16])

$$\int_{-\pi}^{\pi} \frac{\sin ix \sin jx}{\sin^2 x} = (|i+j| - |i-j|)\pi. \quad (70)$$

Now

$$R_{pqr} = B_{pqr} + B_{qrp} + B_{rpq} \quad (71)$$

and with shorthand notation

$$C_{12} \equiv \frac{1}{\sin^2 \frac{1}{2}(\theta_1 - \theta_2)}, \quad (72)$$

our matrix element will contain nine terms

$$\begin{aligned} & \langle B_{stu} + B_{tus} + B_{ust} | C_{12} | B_{pqr} + B_{qrp} + B_{rpq} \rangle_{(2\pi)^3} = \\ & (2\pi)^3 \cdot 4\pi [\delta_{p+q, s+t} nn(p-q, s-t) \delta_{ru} \\ & + \delta_{q+r, s+t} nn(q-r, s-t) \delta_{pu} + \delta_{r+p, s+t} nn(r-p, s-t) \delta_{qu} \\ & + \delta_{p+q, t+u} nn((p-q, t-u) \delta_{rs} + \delta_{q+r, t+u} nn(q-r, t-u) \delta_{ps} \\ & + \delta_{r+p, t+u} nn(r-p, t-u) \delta_{qs} + \delta_{p+q, u+s} nn(p-q, u-s) \delta_{rt} \\ & + \delta_{q+r, u+s} nn(q-r, u-s) \delta_{pt} + \delta_{r+q, u+s} nn(r-q, u-s) \delta_{qt}]. \end{aligned} \quad (73)$$

For the *interaction integrals* we apply a similar technique, now pairing θ_j and θ'_j for variable shifts to $(\theta_j - \theta'_j)/2$ and $(\theta_j + \theta'_j)/2$ and use the shorthand notation

$$Iw(p, s) = \begin{cases} \frac{\pi^3}{3}, & \text{for } p = s, \\ 32\pi \frac{\cos((p-s)\frac{\pi}{2})}{(p-s)^2}, & \text{for } p \neq s, \end{cases} \quad (74)$$

for the integral

$$Iw(p, s) = \int_{-\pi}^{\pi} 2dz e^{-isz} 2w(2z) e^{ipz}, \quad z = (\theta_j - \theta'_j)/2. \quad (75)$$

We then have the full *interaction term*

$$\begin{aligned} & \left\langle R_{stu}R'_{mno} \left| \sum_{j=1}^3 2w(\theta_j - \theta'_j) \right. R_{pqr}R'_{hkl} \right\rangle_{(2\pi)^6} = \\ & 3 \cdot \frac{1}{2^4} (2\pi)^5 \cdot 6 [Iw(p-s, h-m) \delta_{p+s, h+m} \delta_{qt} \delta_{ru} \delta_{kn} \delta_{lo} \\ & + Iw(p-s, l-m) \delta_{p+s, l+m} \delta_{qt} \delta_{ru} \delta_{hn} \delta_{ko} \\ & - Iw(p-s, k-m) \delta_{p+s, k+m} \delta_{qt} \delta_{ru} \delta_{hn} \delta_{lo} \\ & + Iw(r-s, h-m) \delta_{r+s, h+m} \delta_{pt} \delta_{qu} \delta_{kn} \delta_{lo} \\ & + Iw(r-s, l-m) \delta_{r+s, l+m} \delta_{pt} \delta_{qu} \delta_{hn} \delta_{ko} \\ & - Iw(r-s, k-m) \delta_{r+s, k+m} \delta_{pt} \delta_{qu} \delta_{hn} \delta_{lo} \\ & - Iw(q-s, h-m) \delta_{q+s, h+m} \delta_{pt} \delta_{ru} \delta_{kn} \delta_{lo} \\ & - Iw(q-s, l-m) \delta_{q+s, l+m} \delta_{pt} \delta_{ru} \delta_{hn} \delta_{ko} \\ & + Iw(q-s, k-m) \delta_{q+s, k+m} \delta_{pt} \delta_{ru} \delta_{hn} \delta_{lo}]. \end{aligned} \quad (76)$$

In both (73) and (76) it is understood that the index triples fulfil

$$p < q < r, \quad h < k < l, \quad s < t < u, \quad m < n < o \in \mathbb{Z}. \quad (77)$$

This greatly reduces the number of terms in both expressions but leaves open a completeness issue. Results are shown in table 1. All results follow from a single energy scale, $\Lambda = \hbar c/a$ with the length scale a given by $\pi a = r_e$ as for the baryon spectrum (1). No quark mass parameters are involved and no fitting has been undertaken. The π -mesons are treated separately elsewhere as revived Goldstone bosons from a slightly misaligned Higgs mechanism in [26].

Conclusion. – We have examined a model for mesons as interaction states between baryons in $U(3)$ configurations. The model eigenstates have masses comparable to the observed unflavoured mesons above the pions. However, direct identification remains preliminary. In particular both η and η' lie low —perhaps because of missing strange flavour contributions and underestimation of the off-toroidal contribution. On the other hand, f_0 and $f_0(980)$ are high and low, respectively, which may be seen as missing parity splitting. Spin and isospin coupling should be specified by expansion on D -functions for off-toroidal degrees of freedom and a way to average over these for the interaction potential be sought.

* * *

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