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Explicit construction of frames and pairs of dual frames on locally compact abelian groups

Ole Christensen and Say Song Goh

Abstract

Locally compact abelian (LCA) groups form a natural setting for frame theory in signal processing: while classical frame theory was developed in the setting of $L^2(\mathbb{R})$, concrete applications have to be implemented on $\mathbb{R}^d$, two scenarios that are covered under the setup of LCA groups. Due to rapid progress over the past few years, it is known within the mathematical community that large parts of the standard frame theory for $L^2(\mathbb{R})$ can be carried over to the setting of LCA groups. We show that this also holds on the level of explicit constructions, which is a crucial issue for applications within signal processing; in fact we provide explicit constructions of frames and pairs of dual frames on LCA groups. Special attention will be given to the so-called elementary LCA groups, i.e., groups that are tensor products of $\mathbb{R}$, $\mathbb{Z}$, the torus group $\mathbb{T}$, and the finite group $\mathbb{Z}_N$. The frames will have the form of a Fourier-like system, i.e., the Fourier transform of a generalized shift-invariant system. The constructions are based on a partition of unity; in particular we provide a general and flexible approach for constructing a partition of unity that makes it easy to control key features of the underlying functions, e.g., support size and continuity. The results apply to arbitrary LCA groups with a supply of "sufficiently fine" lattices, a condition that is satisfied for all elementary LCA groups. Concrete constructions of pairs of dual frames for several elementary LCA groups are provided as illustration. In the case of a system of functions generated by a single lattice, such as a Gabor system, the general results become particularly transparent.

Keywords. Partitions of unity, frames, dual frames, explicit constructions, LCA groups
Mathematics Subject Classification. 42C15, 42C40, 43A70

1 Introduction

During the last few years, frame theory on locally compact abelian (LCA) groups has developed rapidly. By now, it is well-known that the main body of the theory for generalized shift-invariant (GSI) systems and in particular Gabor systems can be formulated in this framework, and often the formulation of the results carries over word by word from the setting of the real line $\mathbb{R}$ to LCA groups.

The purpose of the current paper is to provide constructions of frames and pairs of dual frames in the setting of LCA groups. Throughout the paper we will refer to these constructions as "explicit": indeed, they generalize completely explicit constructions in the setting of elementary LCA groups and they are as explicit "as it gets" in the general setting. They emphasize a balanced interplay between general constructive approaches in abstract context and concrete constructions in specific situations of practical interest. The frames constructed will have the structure of a Fourier-like system, i.e., the Fourier transform of a GSI system. GSI systems were introduced in $L^2(\mathbb{R}^d)$ in the papers [9] and [16], and generalized to the setting of LCA groups in [14]; the name Fourier-like system was introduced in [5]. Technically the results will be built upon a general method for obtaining partitions of unity on LCA groups based on certain tilings of the group. This general way of obtaining the partition of unity together with its subsequent extension encompass various cases of interest, including the well-studied special case on $\mathbb{R}$ of systems formed by integer-translates of a fixed window function. Several key properties of the functions generating the partition of unity, like support size and continuity, can easily be controlled via the construction. Whereas tilings are often used to construct bases (e.g., in the wavelet context), we will use the resulting partitions of unity to provide general constructions of frames and dual pairs of frames for $L^2(\widehat{G})$, where $\widehat{G}$ is the dual group of an LCA group G. The condition for applying these results to concrete groups basically consists in the requirement that the group G has "sufficiently fine" lattices, a condition that is automatically satisfied for all elementary LCA groups. We will provide examples to show how a number of results, including some known ones on $\mathbb{R}$, can be derived from our general setup.

The paper is organized as follows. In the rest of the introduction we set the stage by formulating certain key definitions and results from the literature. In particular, we refer to the so-called elementary LCA groups that appear often in applications within signal processing; the general results proved in the paper always apply to such groups. In Section 2 we discuss a general way of obtaining a partition of unity based on a tiling of the underlying group; in particular, the results apply to groups having lattices. Then we demonstrate how this approach can be further extended and enhanced, thereby capturing different types of partitions of unity. Section 3 provides the general results for obtaining frames and pairs of dual frames with the structure of a Fourier-like system (which corresponds to a GSI system via the inverse Fourier transform), given any partition of unity. The results are applied to several elementary LCA groups, giving concrete constructions of pairs of dual frames. In Section 4 we specialize to Fourier-like frames generated by a single lattice, which cover the important case of Gabor systems, and also present a construction based on tilings.

As standard references to harmonic analysis on LCA groups we refer to [17] by Rudin and [10] by Hewitt and Ross. Within the particular area of frame expansions we mention the papers [14] by Kutyniok and Labate, [3] by Cabrelli and Paternostro, [2] by Bownik and Ross, [11] by Jakobsen and Lemvig, and [5, 7] by the authors.

Let G denote an LCA group, with the group composition denoted by the symbol " $+$ "

and neutral element 0. Assume that G is equipped with a Hausdorff topology, and that G is a countable union of compact sets and metrizable. A *character* on G is a function $\gamma : G \to \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$, for which $\gamma(x+y) = \gamma(x)\gamma(y),\ \forall x, y \in G$. The set of continuous characters is denoted by $\widehat{G}$, and also forms an LCA group, called the *dual group* of G, when equipped with an appropriate topology and the composition $(\gamma+\gamma')(x) := \gamma(x)\gamma'(x)$, $\gamma, \gamma' \in \widehat{G}, x \in G$. The assumptions on G imply that $\widehat{G}$ is also a countable union of compact sets and metrizable. One can prove that $\widehat{\widehat{G}} = G$. We will therefore from now on use the more symmetric notation $(x, \gamma) := \gamma(x),\ x \in G,\ \gamma \in \widehat{G}$.

A (uniform) *lattice* in the LCA group G is a discrete subgroup Λ for which G/Λ is compact; the *annihilator* $\Lambda^\perp$ of Λ is defined by $\Lambda^\perp := \{\gamma \in \widehat{G} \mid (x,\gamma) = 1,\ \forall x \in \Lambda\}$. The annihilator $\Lambda^\perp$ is a closed subgroup of $\widehat{G}$. Lattices are known explicitly in most of the classical LCA groups; however, there also exist LCA groups without lattices, see, e.g., [12] and [13].

Given any LCA group G with a lattice Λ, there exists a Borel measurable relatively compact set $Q \subset G$ such that

$$G = \bigcup_{\lambda\in\Lambda} (\lambda + Q), \quad (\lambda+Q) \cap (\lambda'+Q) = \emptyset \text{ for } \lambda \neq \lambda',\ \lambda, \lambda' \in \Lambda. \tag{1.1}$$

The set Q is called a *fundamental domain* associated with the lattice Λ. The set $\Lambda^\perp$ is a lattice in $\widehat{G}$, and thus there exists a Borel measurable relatively compact set $V \subset \widehat{G}$ such that

$$\widehat{G} = \bigcup_{\omega\in\Lambda^\perp} (\omega + V), \quad (\omega+V) \cap (\omega'+V) = \emptyset \text{ for } \omega \neq \omega',\ \omega, \omega' \in \Lambda^\perp,$$

i.e., V is a fundamental domain associated with $\Lambda^\perp$.

Fixing a Haar measure μ_G on G, we can define the spaces $L^1(G)$ and $L^2(G)$ in the usual way. The space $L^2(G)$ is a Hilbert space; furthermore, our assumption of G being a countable union of compact sets and metrizable implies (and is, in fact, equivalent to) $L^2(G)$ being separable.

The *Fourier transform* is defined as the operator

$$\mathcal{F} : L^1(G) \to C_0(\widehat{G}),\ \mathcal{F}f(\gamma) := \int_G f(x)(-x,\gamma)\, d\mu_G(x).$$

We will use the usual notation $\widehat{f} := \mathcal{F}f$. The *inversion theorem* states that with appropriate normalization of the Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$, for $f \in L^1(G)$ such that $\widehat{f} \in L^1(\widehat{G})$, it holds that

$$f(x) = \int_{\widehat{G}} \widehat{f}(\gamma)\,(x,\gamma) d\mu_{\widehat{G}}(\gamma),\ x \in G. \tag{1.2}$$

We will always choose the Haar measure on $\widehat{G}$ such that the inversion formula (1.2) holds for the pair G and $\widehat{G}$. The Fourier transform can be extended to a surjective isometry

$\mathcal{F}: L^2(G) \to L^2(\widehat{G})$, and the normalization of the Haar measure that makes the inversion formula work leads to the Plancherel theorem (see [17]),

$$\int_G f(x)\overline{g(x)}\, d\mu_G(x) = \int_{\widehat{G}} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)}\, d\mu_{\widehat{G}}(\gamma),\ f, g \in L^2(G).$$

Given any $\lambda \in G$, consider the *generalized modulation operator*

$$\mathcal{M}_\lambda : L^2(\widehat{G}) \to L^2(\widehat{G}),\ (\mathcal{M}_\lambda f)(\gamma) := (\lambda, \gamma)\, f(\gamma).$$

It is easy to see that $\mathcal{M}_\lambda$ is a unitary operator. We will consider actions by a class of operators $\mathcal{M}_\lambda$ on a countable collection of functions $\{\Phi_k\}_{k\in I}$ in $L^2(\widehat{G})$, with λ belonging to lattices Λ_k that depend on $k \in I$. The resulting class of functions in $L^2(\widehat{G})$ is then of the form $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in I}$, which we refer to as a *Fourier-like system*. This system is the Fourier transform of the GSI system $\{\mathcal{F}^{-1}\Phi_k(\cdot - \lambda)\}_{\lambda\in\Lambda_k,k\in I}$.

Technically our explicit constructions are based on results from [5] which we will introduce now. Let I denote a countable index set, and let $\{\Phi_k\}_{k\in I}$ and $\{\widetilde{\Phi_k}\}_{k\in I}$ be two collections of functions in $L^2(\widehat{G})$. The following is Corollary 3.5 in [5], where for each $k \in I$, the set $V_k \subset \widehat{G}$ denotes a fundamental domain associated with the lattice $\Lambda_k^\perp$.

Lemma 1.1 *In addition to the above setup, assume that for each $k \in I$, the function Φ_k satisfies that*

$$\text{supp}\, \Phi_k \cap \text{supp}\, \Phi_k(\cdot + \omega) = \emptyset,\ \forall\, \omega \in \Lambda_k^\perp \setminus \{0\} \tag{1.3}$$

(*up to a set of measure zero in $\widehat{G}$*). *Then the following hold:*

(i) $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ *is a Bessel sequence in $L^2(\widehat{G})$ if and only if*

$$B := \operatorname*{ess\,sup}_{\gamma\in\widehat{G}} \sum_{k\in I} \mu_{\widehat{G}}(V_k)|\Phi_k(\gamma)|^2 < \infty.$$

(ii) *If* (i) *holds, then $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ is a frame for $L^2(\widehat{G})$ if and only if*

$$A := \operatorname*{ess\,inf}_{\gamma\in\widehat{G}} \sum_{k\in I} \mu_{\widehat{G}}(V_k)|\Phi_k(\gamma)|^2 > 0.$$

Another result that we need is Theorem 3.11 in [5], which we state below.

Theorem 1.2 *In addition to the above setup, assume that for each $k \in I$,*

$$\text{supp}\, \Phi_k \cap \text{supp}\, \widetilde{\Phi_k}(\cdot + \omega) = \emptyset,\ \forall\, \omega \in \Lambda_k^\perp \setminus \{0\} \tag{1.4}$$

(*up to a set of measure zero in $\widehat{G}$*). *If $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in I}$ are Bessel sequences in $L^2(\widehat{G})$, they are dual frames for $L^2(\widehat{G})$ if and only if*

$$\sum_{k\in I} \mu_{\widehat{G}}(V_k)\, \overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\gamma) = 1,\ a.e.\, \gamma \in \widehat{G}. \tag{1.5}$$

We will provide a number of explicit constructions for the LCA groups $\mathbb{R}$, the torus $\mathbb{T}$, and the finite groups $\mathbb{Z}_N := \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$; here, we use the notation $\mathbb{Z}_N$ to denote the additive group of integers modulo N as well as the set of consecutive integers $\{0, 1, \ldots, N-1\}$. More generally, the results apply to any *elementary LCA group* [15], i.e., a group of the form $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2} \times \mathbb{T}^{d_3} \times F$, where $d_1, d_2, d_3 \in \mathbb{N}_0$ and F is a finite abelian group. It is well-known that any finite abelian group is isomorphic to a direct product of cyclic groups of prime-power order, and any cyclic group of order N is isomorphic to $\mathbb{Z}_N$; for this reason (and because all the other groups indeed deal with scalars), we will restrict our attention to LCA groups of the form $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2} \times \mathbb{T}^{d_3} \times \mathbb{Z}_N$, where $d_1, d_2, d_3 \in \mathbb{N}_0$ and $N \in \mathbb{N}$.

2 Partition of unity on LCA groups

In this section we will describe various ways of obtaining explicitly given partitions of unity on an LCA group. All the constructions will be based on the choice of a certain integrable function on the group, and several properties of the resulting partition of unity can be controlled by the properties of that function.

Recall that a countable set $\{x_k\}_{k\in I} \subset G$ is *separated* if there exists a relatively compact neighborhood U of 0 such that the sets $x_k + U$, $k \in I$, are disjoint. The set $\{x_k\}_{k\in I}$ is *relatively separated* if it is a finite union of separated sets. In [8] it was proved that $\{x_k\}_{k\in I}$ is relatively separated if and only if, for any relatively compact set $W \subset G$ it holds that

$$\sup_{k\in I} \sharp\{\ell \in I \big| \, (x_k + W) \cap (x_\ell + W) \neq \emptyset\} < \infty, \tag{2.1}$$

where for any countable set J, the notation $\sharp J$ denotes its cardinality.

We will say that a countable collection of measurable subsets $\{Q_k\}_{k\in I}$ of G forms a *tiling* of G if

$$G = \bigcup_{k\in I} Q_k, \quad Q_k \cap Q_{k'} = \emptyset \text{ for } k \neq k', \; k, k' \in I.$$

Note that the term "tiling" often is associated with sets formed by translation of a fixed set; it will not give rise to confusion that we use the same word in the more general context described here. Any tiling of G leads to a large class of partitions of unity, with properties that can be controlled via the choice of a certain function $u \in L^1(G)$:

Proposition 2.1 *Consider a countable tiling $\{Q_k\}_{k\in I}$ of G consisting of measurable sets Q_k. Let $u \in L^1(G)$ be chosen such that $\int_G u(t)\, d\mu_G(t) = 1$, and define the functions $g_k : G \to \mathbb{C}$, $k \in I$, by*

$$g_k(x) := \int_{x-Q_k} u(t)\, d\mu_G(t), \; x \in G. \tag{2.2}$$

Then the following hold:

(i) *The partition of unity condition*

$$\sum_{k\in I} g_k(x) = 1,\ x \in G,$$

is satisfied, where the infinite series converges unconditionally.

(ii) *If u is nonnegative, then g_k is nonnegative for all $k \in I$.*

(iii) *The functions g_k, $k \in I$, are continuous and* $\operatorname{supp} g_k \subseteq \overline{\operatorname{supp} u + Q_k}$ *for all $k \in I$.*

(iv) *If there exist a compact set $Q \subset G$ and a sequence $\{x_k\}_{k\in I} \subset G$ such that*

$$Q_k \subseteq x_k + Q,\ \forall k \in I, \tag{2.3}$$

then $\operatorname{supp} g_k \subseteq x_k + \overline{\operatorname{supp} u + Q}$ *for all $k \in I$.*

Proof. The result in (i) is a direct consequence of the property that $\{Q_k\}_{k\in I}$ tiles G, combined with the Lebesgue dominated convergence theorem. The property (ii) is clear, (iii) follows by direct calculation, and (iv) is a consequence of (iii).

Corollary 2.2 *Consider a countable tiling $\{Q_k\}_{k\in I}$ of G consisting of measurable sets Q_k, and assume that there exist a compact set $Q \subset G$ and a sequence $\{x_k\}_{k\in I} \subset G$ such that* (2.3) *holds. Then, given any open set U containing the set Q, the function u in Proposition* 2.1 *can be chosen such that* $\operatorname{supp} g_k \subseteq x_k + U$ *for all $k \in I$, where g_k is defined as in* (2.2).

Proof. Note that by Theorem 2.7 in [18] there exists an open set $U_1 \subset G$ such that $\overline{U_1}$ is compact and

$$Q \subset U_1 \subset \overline{U_1} \subset U. \tag{2.4}$$

The function $f : G \times G \to G$ given by $f(x, y) := x + y$ is continuous. For any $y \in Q$, we have $f(0, y) = y \in U_1$. Since U_1 is open, there exist a neighborhood V_y about $0 \in G$ and a neighborhood W_y about y such that $f(V_y \times W_y) \subset U_1$. The sets W_y, $y \in Q$, form an open cover of the compact set Q, so we can choose $y_1, \ldots, y_L \in Q$ such that $Q \subseteq \bigcup_{\ell=1}^{L} W_{y_\ell}$. Now, let $V := \bigcap_{\ell=1}^{L} V_{y_\ell}$. Fix any $x_0 \in V$ and choose by Urysohn's lemma (Lemma 2.12 in [18]) the function $u \in C_c(G)$ such that $u(x_0) = 1$, $0 \le u(x) \le 1$ for all $x \in G$, and $\operatorname{supp} u \subset V$. Then $\int_G u(t)\, d\mu_G(t)$ is strictly positive and finite, so by an appropriate normalization we obtain that $\int_G u(t)\, d\mu_G(t) = 1$.

We will now prove that with this choice of the function u, the functions g_k, $k \in I$, satisfy the required condition; by Proposition 2.1 (iv) it is enough to show that $\overline{\operatorname{supp} u + Q} \subset U$. To prove this, by (2.4), it is enough to show that $\operatorname{supp} u + Q \subset U_1$. Now, for $x \in \operatorname{supp} u$ and $y \in Q$, take $\ell_0 \in \{1, 2, \ldots, L\}$ such that $y \in W_{y_{\ell_0}}$. Since $x \in \operatorname{supp} u \subset V = \bigcap_{\ell=1}^{L} V_{y_\ell}$, we have $(x, y) \in V_{y_{\ell_0}} \times W_{y_{\ell_0}}$, and hence $x + y = f(x, y) \in U_1$, as desired. □

Note that if Λ is a lattice in G with an associated fundamental domain $Q \subset G$, then (1.1) shows that the sets $Q_\lambda := \lambda + Q$, $\lambda \in \Lambda$, form a tiling of G. In this case the collection of functions constituting the partition of unity consists of translates of a fixed function along the lattice Λ:

Corollary 2.3 *Suppose that Λ is a lattice in G with fundamental domain Q. Assume that $u \in L^1(G)$ and that $\int_G u(t)\,d\mu_G(t) = 1$. Then the following hold:*

(i) *The function*

$$g(x) := \int_{x-Q} u(t)\,d\mu_G(t),\ x \in G, \tag{2.5}$$

is continuous and satisfies the partition of unity condition

$$\sum_{\lambda \in \Lambda} g(x-\lambda) = 1,\ x \in G.$$

(ii) *If u is nonnegative, then g is nonnegative.*

(iii) *If $\operatorname{supp} u$ is compact, then g has compact support contained in $\overline{\operatorname{supp} u + Q}$.*

Proof. The result is a direct consequence of Proposition 2.1 and the fact that for $\lambda \in \Lambda$,

$$g_\lambda(x) = \int_{x-\lambda-Q} u(t)\,d\mu_G(t) = g(x-\lambda),\ x \in G.$$

□

Note that choosing the lattice Λ to be very fine, the support of the function g in (2.5) generating the partition of unity will be close to the support of the chosen function u. For example, if $G = \mathbb{R}^d$ and $\Lambda = \frac{1}{N}\mathbb{Z}^d$ for some $N \in \mathbb{N}$, we can take $Q = [-\frac{1}{2N}, \frac{1}{2N})^d$, and then $\operatorname{supp} g \subseteq \overline{\operatorname{supp} u + [-\frac{1}{2N}, \frac{1}{2N})^d}$.

Returning to Proposition 2.1, we note that the flexible method provided by it to generate partitions of unity can be further generalized and enhanced. The starting point is again a tiling $\{Q_k\}_{k\in I}$ of G.

Theorem 2.4 *Consider a countable tiling $\{Q_k\}_{k\in I}$ of G consisting of measurable sets Q_k. For $n \in \mathbb{N}$, let $u_1, \ldots, u_n \in L^1(G)$ be chosen such that $\int_G u_s(t)\,d\mu_G(t) = 1$ for every $s = 1, 2, \ldots, n$. Take any n measurable functions $K_s : G \times G \to G$, $s = 1, 2, \ldots, n$. For $s = 0, 1, \ldots, n$, define the functions $g_k^{(s)} : G \to \mathbb{C}$, $k \in I$, by setting*

$$g_k^{(0)}(x) := \chi_{Q_k}(x),\ x \in G, \tag{2.6}$$

and for $s = 1, 2, \ldots, n$,

$$g_k^{(s)}(x) := \int_G g_k^{(s-1)}(K_s(x,t))u_s(t)\,d\mu_G(t),\ x \in G. \tag{2.7}$$

Then the following hold for every $s = 0, 1, \ldots, n$:

(i) *There exists a constant* $C_s > 0$ *such that*

$$\sum_{k\in I} |g_k^{(s)}(x)| \le C_s,\ x \in G. \tag{2.8}$$

(ii) *The partition of unity condition*

$$\sum_{k\in I} g_k^{(s)}(x) = 1,\ x \in G, \tag{2.9}$$

is satisfied, where the infinite series converges unconditionally.

(iii) *For* $s \in \{1, 2, \ldots, n\}$, *if* u_s *is nonnegative, then* $g_k^{(s)}$ *is nonnegative for all* $k \in I$.

Proof. We shall establish the result by induction on $s \in \{0, 1, \ldots, n\}$. For $s = 0$, (2.8) and (2.9) are clearly satisfied by the functions $g_k^{(0)}$ defined in (2.6). Next, for $s \in \{1, 2, \ldots, n\}$, it follows from (2.7) that for $x \in G$,

$$\sum_{k\in I} |g_k^{(s)}(x)| \le \sum_{k\in I} \int_G |g_k^{(s-1)}(K_s(x,t))|\, |u_s(t)|\, d\mu_G(t).$$

Since the induction hypothesis on (2.8) implies that

$$\int_G \sum_{k\in I} |g_k^{(s-1)}(K_s(x,t))|\, |u_s(t)|\, d\mu_G(t) \le C_{s-1} \|u_s\|_{L^1(G)} < \infty$$

for some constant $C_{s-1} > 0$, we conclude from the Lebesgue dominated convergence theorem that (i) holds. In addition, applying the induction hypothesis on (2.9) to (2.7) gives

$$\sum_{k\in I} g_k^{(s)}(x) = \int_G \sum_{k\in I} g_k^{(s-1)}(K_s(x,t)) u_s(t)\, d\mu_G(t) = \int_G u_s(t)\, d\mu_G(t) = 1,\ x \in G,$$

which establishes (ii). As for (iii), it also follows by induction from the definition (2.7). □

Compared with Proposition 2.1, being able to choose desired functions $K_s : G \times G \to G$ in Theorem 2.4 provides additional flexibility in constructing different types of partitions of unity. The examples below demonstrate how the general setup of Theorem 2.4 unifies some of the existing constructions.

Example 2.5 Let $n \in \mathbb{N}$ and consider the functions $K_s : G \times G \to G$, $s = 1, 2, \ldots, n$, defined by $K_s(x, t) := x - t$ for $x, t \in G$. Based on a tiling $\{Q_k\}_{k\in I}$ of G and functions $u_s : G \to G$, $s = 1, 2, \ldots, n$, satisfying the conditions of Theorem 2.4, we construct functions $g_k^{(s)} : G \to G$ as in (2.6) and (2.7). For $s = 1$, this gives

$$g_k^{(1)}(x) = \int_G \chi_{Q_k}(x - t) u_1(t)\, d\mu_G(t) = \int_{x-Q_k} u_1(t)\, d\mu_G(t),\ x \in G,$$

which is of the form (2.2) in Proposition 2.1. Furthermore, for $s = 1, 2, \ldots, n$, we obtain the convolution formula

$$g_k^{(s)}(x) = \int_G g_k^{(s-1)}(x-t) u_s(t)\, d\mu_G(t) = (g_k^{(s-1)} * u_s)(x),\ x \in G.$$

Using (2.6), this implies that for $k \in I$,

$$g_k^{(n)} = \chi_{Q_k} * u_1 * \cdots * u_n. \tag{2.10}$$

Now, let us specialize the tiling $\{Q_k\}_{k\in I}$ of G to one that is of the form $\{\lambda + Q\}_{\lambda\in\Lambda}$, where Λ is a lattice in G and Q is a fundamental domain associated with Λ. In the formula (2.10), we take $u_s := w_s \chi_Q$, $s = 1, 2, \ldots, n$, where w_s are functions in $L^2(Q)$ satisfying $\int_Q w_s(t)\, d\mu_G(t) = 1$. Thus when $\lambda = 0$, we arrive at the function $g_0^{(n)} = \chi_Q * w_1\chi_Q * \cdots * w_n\chi_Q$. This is the weighted B-spline of order $n+1$ considered in Lemma 3.7 of [5], which unifies the classical uniform B-splines on $\mathbb{R}$ and the exponential B-splines. □

Example 2.6 Take $G = \mathbb{R}^d$ and for $n \in \mathbb{N}$, define the functions $K_s : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $s = 1, 2, \ldots, n$, by

$$K_s(x,t) := \begin{cases} \frac{x}{\|t\|}, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0, \end{cases}$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^d$. Then the formula (2.7) in Theorem 2.4 becomes

$$g_k^{(s)}(x) = \int_{\mathbb{R}^d} g_k^{(s-1)}\Big(\frac{x}{\|t\|}\Big) u_s(t)\, dt,\ x \in \mathbb{R}^d,$$

and this includes the setup in [6] for constructing scaling partitions of unity.

For the one-dimensional case of $G = \mathbb{R}$, as an alternative to the above approach, we may apply Proposition 2.1 to the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ to obtain scaling partitions of unity. This will be elaborated in Example 3.9 later. □

In this paper, our explicit constructions of frames and dual frames on LCA groups will be based on functions that satisfy the partition of unity condition. Since Proposition 2.1 already gives a rich and flexible setting for generating partitions of unity, henceforth we will focus on using it, instead of Theorem 2.4, for frame construction. The further generality provided by Theorem 2.4, especially on the choice of the functions $K_s : G \times G \to G$, could form the basis of a future study.

3 Fourier-like frames on LCA groups

The purpose of this section is to provide a number of general results for constructing frames and dual pairs of frames on LCA groups. In order to formulate the results as general as possible we will use the terminology outlined in the following definition.

Definition 3.1 *Let G denote an LCA group.*

(i) *A countable sequence $\{\Omega_k\}_{k\in I}$ of measurable and relatively compact sets in $\widehat{G}$ is finitely overlapping if*

$$\sup_{k\in I} \sharp\{\ell \in I \,\big|\, \mu_{\widehat{G}}(\Omega_k \cap \Omega_\ell) > 0\} < \infty. \tag{3.1}$$

(ii) *Given two measurable sets Ω, Ω' in $\widehat{G}$, a lattice Λ in G is said to be sufficiently fine with respect to Ω and Ω' if*

$$\Omega \bigcap (\omega + \Omega') = \emptyset, \ \forall \omega \in \Lambda^\perp \setminus \{0\} \tag{3.2}$$

(*up to a set of measure zero in $\widehat{G}$*).

(iii) *If (3.2) holds with $\Omega' = \Omega$, the lattice Λ is simply said to be sufficiently fine with respect to Ω.*

(iv) *Given two countable sequences of measurable sets $\{\Omega_k\}_{k\in I}, \{\Omega'_k\}_{k\in I}$ in $\widehat{G}$, a sequence of lattices $\{\Lambda_k\}_{k\in I}$ in G is said to be sufficiently fine with respect to $\{\Omega_k\}_{k\in I}$ and $\{\Omega'_k\}_{k\in I}$ if, for all $k \in I$,*

$$\Omega_k \bigcap (\omega + \Omega'_k) = \emptyset, \ \forall \omega \in \Lambda_k^\perp \setminus \{0\} \tag{3.3}$$

(*up to a set of measure zero in $\widehat{G}$*).

(v) *If (3.3) holds with $\Omega'_k = \Omega_k$, the sequence of lattices $\{\Lambda_k\}_{k\in I}$ is simply said to be sufficiently fine with respect to the sets $\{\Omega_k\}_{k\in I}$.*

Note that a lattice Λ is sufficiently fine with respect to a measurable set Ω if we can find a fundamental domain V for the annihilator $\Lambda^\perp$ such that $\Omega \subseteq V$; this condition resembles the one that was used in the construction of spline-based frames in [5]. The concept of finitely overlapping sets is related to the idea of admissible coverings used by Feichtinger and Gröbner in [8]; however, since the technical conditions are not precisely the same we will use a different terminology here. It is also related to the covering index defined in [1] by Aldroubi, Cabrelli and Molter; indeed, $\{\Omega_k\}_{k\in I}$ being finitely overlapping means that its covering index is finite. In our definition of finitely overlapping sets in (3.1), by removing a countable collection of sets of measure zero, we can obtain sets $\{\Omega_k\}_{k\in I}$ that satisfy the simpler condition

$$\sup_{k\in I} \sharp\{\ell \in I \,\big|\, \Omega_k \cap \Omega_\ell \neq \emptyset\} < \infty.$$

However, we do not use this simpler definition of finitely overlapping sets as it does not cover fully some of the subsequent results and concrete examples on Fourier-like frames, such as that in Example 3.9.

We first notice that lattices as in Definition 3.1 always exist in the case of elementary LCA groups.

Lemma 3.2 *Let G denote an elementary LCA group. Then, for arbitrary relatively compact measurable sets Ω, Ω' in $\widehat{G}$, there exists a lattice Λ in G which is sufficiently fine with respect to Ω and Ω'.*

Proof. For convenience we identify the torus $\mathbb{T}$ with the interval $[0,1)$. If $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2} \times \mathbb{T}^{d_3} \times \mathbb{Z}_N$, where $d_1, d_2, d_3 \in \mathbb{N}_0$ and $N \in \mathbb{N}$, then the dual group is $\widehat{G} = \mathbb{R}^{d_1} \times \mathbb{T}^{d_2} \times \mathbb{Z}^{d_3} \times \mathbb{Z}_N$. Consider the following collection of lattices Λ_L in G, indexed by the parameter $L \in 2\mathbb{N}$:

$$\Lambda_L = \frac{1}{L}\mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} \times \frac{1}{L}\mathbb{Z}_L^{d_3} \times \mathbb{Z}_N. \tag{3.4}$$

The lattice Λ_L has the fundamental domain $[-\frac{1}{2L}, \frac{1}{2L})^{d_1} \times \{0\}^{d_2} \times [0, \frac{1}{L})^{d_3} \times \{0\}$. The annihilator is $\Lambda_L^{\perp} = L\mathbb{Z}^{d_1} \times \{0\}^{d_2} \times L\mathbb{Z}^{d_3} \times \{0\}$, with the fundamental domain

$$V_L = \Big[-\frac{L}{2}, \frac{L}{2}\Big)^{d_1} \times \mathbb{T}^{d_2} \times \Big\{-\frac{L}{2}, -\frac{L}{2}+1, \dots, \frac{L}{2}-1\Big\}^{d_3} \times \mathbb{Z}_N. \tag{3.5}$$

The set $\Omega \cup \Omega'$ is relatively compact in $\widehat{G}$, and hence contained in the sets V_L of the form (3.5) whenever $L \in 2\mathbb{N}$ and $L \geq L_0$ for some $L_0 \in \mathbb{N}$. Since V_L is a fundamental domain associated with the annihilator $\Lambda_L^{\perp}$ of the lattice Λ_L in (3.4), we see that for all $\omega \in \Lambda^{\perp} \backslash \{0\}$,

$$(\Omega \cup \Omega') \bigcap (\omega + (\Omega \cup \Omega')) = \emptyset.$$

It follows that (3.2) holds for $L \in 2\mathbb{N}, L \geq L_0$. Then the result for arbitrary elementary LCA groups is obtained as a consequence. $\square$

The following result yields a general construction of a frame $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda \in \Lambda_k, k \in I}$ for $L^2(\widehat{G})$ having the structure of a Fourier-like system. The result is based on a collection of continuous compactly supported functions g_k, $k \in I$, forming a partition of unity and a corresponding collection of finitely overlapping sets Ω_k such that $\operatorname{supp} g_k \subseteq \Omega_k$. The obvious choice is to take $\Omega_k := \operatorname{supp} g_k$, $k \in I$; however, the possibility of taking Ω_k to be a larger set will be needed in Theorem 4.1 later. Note that due to the structure of the result, the partition of unity takes place on the dual group $\widehat{G}$.

Theorem 3.3 *Let I denote a countable index set and consider nonnegative functions $g_k \in C_c(\widehat{G})$ and relatively compact measurable sets Ω_k, $k \in I$, in $\widehat{G}$ such that the following hold:*

(i) $\sum_{k \in I} g_k(\gamma) = 1$, *a.e.* $\gamma \in \widehat{G}$.

(ii) *The sets $\{\Omega_k\}_{k \in I}$ are finitely overlapping.*

(iii) $\operatorname{supp} g_k \subseteq \Omega_k$, $\forall k \in I$.

(iv) *$\{\Lambda_k\}_{k \in I}$ is a family of sufficiently fine lattices in G with respect to the sets $\{\Omega_k\}_{k \in I}$.*

For $k \in I$, let V_k denote a measurable fundamental domain associated with $\Lambda_k^\perp$, and define

$$\Phi_k(\gamma) := \frac{1}{\sqrt{\mu_{\widehat{G}}(V_k)}}\, g_k(\gamma),\ \gamma \in \widehat{G}. \tag{3.6}$$

Then the Fourier-like system $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ is a frame for $L^2(\widehat{G})$.

Proof. Since $\operatorname{supp}\Phi_k \subseteq \Omega_k$ for every $k \in I$, the assumption (iv) implies that for $k \in I$ and $\omega \in \Lambda_k^\perp \setminus \{0\}$,

$$\operatorname{supp}\Phi_k \cap \operatorname{supp}\Phi_k(\cdot + \omega) \subseteq \Omega_k \cap (-\omega + \Omega_k) = \emptyset$$

(up to a set of measure zero in $\widehat{G}$). Thus we can apply Lemma 1.1. Then

$$B := \operatorname*{ess\,sup}_{\gamma\in\widehat{G}} \sum_{k\in I} \mu_{\widehat{G}}(V_k)|\Phi_k(\gamma)|^2 = \operatorname*{ess\,sup}_{\gamma\in\widehat{G}} \sum_{k\in I} |g_k(\gamma)|^2 \ \ \leq \ \ \operatorname*{ess\,sup}_{\gamma\in\widehat{G}} \sum_{k\in I} g_k(\gamma) = 1,$$

implying that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ is a Bessel sequence. Now, since the sequence $\{\Omega_k\}_{k\in I}$ is finitely overlapping, it follows from (iii) that

$$N := \operatorname*{ess\,sup}_{\gamma\in\widehat{G}} \sharp\{k \in I \,\big|\, g_k(\gamma) \neq 0\} \leq \operatorname*{ess\,sup}_{\gamma\in\widehat{G}} \sharp\{k \in I \,\big|\, \gamma \in \Omega_k\} < \infty.$$

Note that the set

$$\mathcal{N} := \bigcup_{k\in I}\bigcup_{\ell\in I} \{\Omega_k \cap \Omega_\ell \,|\, \mu_{\widehat{G}}(\Omega_k \cap \Omega_\ell) = 0\} \tag{3.7}$$

is of measure zero. Thus, given $\gamma \in \widehat{G}\backslash\mathcal{N}$ for which (i) holds, there exists $k \in I$ such that $|g_k(\gamma)| \geq 1/N$. Therefore

$$A := \operatorname*{ess\,inf}_{\gamma\in\widehat{G}} \sum_{k\in I} \mu_{\widehat{G}}(V_k)|\Phi_k(\gamma)|^2 = \operatorname*{ess\,inf}_{\gamma\in\widehat{G}} \sum_{k\in I} |g_k(\gamma)|^2 \geq \frac{1}{N^2}. \tag{3.8}$$

It follows now from Lemma 1.1 that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ indeed is a frame for $L^2(\widehat{G})$. □

Note that the assumptions in Theorem 3.3 imply that the sets $\{\Omega_k\}_{k\in I}$ form a cover of the group $\widehat{G}$. In [1], covers formed by sets Ω_k, $k \in \mathbb{Z}$, that are scaled versions of a fixed set Q are used to construct wavelet frames for $L^2(\mathbb{R}^d)$.

Note also that by Lemma 3.2 the key condition in Theorem 3.3 regarding the choice of the lattices Λ_k can always be satisfied in the case of an elementary LCA group. We also notice that Theorem 3.3 only requires the partition of unity condition to hold almost everywhere.

In the full generality of Theorem 3.3 we will now show how a general construction of dual pairs of frames can be obtained, just by imposing a stronger condition on the lattices.

Theorem 3.4 *Under the assumptions in Theorem 3.3, let*

$$\Delta_k := \{\ell \in I \,|\, \mu_{\widehat{G}}(\Omega_k \cap \Omega_\ell) > 0\},\ k \in I.$$

For $k \in I$, let

$$\Omega'_k := \bigcup_{\ell \in \Delta_k} \Omega_\ell, \tag{3.9}$$

and assume that the sequence of lattices $\{\Lambda_k\}_{k\in I}$ in G is sufficiently fine with respect to the sets $\{\Omega'_k\}_{k\in I}$. Then the functions $\widetilde{\Phi_k}$, $k \in I$, defined by

$$\widetilde{\Phi_k}(\gamma) := \frac{1}{\sqrt{\mu_{\widehat{G}}(V_k)}} \sum_{\ell \in \Delta_k} g_\ell(\gamma),\ \gamma \in \widehat{G}, \tag{3.10}$$

are continuous with compact support, and $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k, k\in I}$ are dual frames for $L^2(\widehat{G})$.

Proof. We will apply Theorem 1.2. Since the sets $\{\Omega_k\}_{k\in I}$ are finitely overlapping,

$$M := \sup_{k\in I} \sharp\{\ell \in I \,|\, \mu_{\widehat{G}}(\Omega_k \cap \Omega_\ell) > 0\} < \infty.$$

Thus the cardinality of the set Δ_k is finite and uniformly bounded by M, independently of $k \in I$. Thus (3.10) shows that $\widetilde{\Phi_k} \in C_c(\widehat{G})$. Furthermore,

$$\operatorname{supp} \widetilde{\Phi_k} \subseteq \bigcup_{\ell\in\Delta_k} \operatorname{supp} g_\ell \subseteq \bigcup_{\ell\in\Delta_k} \Omega_\ell = \Omega'_k.$$

Thus, since the sequence of lattices $\{\Lambda_k\}_{k\in I}$ is sufficiently fine with respect to the sets $\{\Omega'_k\}_{k\in I}$,

$$\operatorname{supp} \Phi_k \cap \operatorname{supp} \widetilde{\Phi_k}(\cdot + \omega) = \emptyset,\ \forall \omega \in \Lambda_k^\perp \setminus \{0\}$$

(up to a set of measure zero in $\widehat{G}$), i.e., the condition (1.4) in Theorem 1.2 is satisfied.

We will now verify that $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k, k\in I}$ is a Bessel sequence. Recall that the set $\mathcal{N}$ defined in (3.7) is of measure zero. Consider any $\gamma \in \widehat{G}\backslash\mathcal{N}$ for which the partition of unity condition is satisfied; then $\gamma \in \Omega_\nu$ for some $\nu \in I$. Let us write $\Delta_\nu = \{\ell_{\nu,1}, \dots, \ell_{\nu,J_\nu}\}$, where $J_\nu \le M$ and $\ell_{\nu,1} = \nu$. Also, let

$$I_\nu := \bigcup_{j=1}^{J_\nu} \Delta_{\ell_{\nu,j}}.$$

As a first step to prove the Bessel condition, we will show that

$$\gamma \in \Omega_\nu \Rightarrow g_\ell(\gamma) = 0 \text{ for all } \ell \in \Delta_k,\ k \in I \setminus I_\nu. \tag{3.11}$$

To this end, let $k \in I \setminus I_\nu$. Then $k \notin \Delta_{\ell_{\nu,j}}$ for all $j = 1, 2, \dots, J_\nu$. Now, take any $\ell \in \Delta_k$. We note that $\ell \notin \Delta_\nu$. Indeed, if $\ell \in \Delta_\nu$, it follows that $\ell = \ell_{\nu,j}$ for some $j \in \{1, 2, \dots, J_\nu\}$.

Since $\mu_{\widehat{G}}(\Omega_k \cap \Omega_\ell) > 0$ by the assumption that $\ell \in \Delta_k$, we have $\mu_{\widehat{G}}(\Omega_k \cap \Omega_{\ell_{\nu,j}}) > 0$; this means that $k \in \Delta_{\ell_{\nu,j}} \subseteq I_\nu$, which contradicts the assumption that $k \in I \setminus I_\nu$. Now, using that $\ell \notin \Delta_\nu$, we obtain $\mu_{\widehat{G}}(\Omega_\nu \cap \Omega_\ell) = 0$. Thus, for $\gamma \in \Omega_\nu$, as $\gamma \in \widehat{G} \backslash \mathcal{N}$, we have $\gamma \notin \Omega_\ell$ and hence $\gamma \notin \operatorname{supp} g_\ell$, which proves (3.11). It follows that for $\gamma \in \Omega_\nu$,

$$\begin{aligned}\sum_{k\in I} \mu_{\widehat{G}}(V_k)|\widetilde{\Phi_k}(\gamma)|^2 &= \sum_{k\in I\setminus I_\nu} \left|\sum_{\ell\in\Delta_k} g_\ell(\gamma)\right|^2 + \sum_{k\in I_\nu} \left|\sum_{\ell\in\Delta_k} g_\ell(\gamma)\right|^2 \\ &\leq 0 + \sharp I_\nu \leq J_\nu M \leq M^2.\end{aligned}$$

Thus, by Lemma 1.1 $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k, k\in I}$ is a Bessel sequence, as desired.

Now, by Theorem 3.3 we know that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ is a Bessel sequence, so in order to apply Theorem 1.2 we only need to verify the condition (1.5). By construction, for a.e. $\gamma \in \operatorname{supp} g_k$,

$$\widetilde{\Phi_k}(\gamma) = \frac{1}{\sqrt{\mu_{\widehat{G}}(V_k)}} \sum_{\ell\in\Delta_k} g_\ell(\gamma) = \frac{1}{\sqrt{\mu_{\widehat{G}}(V_k)}} \sum_{\ell\in I} g_\ell(\gamma) = \frac{1}{\sqrt{\mu_{\widehat{G}}(V_k)}}.$$

It follows that for a.e. $\gamma \in \widehat{G}$, we have $\overline{g_k(\gamma)}\widetilde{\Phi_k}(\gamma) = \frac{1}{\sqrt{\mu_{\widehat{G}}(V_k)}} \overline{g_k(\gamma)}$, and hence

$$\sum_{k\in I} \mu_{\widehat{G}}(V_k)\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\gamma) = \sum_{k\in I} \sqrt{\mu_{\widehat{G}}(V_k)}\overline{g_k(\gamma)}\widetilde{\Phi_k}(\gamma) = \sum_{k\in I} \overline{g_k(\gamma)} = 1.$$

Consequently, Theorem 1.2 implies that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k, k\in I}$ are dual frames, as claimed. □

Remark 3.5 As noticed by an anonymous reviewer, the assumptions in Theorem 3.4 also imply that the functions

$$\widetilde{\Phi_k} := \frac{1}{\sqrt{\mu_{\widehat{G}}(V_k)}} \chi_{\operatorname{supp} g_k},\ k \in I,$$

generate a dual frame; they are simpler than the ones in (3.10), at the price of sacrificing the continuity requirement.

Remark 3.6 The proof that $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k, k\in I}$ is a Bessel sequence in Theorem 3.4 uses the assumption that the sequence of lattices $\{\Lambda_k\}_{k\in I}$ is sufficiently fine with respect to the sets $\{\Omega_k'\}_{k\in I}$, i.e., for every $k \in I$ and all $\omega \in \Lambda_k^\perp \setminus \{0\}$,

$$\Omega_k' \bigcap (\omega + \Omega_k') = \emptyset \tag{3.12}$$

(up to a set of measure zero in $\widehat{G}$). Frequently, the Bessel condition can be verified by other means; in such cases the lattice condition (3.12) might be replaced by weaker or more convenient conditions. Indeed, for the rest of the proof of Theorem 3.4, it is enough to assume that the sequence of lattices $\{\Lambda_k\}_{k\in I}$ is sufficiently fine with respect to the sets $\{\Omega_k\}_{k\in I}$ and $\{\Omega'_k\}_{k\in I}$, i.e., for every $k \in I$ and all $\omega \in \Lambda_k^\perp \setminus \{0\}$,

$$\Omega_k \bigcap (\omega + \Omega'_k) = \emptyset \tag{3.13}$$

(up to a set of measure zero in $\widehat{G}$). Thus, in case the Bessel condition is available for other reasons, we can replace the assumption (3.12) by (3.13) in Theorem 3.4, and thereby obtain a more general result. We will apply this observation in Theorem 4.3.

We will now illustrate our results with applications to several elementary LCA groups. Our approach is as follows. We begin with a nonnegative compactly supported function $u \in L^1(\widehat{G})$ such that $\int_{\widehat{G}} u(t)\, d\mu_{\widehat{G}}(t) = 1$, and a countable tiling $\{Q_k\}_{k\in I}$ of $\widehat{G}$ consisting of measurable sets Q_k. Then Proposition 2.1 is applied to define functions $g_k \in C_c(\widehat{G})$, $k \in I$, which satisfy the condition $\operatorname{supp} g_k \subseteq \overline{\operatorname{supp} u + Q_k} \subseteq \Omega_k$ for all $k \in I$, where $\{\Omega_k\}_{k\in I}$ are finitely overlapping sets in $\widehat{G}$. Finally, by choosing sufficiently fine lattices $\{\Lambda_k\}_{k\in I}$ in G with respect to the corresponding sets $\{\Omega'_k\}_{k\in I}$ in (3.9), it follows from Theorem 3.4 that the functions Φ_k and $\widetilde{\Phi_k}$, $k \in I$, as defined in (3.6) and (3.10) generate dual frames $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k, k\in I}$ for $L^2(\widehat{G})$.

We first consider the torus group, $G = \mathbb{T}$.

Example 3.7 Take $G = \mathbb{T}$; then $\widehat{G} = \mathbb{Z}$. Let $u \in \ell^1(\mathbb{Z})$ be a nonnegative finitely supported sequence such that $\sum_{m\in\mathbb{Z}} u(m) = 1$ and $\operatorname{supp} u \subseteq \{\alpha, \alpha+1, \dots, \beta\}$ for some $\alpha, \beta \in \mathbb{Z}$, $\alpha < \beta$. Let $\{q_k\}_{k\in\mathbb{Z}}$ denote a strictly increasing sequence in $\mathbb{Z}$ and assume that

$$\inf_{k\in\mathbb{Z}} (q_{k+1} - q_k) \geq \beta - \alpha. \tag{3.14}$$

Consider the sets $Q_k := \{q_k, q_k + 1, \dots, q_{k+1} - 1\}$, $k \in \mathbb{Z}$, which clearly form a tiling of $\mathbb{Z}$. Applying Proposition 2.1, we see that the sequences g_k, $k \in \mathbb{Z}$, given by

$$g_k(j) := \sum_{m=j-q_{k+1}+1}^{j-q_k} u(m), \;\; j \in \mathbb{Z},$$

are nonnegative and satisfy the partition of unity condition $\sum_{k\in\mathbb{Z}} g_k(j) = 1, j \in \mathbb{Z}$; furthermore

$$\operatorname{supp} g_k \subseteq \operatorname{supp} u + Q_k \subseteq \Omega_k := \{\alpha + q_k, \dots, \beta + q_{k+1} - 1\},\; k \in \mathbb{Z}.$$

Now, for $k \in \mathbb{Z}$, observe from (3.14) that

$$\Delta_k := \{\ell \in \mathbb{Z} \,\big|\, \mu_{\mathbb{Z}}(\Omega_k \cap \Omega_\ell) > 0\} = \{k-1, k, k+1\}.$$

Clearly the sets $\{\Omega_k\}_{k\in\mathbb{Z}}$ are finitely overlapping. For $k \in \mathbb{Z}$, since

$$\Omega'_k := \bigcup_{\ell\in\Delta_k} \Omega_\ell = \{\alpha + q_{k-1}, \dots, \beta + q_{k+2} - 1\},$$

choosing the lattice $\Lambda_k = b_k\mathbb{Z}$ for some $b_k > 0$ such that $b_k^{-1} \in \mathbb{Z}$ and

$$b_k \leq (\beta - \alpha + q_{k+2} - q_{k-1})^{-1},$$

it follows that $\Lambda_k^\perp = b_k^{-1}\mathbb{Z}$ and $\Omega'_k \bigcap (\omega + \Omega'_k) = \emptyset$ for $\omega \in \Lambda_k^\perp \setminus \{0\}$. Thus $\{\Lambda_k\}_{k\in\mathbb{Z}}$ is indeed a family of sufficiently fine lattices in $\mathbb{T}$ with respect to the sets $\{\Omega'_k\}_{k\in\mathbb{Z}}$. By Theorem 3.4, the resulting sequences Φ_k and $\widetilde{\Phi_k}$, $k \in \mathbb{Z}$, as defined in (3.6) and (3.10) generate a pair of dual frames $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in\mathbb{Z}}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in\mathbb{Z}}$ for $\ell^2(\mathbb{Z})$. □

As our next application, we consider the finite group $G = \mathbb{Z}_N$. The periodicity of the setting makes the calculations more involved than those for the torus group.

Example 3.8 Fix $N \in \mathbb{N}$ and let $G = \mathbb{Z}_N$. Our goal here is to construct dual frames $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in\mathbb{Z}_M}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in\mathbb{Z}_M}$ for $L^2(\mathbb{Z}_N)$ for some $M \in \mathbb{N}, M \leq N$. For simplicity, we will assume that $M \geq 3$.

First, note that $\widehat{G} = \mathbb{Z}_N$. Let u be a nonnegative N-periodic sequence such that $\sum_{m\in\mathbb{Z}_N} u(m) = 1$, and for some $\beta \in \mathbb{Z}$ with $1 \leq \beta \leq N-1$, on $\mathbb{Z}_N$ we have $\operatorname{supp} u \subseteq \{0, 1, \ldots, \beta\}$. Choose integers $0 =: q_0 < q_1 < \cdots < q_M := N$ such that

$$\min_{k\in\mathbb{Z}_M} (q_{k+1} - q_k) \geq \beta. \tag{3.15}$$

Let $Q_k := \{q_k, q_k + 1, \ldots, q_{k+1} - 1\}$ for $k \in \mathbb{Z}_M$; then $\{Q_k\}_{k\in\mathbb{Z}_M}$ is a tiling of $\mathbb{Z}_N$. By Proposition 2.1, the N-periodic sequences g_k, $k \in \mathbb{Z}_M$, defined by

$$g_k(j) := \sum_{m=j-q_{k+1}+1}^{j-q_k} u(m),\ j \in \mathbb{Z}_N,$$

satisfy the condition $\sum_{k\in\mathbb{Z}_M} g_k(j) = 1$, $j \in \mathbb{Z}_N$. Note that depending on the values of q_k, q_{k+1} and j, the set $\{(j - q_{k+1} + 1) \bmod N, (j - q_{k+1} + 2) \bmod N, \ldots, (j - q_k) \bmod N\}$ appearing in this formula of $g_k(j)$ comprises different one or two sets of consecutive integers in $\{0, 1, \ldots, N-1\}$. For $k \in \mathbb{Z}_M$, we have $\operatorname{supp} g_k \subseteq \operatorname{supp} u + Q_k \subseteq \Omega_k$, where the set Ω_k on $\mathbb{Z}_N$ is given by

$$\Omega_k = \{q_k, (q_k + 1) \bmod N, \ldots, (\beta + q_{k+1} - 1) \bmod N\}.$$

In order to facilitate the subsequent calculations, let us extend the definition of the numbers q_k, $k \in \mathbb{Z}_M$, to $k \in \mathbb{Z}$ in the following way: for $k \in \mathbb{Z}$, write $k = k' + pM$ where $k' \in \mathbb{Z}_M$ and $p \in \mathbb{Z}$, and then set $q_k = q_{k'+pM} := q_{k'} + pN$.

Now, for $k \in \mathbb{Z}_M$, a direct calculation using (3.15) and the assumption that $M \geq 3$ leads to

$$\Delta_k := \{\ell \in \mathbb{Z}_M \,|\, \mu_{\mathbb{Z}_N}(\Omega_k \cap \Omega_\ell) > 0\} = \{(k-1) \bmod M, k, (k+1) \bmod M\}.$$

Thus the sets $\{\Omega_k\}_{k\in\mathbb{Z}_M}$ are finitely overlapping. For $k\in\mathbb{Z}_M$, on $\mathbb{Z}_N$ we have

$$\Omega_k' := \bigcup_{\ell\in\Delta_k}\Omega_\ell = \{q_{k-1} \bmod N, (q_{k-1}+1) \bmod N, \ldots, (\beta+q_{k+2}-1) \bmod N\}. \tag{3.16}$$

To apply Theorem 3.4, we need to choose, for each $k\in\mathbb{Z}_M$, a lattice $\Lambda_k = b_k\,\mathbb{Z}_{N/b_k}$ in $G=\mathbb{Z}_N$ for some $b_k\in\mathbb{N}$ which divides N, such that $\{\Lambda_k\}_{k\in\mathbb{Z}_M}$ is a family of sufficiently fine lattices in G with respect to the sets $\{\Omega_k'\}_{k\in\mathbb{Z}_M}$. For $k\in\mathbb{Z}_M$, suppose first that $\beta+q_{k+2}-1-q_{k-1}\geq N-1$, i.e., the cardinality of the set $\{q_{k-1}, q_{k-1}+1,\ldots,\beta+q_{k+2}-1\}$ is at least N. Since Ω_k' in (3.16) is a set in $\mathbb{Z}_N$, this implies that $\Omega_k'=\mathbb{Z}_N$. In this case we choose $b_k=1$ and obtain the lattice $\Lambda_k=\mathbb{Z}_N$; hence $\Lambda_k^\perp=\{0\}$. Thus we trivially have that $\Omega_k'\bigcap(\omega+\Omega_k')=\emptyset$ for $\omega\in\Lambda_k^\perp\setminus\{0\}$. On the other hand, if

$$1\leq \beta+q_{k+2}-1-q_{k-1} < N-1, \tag{3.17}$$

select b_k that divides N and satisfies that

$$b_k \leq \left\lfloor \frac{N-1}{\beta+q_{k+2}-1-q_{k-1}} \right\rfloor. \tag{3.18}$$

It then follows from (3.16) that if for some $\ell\in\mathbb{Z}_{b_k}\setminus\{0\}$ (i.e., for some $\omega=\frac{N}{b_k}\ell\in\Lambda_k^\perp\setminus\{0\}=\frac{N}{b_k}\mathbb{Z}_{b_k}\setminus\{0\}$) there exists $j\in\Omega_k'\bigcap\left(\frac{N}{b_k}\,\ell+\Omega_k'\right)$, then for some $p_1,p_2\in\mathbb{Z}$ we have $j-p_1N, j-p_2N-\frac{N}{b_k}\ell\in\{q_{k-1},q_{k-1}+1,\ldots,\beta+q_{k+2}-1\}$. This implies that

$$\left|(p_1-p_2)-\frac{\ell}{b_k}\right| \leq \frac{\beta+q_{k+2}-1-q_{k-1}}{N} < \frac{\beta+q_{k+2}-1-q_{k-1}}{N-1}.$$

Using (3.17) this forces that $p_1-p_2\in\{0,1\}$. However, using (3.18) it is easy to see that both cases lead to a contradiction. Thus we again have $\Omega_k'\bigcap(\omega+\Omega_k')=\emptyset$ for $\omega\in\Lambda_k^\perp\setminus\{0\}$. Hence, $\{\Lambda_k\}_{k\in\mathbb{Z}_M}$ is a family of sufficiently fine lattices in $\mathbb{Z}_N$ with respect to the sets $\{\Omega_k'\}_{k\in\mathbb{Z}_M}$. Applying Theorem 3.4, with Φ_k and $\widetilde{\Phi_k}$, $k\in\mathbb{Z}_M$, defined as in (3.6) and (3.10), we obtain a pair of dual frames $\{\mathcal{M}_\lambda\Phi_k\}_{\lambda\in\Lambda_k,k\in\mathbb{Z}_M}$ and $\{\mathcal{M}_\lambda\widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in\mathbb{Z}_M}$ for $L^2(\mathbb{Z}_N)$. □

In our final example we will construct dual pairs of frames for $L^2(\mathbb{R})$ in a rather unusual way. Indeed we will first apply Proposition 2.1 to the *multiplicative group* $\mathbb{R}^* := \mathbb{R}\setminus\{0\}$ and obtain a partition of unity on $\mathbb{R}$; starting from this, we will then apply Theorem 3.4 to the *additive group* $\mathbb{R}$ to construct the desired frames.

Example 3.9 The Haar measure on the multiplicative group $\mathbb{R}^*$ is $d\mu_{\mathbb{R}^*}(t)=|t|^{-1}\,dt$, where dt is the usual Lebesgue measure. Let $u\in L^1(\mathbb{R}^*)$ be a nonnegative function satisfying that $\int_{\mathbb{R}^*}u(t)\,d\mu_{\mathbb{R}^*}(t)=1$ and $\operatorname{supp} u\subseteq[-\beta,-\alpha]\cup[\alpha,\beta]$ for some $\alpha,\beta\in\mathbb{R}$, $0<\alpha<\beta$. Choose a strictly increasing positive sequence $\{q_k\}_{k\in\mathbb{Z}}$ such that

$$\lim_{k\to\infty} q_k=\infty,\quad \lim_{k\to-\infty} q_k = 0, \text{ and } \inf_{k\in\mathbb{Z}}\frac{q_{k+1}}{q_k}\geq\frac{\beta}{\alpha}. \tag{3.19}$$

Now, let

$$Q_k := (-q_{k+1}, -q_k] \cup [q_k, q_{k+1}),\ k \in \mathbb{Z}.$$

Clearly $\{Q_k\}_{k\in\mathbb{Z}}$ is a tiling of $\mathbb{R}^*$. Applying Proposition 2.1 to this tiling and the function u, the functions g_k in (2.2) take the form

$$g_k(x) = \int_{-|x|/q_k}^{-|x|/q_{k+1}} \frac{u(t)}{|t|}\,dt + \int_{|x|/q_{k+1}}^{|x|/q_k} \frac{u(t)}{|t|}\,dt,\ x \in \mathbb{R}^*;$$

these functions are nonnegative and satisfy the conditions

$$\sum_{k\in\mathbb{Z}} g_k(x) = 1,\ x \in \mathbb{R}^*, \tag{3.20}$$

$$\operatorname{supp} g_k \subseteq \overline{\operatorname{supp} u + Q_k} \subseteq \Omega_k := \overline{([-\beta,-\alpha] \cup [\alpha,\beta]) + Q_k},\ k \in \mathbb{Z}. \tag{3.21}$$

Note that in (2.2) and (3.21) the operations "minus" and "plus" now mean division and multiplication respectively. By a direct calculation,

$$\Omega_k = [-\beta q_{k+1}, -\alpha q_k] \cup [\alpha q_k, \beta q_{k+1}],\ k \in \mathbb{Z}. \tag{3.22}$$

Now, with the partition of unity (3.20) being established, we will apply Theorem 3.4 to the additive group $G = \mathbb{R}$; thus $\widehat{G} = \mathbb{R}$. Using (3.22) and the last condition in (3.19), we see that for $k \in \mathbb{Z}$,

$$\Delta_k := \{\ell \in \mathbb{Z} \,|\, \mu_{\mathbb{R}}(\Omega_k \cap \Omega_\ell) > 0\} = \{k-1, k, k+1\}.$$

Thus the sets $\{\Omega_k\}_{k\in\mathbb{Z}}$ are finitely overlapping. Since

$$\Omega'_k := \bigcup_{\ell\in\Delta_k} \Omega_\ell = [-\beta q_{k+2}, -\alpha q_{k-1}] \cup [\alpha q_{k-1}, \beta q_{k+2}] \subseteq [-\beta q_{k+2}, \beta q_{k+2}],\ k \in \mathbb{Z},$$

choosing $\Lambda_k = b_k\,\mathbb{Z}$ with $0 < b_k \leq \frac{1}{2\beta q_{k+2}}$, it follows that $\Lambda_k^\perp = b_k^{-1}\,\mathbb{Z}$ and for every $\omega \in \Lambda_k^\perp \setminus \{0\}$, $\Omega'_k \bigcap (\omega + \Omega'_k) = \emptyset$ (up to a set of measure zero in $\mathbb{R}$). This shows that $\{\Lambda_k\}_{k\in\mathbb{Z}}$ is a family of sufficiently fine lattices in $\mathbb{R}$ with respect to the sets $\{\Omega'_k\}_{k\in\mathbb{Z}}$. By Theorem 3.4, the functions Φ_k and $\widetilde{\Phi_k}$, $k \in \mathbb{Z}$, as defined in (3.6) and (3.10) therefore generate a pair of dual frames $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in\mathbb{Z}}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k, k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. The interested reader can verify that by choosing $q_k = a^k$ and $b_k = a^{-k}b$ for some $a > 1$ and $b > 0$, and $0 < \alpha < \beta$ such that $\alpha a \geq \beta$, this leads to classical wavelet frame constructions on $\mathbb{R}$ when b satisfies the inequality $b \leq \frac{1}{2\beta a^2}$. □

4 Frames generated by a single lattice

The theoretical results in Section 3 are based on a family of lattices $\{\Lambda_k\}_{k\in I}$ in the LCA group G. In this section we will provide various methods for constructing explicitly given frames and pairs of dual frames from a single lattice. All the constructions are based on certain partitions of unity on the dual group $\widehat{G}$. We will first state a general version of the results in Subsection 4.1, with minimal requirements on the partition of unity. In Subsection 4.2 we will specialize to the case of partitions of unity obtained via lattices in the underlying group, in which case the frames will have the structure of a Gabor system. In Subsection 4.3 we then consider partitions of unity that are constructed via tilings as in Section 2.

4.1 Frame constructions based on general partitions of unity

We will first formulate a general result concerning construction of a Fourier-like frame from a partition of unity. It is based on Theorem 3.3 but is much easier to apply in concrete cases because it only requires choosing a single lattice that is sufficiently fine.

Theorem 4.1 *Let I denote a countable index set and consider nonnegative functions $g_k \in C_c(\widehat{G})$, $k \in I$, such that the following hold:*

(i) $\sum_{k\in I} g_k(\gamma) = 1$, *a.e.* $\gamma \in \widehat{G}$.

(ii) *There exists a relatively separated set $\{x_k\}_{k\in I} \subset \widehat{G}$ and a relatively compact measurable set $\Omega \subseteq \widehat{G}$ such that*

$$\operatorname{supp} g_k \subseteq x_k + \Omega, \ \forall k \in I.$$

Then for every lattice $\Lambda \subset G$ which is sufficiently fine with respect to the set Ω, the Fourier-like system $\{\mathcal{M}_\lambda g_k\}_{\lambda\in\Lambda,k\in I}$ is a frame for $L^2(\widehat{G})$.

Proof. Let $\Omega_k := x_k + \Omega$, $k \in I$; then $\operatorname{supp} g_k \subseteq \Omega_k$, $\forall k \in I$. Since $\{x_k\}_{k\in I}$ is relatively separated and Ω is relatively compact,

$$\sup_{k\in I} \sharp\{\ell \in I \big|\, \mu_{\widehat{G}}(\Omega_k \cap \Omega_\ell) > 0\} \leq \sup_{k\in I} \sharp\{\ell \in I \big|\, (x_k + \Omega) \cap (x_\ell + \Omega) \neq \emptyset\} < \infty,$$

i.e., the sets $\{\Omega_k\}_{k\in I}$ are finitely overlapping. Now, considering any lattice Λ which is sufficiently fine with respect to Ω, we have that for all $\omega \in \Lambda^\perp \setminus \{0\}$, $\Omega \bigcap (\omega + \Omega) = \emptyset$, and therefore $(x_k+\Omega) \bigcap (\omega + (x_k + \Omega)) = \emptyset$ (up to a set of measure zero in $\widehat{G}$); thus the lattice Λ is sufficiently fine with respect to any of the sets Ω_k. Then the result follows from Theorem 3.3 with $\Lambda_k = \Lambda$. Indeed, as Λ_k is now independent of $k \in I$, the function Φ_k in (3.6) takes the form $\Phi_k(\gamma) := \frac{1}{\sqrt{\mu_{\widehat{G}}(V)}}\, g_k(\gamma)$, where V is a fundamental domain associated with the lattice $\Lambda^\perp$; hence, since $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda,k\in I}$ is a frame for $L^2(\widehat{G})$, the family $\{\mathcal{M}_\lambda g_k\}_{\lambda\in\Lambda,k\in I}$ is a frame as well. □

As in Section 3, the key to obtain a construction of a pair of dual frames, rather than just a frame, is to choose a finer lattice Λ:

Theorem 4.2 *Under the assumptions in Theorem* 4.1*, let*

$$\Delta_k := \{\ell \in I \,|\, \mu_{\widehat{G}}((x_k + \Omega) \cap (x_\ell + \Omega)) > 0\},\ k \in I. \tag{4.1}$$

Let $\Omega' := \Omega - \Omega + \Omega$*, and assume that the lattice* Λ *is sufficiently fine with respect to the set* Ω'*. Then the functions* h_k*,* $k \in I$*, given by*

$$h_k(\gamma) := \frac{1}{\mu_{\widehat{G}}(V)} \sum_{\ell \in \Delta_k} g_\ell(\gamma),\ \gamma \in \widehat{G}, \tag{4.2}$$

are continuous with compact support, and $\{\mathcal{M}_\lambda g_k\}_{\lambda \in \Lambda, k \in I}$ *and* $\{\mathcal{M}_\lambda h_k\}_{\lambda \in \Lambda, k \in I}$ *are dual frames for* $L^2(\widehat{G})$*.*

Proof. Let $\Omega_k := x_k + \Omega$, $k \in I$. We will apply Theorem 3.4, and we only have to show that the stated condition on the lattice Λ implies that the sequence of lattices $\{\Lambda_k\}_{k \in I}$, defined by $\Lambda_k := \Lambda$ for all $k \in I$, is sufficiently fine with respect to the sets $\{\Omega_k'\}_{k \in I}$ given by (3.9). First, if $\ell \in \Delta_k$, there exist $\gamma_1, \gamma_2 \in \Omega$ such that $x_k + \gamma_1 = x_\ell + \gamma_2$, i.e., $x_\ell = x_k + \gamma_1 - \gamma_2 \in x_k + \Omega - \Omega$. It follows that for $\ell \in \Delta_k$,

$$x_\ell + \Omega \subset x_k + \Omega - \Omega + \Omega = x_k + \Omega'.$$

Thus the condition (3.9) implies that $\Omega_k' \subset x_k + \Omega'$ for all $k \in I$. Observe that $\{\Lambda_k\}_{k \in I}$ is sufficiently fine with respect to $\{\Omega_k'\}_{k \in I}$ if it is sufficiently fine with respect to the sets $\{x_k + \Omega'\}_{k \in I}$, i.e., for all $k \in I$,

$$(x_k + \Omega') \bigcap (\omega + (x_k + \Omega')) = \emptyset,\ \forall \omega \in \Lambda_k^\perp \setminus \{0\} \tag{4.3}$$

(up to a set of measure zero in $\widehat{G}$). Applying a simple shift and recalling the definition of Λ_k, (4.3) is equivalent to

$$\Omega' \bigcap (\omega + \Omega') = \emptyset,\ \forall \omega \in \Lambda^\perp \setminus \{0\}$$

(up to a set of measure zero in $\widehat{G}$), a condition that is satisfied due to the assumption on the lattice Λ. The result now follows from Theorem 3.4, using the relation between the functions Φ_k and g_k given in (3.6). □

4.2 Explicit Gabor frame constructions

In this subsection we will consider partitions of unity formed by translates of a fixed function g along a lattice Λ; in this case the approach in Subsection 4.1 turns out to yield a Gabor system. As usual, for $k \in \widehat{G}$, we consider the *translation operator*

$$T_k : L^2(\widehat{G}) \to L^2(\widehat{G}),\ (T_k f)(\gamma) := f(\gamma - k).$$

The following result combines the information from Theorem 4.1 and Theorem 4.2 in this particular situation.

Theorem 4.3 *Let Γ denote a lattice in $\widehat{G}$. Consider a nonnegative, continuous, and compactly supported function $g : \widehat{G} \to \mathbb{R}$ such that*

$$\sum_{k\in\Gamma} g(\gamma - k) = 1,\ \gamma \in \widehat{G},$$

and let $\Omega := \operatorname{supp} g$. Then the following hold:

(i) *For any lattice Λ in G that is sufficiently fine with respect to the set Ω, the family $\{\mathcal{M}_\lambda T_k g\}_{\lambda\in\Lambda,k\in\Gamma}$ is a frame for $L^2(\widehat{G})$.*

(ii) *Let $\Delta := \{\ell \in \Gamma \,|\, \mu_{\widehat{G}}(\operatorname{supp} g \cap (\ell + \operatorname{supp} g)) > 0\}$, and put $\Omega' := \Omega - \Omega + \Omega$. Assume that the lattice Λ in G is sufficiently fine with respect to the sets Ω and Ω'. Letting V denote a fundamental domain associated with the lattice $\Lambda^\perp$, the function*

$$h(\gamma) := \frac{1}{\mu_{\widehat{G}}(V)} \sum_{\ell\in\Delta} g(\gamma - \ell),\ \gamma \in \widehat{G}, \tag{4.4}$$

belongs to $C_c(\widehat{G})$ and generates a dual frame $\{\mathcal{M}_\lambda T_k h\}_{\lambda\in\Lambda,k\in\Gamma}$ of $\{\mathcal{M}_\lambda T_k g\}_{\lambda\in\Lambda,k\in\Gamma}$ for $L^2(\widehat{G})$.

Proof. By assumption the functions $g_k := T_k g,\ k \in \Gamma$, satisfy the condition (i) in Theorem 4.1. Also, $\operatorname{supp} g_k = k + \operatorname{supp} g = k + \Omega$; since a lattice is relatively separated, the condition (ii) in Theorem 4.1 is satisfied as well. Thus (i) in this result follows directly from Theorem 4.1.

For the proof of (ii), we will use the proof of Theorem 4.2. The set Δ_k in (4.1) is now

$$\begin{aligned} \Delta_k &= \{\ell \in \Gamma \,|\, \mu_{\widehat{G}}((k + \operatorname{supp} g) \cap (\ell + \operatorname{supp} g)) > 0\} \\ &= k + \{\ell \in \Gamma \,|\, \mu_{\widehat{G}}(\operatorname{supp} g \cap (\ell + \operatorname{supp} g)) > 0\} = k + \Delta, \end{aligned}$$

where we note that Δ is a set with finite cardinality. Thus, the functions h_k in (4.2) take the form

$$h_k(\gamma) = \frac{1}{\mu_{\widehat{G}}(V)} \sum_{\ell\in\Delta_k} T_\ell g(\gamma) = T_k\Big(\frac{1}{\mu_{\widehat{G}}(V)} \sum_{\ell\in\Delta} T_\ell g\Big)(\gamma) = (T_k h)(\gamma),$$

with the function h as defined in (4.4). As a finite linear combination of Bessel sequences, the collection $\{\mathcal{M}_\lambda T_k h\}_{\lambda\in\Lambda,k\in\Gamma}$ is clearly a Bessel sequence. Letting $\Omega_k := k + \Omega$ and referring to Remark 3.6, it is therefore sufficient to apply the weaker lattice condition

$$\Omega_k \bigcap (\omega + \Omega_k') = \emptyset,\ \forall k \in \Gamma,\ \omega \in \Lambda^\perp \setminus \{0\}$$

(up to a set of measure zero in $\widehat{G}$), which is satisfied if

$$\Omega \bigcap (\omega + \Omega') = \emptyset, \ \forall \omega \in \Lambda^{\perp} \setminus \{0\}$$

(up to a set of measure zero in $\widehat{G}$); this condition holds if the lattice Λ is sufficiently fine with respect to the sets Ω and Ω', as claimed. This proves that $\{\mathcal{M}_\lambda T_k g\}_{\lambda\in\Lambda, k\in\Gamma}$ and $\{\mathcal{M}_\lambda T_k h\}_{\lambda\in\Lambda, k\in\Gamma}$ are dual frames for $L^2(\widehat{G})$. □

Note that Theorem 4.3 is formulated in an operative way with respect to applications of Corollary 2.3. Combining the two results, we arrive at the following explicit construction of a Gabor frame $\{\mathcal{M}_\lambda T_k g\}_{\lambda\in\Lambda, k\in\Gamma}$ for $L^2(\widehat{G})$:

Corollary 4.4 *Consider a lattice Γ in $\widehat{G}$ with a corresponding fundamental domain Q, and choose a nonnegative compactly supported function $u \in L^1(\widehat{G})$ such that $\int_{\widehat{G}} u(t)\, d\mu_{\widehat{G}}(t) = 1$. Define the function*

$$g(\gamma) := \int_{\gamma - Q} u(t)\, d\mu_{\widehat{G}}(t), \ \gamma \in \widehat{G}.$$

Let $\Omega := \overline{\operatorname{supp} u + Q}$. Then the following hold:

(i) *For any lattice Λ in G that is sufficiently fine with respect to the set Ω, the Gabor system $\{\mathcal{M}_\lambda T_k g\}_{\lambda\in\Lambda, k\in\Gamma}$ is a frame for $L^2(\widehat{G})$.*

(ii) *Define the set Δ and the function h as in Theorem 4.3, and assume that Λ is sufficiently fine with respect to the sets Ω and $\Omega' := \Omega - \Omega + \Omega$. Then the Gabor systems $\{\mathcal{M}_\lambda T_k g\}_{\lambda\in\Lambda, k\in\Gamma}$ and $\{\mathcal{M}_\lambda T_k h\}_{\lambda\in\Lambda, k\in\Gamma}$ are dual frames for $L^2(\widehat{G})$.*

It is not difficult to show that Theorem 4.3 generalizes the classical results for obtaining Gabor frames for $L^2(\mathbb{R})$ based on B-splines (the results known in the literature as the "painless case") and the corresponding explicit construction of dual pairs of Gabor frames, see Theorem 12.5.1 in [4].

4.3 Frame constructions based on tilings

Theorem 4.1 and Theorem 4.2 are based on general partitions of unity. In this subsection we will consider frame constructions based on partitions of unity obtained via tilings as in Section 2. As we have seen in Proposition 2.1, this allows us to control key features of the functions g_k via the choice of the auxiliary function u.

Corollary 4.5 *Consider a countable tiling $\{Q_k\}_{k\in I}$ of $\widehat{G}$ consisting of measurable sets Q_k for which $Q_k \subseteq x_k + Q$ for a compact measurable set $Q \subset \widehat{G}$ and a relatively separated* *sequence* $\{x_k\}_{k\in I} \subset \widehat{G}$. *Choose a nonnegative compactly supported function* $u \in L^1(\widehat{G})$ *such that* $\int_{\widehat{G}} u(t)\, d\mu_{\widehat{G}}(t) = 1$, *and define the functions* $g_k \in C_c(\widehat{G})$, $k \in I$, *by*

$$g_k(\gamma) := \int_{\gamma - Q_k} u(t) d\mu_{\widehat{G}}(t),\ \gamma \in \widehat{G}. \tag{4.5}$$

Let $\Omega := \overline{\operatorname{supp} u + Q}$. *Then the following hold:*

(i) *For any lattice* Λ *in* G *that is sufficiently fine with respect to the set* Ω, *the system* $\{\mathcal{M}_\lambda g_k\}_{\lambda\in\Lambda, k\in I}$ *is a frame for* $L^2(\widehat{G})$.

(ii) *Let* $\Delta_k := \{\ell \in I \,|\, \mu_{\widehat{G}}((x_k + \Omega) \cap (x_\ell + \Omega)) > 0\}$, $k \in I$, *and* $\Omega' := \Omega - \Omega + \Omega$, *and assume that the lattice* Λ *is sufficiently fine with respect to the set* Ω'. *Then, with the functions* h_k *in* (4.2), $\{\mathcal{M}_\lambda g_k\}_{\lambda\in\Lambda, k\in I}$ *and* $\{\mathcal{M}_\lambda h_k\}_{\lambda\in\Lambda, k\in I}$ *are dual frames for* $L^2(\widehat{G})$.

Proof. By Proposition 2.1 the functions g_k, $k \in I$, satisfy the partition of unity condition in Theorem 4.1(i). Also, by Proposition 2.1 we know that $\operatorname{supp} g_k \subseteq x_k + \overline{\operatorname{supp} u + Q}$. Thus, (i) follows from Theorem 4.1; (ii) follows from Theorem 4.2. □

Intuitively, the assumption of $\{x_k\}_{k\in I}$ being relatively separated in Corollary 4.5 prevents the sets Q_k from being too small and clustered together. The example below indeed shows that the conclusion in Corollary 4.5 might fail without that condition.

Example 4.6 We consider the group $\widehat{G} = \mathbb{R}$, and for notational convenience, we will formulate the following construction for a tiling indexed as $\{Q_{n,k}\}_{n\in\mathbb{Z}, k\in I_n}$ for certain finite sets I_n, $n \in \mathbb{Z}$. For $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n-1\}$, let $Q_{n,k} := [n + \frac{k}{n}, n + \frac{k+1}{n})$; and for $n \in \{0, -1, \ldots\}$, let $Q_{n,0} := [n, n+1)$. Then the defined sets yield a tiling $\{Q_{n,k}\}_{n\in\mathbb{Z}, k\in I_n}$ of $\mathbb{R}$, where I_n equals $\{0, 1, \ldots, n-1\}$ if $n \in \mathbb{N}$, and $\{0\}$ otherwise. Note that for $n \in \mathbb{Z}$ and $k \in I_n$, we have $Q_{n,k} \subseteq x_{n,k} + [0, 1]$, where

$$x_{n,k} := \begin{cases} n + \frac{k}{n}, & \text{if } n \in \mathbb{N} \text{ and } k \in \{0, 1, \ldots, n-1\}, \\ n, & \text{if } n \in \{0, -1, \ldots\} \text{ and } k = 0. \end{cases}$$

The characterization in (2.1) shows that the sequence $\{x_{n,k}\}_{n\in\mathbb{Z}, k\in I_n}$ is not relatively separated.

Now, consider the function $u := \chi_{[-1,0)}$. Based on the tiling $\{Q_{n,k}\}_{n\in\mathbb{Z}, k\in I_n}$, define the functions $g_{n,k}$ as in (4.5). In particular, for $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n-1\}$,

$$g_{n,k}(\gamma) = \int_{\gamma - n - \frac{k+1}{n}}^{\gamma - n - \frac{k}{n}} u(t)\, dt,\ \gamma \in \mathbb{R}. \tag{4.6}$$

As noted in Proposition 2.1(iii), for $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n-1\}$, we have $\operatorname{supp} g_{n,k} \subseteq [n - 1 + \frac{k}{n}, n + \frac{k+1}{n}] \subseteq [n-1, n+1]$, while for $n \in \{0, -1, \ldots\}$, $\operatorname{supp} g_{n,0} \subseteq [n-1, n+1]$.

In the following argument, we fix some integer $n_0 \geq 2$ and compare the interval $(n_0 - \frac{1}{n_0}, n_0)$ with the sets that contain $\operatorname{supp} g_{n,k}$. Indeed, observe that for $n \in \mathbb{N}$ and $k \in$

$\{0,1,\ldots,n-1\}$, $[n-1+\frac{k}{n}, n+\frac{k+1}{n}] \cap (n_0-\frac{1}{n_0}, \frac{1}{n_0}) \neq \emptyset$ if and only if, either $n = n_0$ and $k \in \{0,1,\ldots,n_0-1\}$, or $n = n_0 - 1$ and $k = n_0 - 2$. In addition, for $n \in \{0,-1,\ldots\}$, the sets $[n-1, n+1]$ and $(n_0 - \frac{1}{n_0}, \frac{1}{n_0})$ are always disjoint. Thus we consider exclusively $\gamma \in (n_0 - \frac{1}{n_0}, n_0)$. First we note that for $n = n_0$ and $k \in \{0,1,\ldots,n_0-2\}$, $\gamma - n - \frac{k+1}{n} > -1$ and $\gamma - n - \frac{k}{n} < 0$. Thus, by (4.6),

$$g_{n_0,k}(\gamma) = \int_{\gamma-n_0-\frac{k+1}{n_0}}^{\gamma-n_0-\frac{k}{n_0}} dt = \frac{1}{n_0}. \tag{4.7}$$

For $n = n_0$ and $k = n_0 - 1$, we still have $\gamma - n - \frac{k}{n} < 0$, but $\gamma - n - \frac{k+1}{n} < -1$; thus, again by (4.6),

$$g_{n_0,n_0-1}(\gamma) = \int_{-1}^{\gamma-n_0-\frac{n_0-1}{n_0}} dt = \gamma - n_0 + \frac{1}{n_0}. \tag{4.8}$$

Finally, for $n = n_0 - 1$ and $k = n_0 - 2$, we see that $\gamma - n - \frac{k+1}{n} > -1$, while $\gamma - n - \frac{k}{n} > 0$. This gives

$$g_{n_0-1,n_0-2}(\gamma) = \int_{\gamma-(n_0-1)-\frac{(n_0-2)+1}{n_0-1}}^{0} dt = n_0 - \gamma. \tag{4.9}$$

It then follows from (4.7)–(4.9) that

$$\sum_{n\in\mathbb{Z}} \sum_{k\in I_n} |g_{n,k}(\gamma)|^2 = \sum_{k=0}^{n_0-2} \left(\frac{1}{n_0}\right)^2 + \left(\gamma - n_0 + \frac{1}{n_0}\right)^2 + (n_0-\gamma)^2 \leq \frac{n_0+1}{n_0^2} \to 0 \text{ as } n_0 \to \infty.$$

Thus, for the bound in (3.8) we get $A = 0$. Since strict positivity of A is a necessary condition for the frame property under the assumption (1.3) in Lemma 1.1, we conclude that the frame condition is violated for sufficiently fine lattices with respect to the set $\operatorname{supp} u + [0,1] = [-1,1]$.

□

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