An improved tensorial implementation of the incremental harmonic balance method for frequency-domain stability analysis

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- Applications in dynamic systems
  - SDOF and MDOF nonlinear dynamic systems
  - Finite element models of nonlinear mechanical systems
- Analyses
  - Time-periodic response analysis
    - Path-following techniques
  - Stability analysis
    - Monodromy matrix
  - Bifurcation
- Structural optimization
  - Nonlinear forced resonances
  - Nonlinear normal modes

The governing equation of finite element model of nonlinear mechanical system with geometric nonlinearity can be written as

$$\mathsf{M}\frac{\mathrm{d}^2\mathsf{u}}{\mathrm{d}t^2} + \mathsf{C}\frac{\mathrm{d}\mathsf{u}}{\mathrm{d}t} + \mathsf{g}(\mathsf{u}) = \mathsf{f}(\omega t) \tag{1}$$

- M : the mass matrix
- C : the linear damping matrix
- g : the nonlinear internal force vector (e.g. quadratic/cubic polynomial function)
- f : the time-periodic external force vector
- ω : the circular frequency $\omega = \frac{2\pi}{T}$, where T is the time period

- Step 1: Introduce a new time scale τ

$$\tau = \omega t = 2\pi(t/T) \tag{2}$$

We have

$$\frac{\mathrm{d}\mathsf{u}}{\mathrm{d}t} = \omega \frac{\mathrm{d}\mathsf{u}}{\mathrm{d}\tau}, \quad \frac{\mathrm{d}^2\mathsf{u}}{\mathrm{d}t^2} = \omega^2 \frac{\mathrm{d}^2\mathsf{u}}{\mathrm{d}\tau^2} \tag{3}$$

Equation (1) can be written as

$$\omega^2 \mathsf{M} \frac{\mathrm{d}^2\mathsf{u}}{\mathrm{d}\tau^2} + \omega\, \mathsf{C} \frac{\mathrm{d}\mathsf{u}}{\mathrm{d}\tau} + \mathsf{g}(\mathsf{u}) = \mathsf{f}(\tau) \tag{4}$$

- Step 2: Fourier expansion of time-periodic response

$$\mathsf{u}(\tau) = \mathsf{s}(\tau)\, \mathsf{q} \tag{5}$$

where $\mathsf{s}(\tau)$ is Fourier basis (i.e. cosine, sine), q is a vector of Fourier coefficients. Substituting equation (5) into equation (4) and applying the Galerkin method, *the governing equation in the frequency domain* is

$$\left(\omega^2\, \overline{\mathsf{M}} + \omega\, \overline{\mathsf{C}}\,\right)\, \mathsf{q} + \overline{\mathsf{g}}\,(\,\mathsf{q}\,) = \overline{\mathsf{f}} \tag{6}$$

The governing equation can also be written in a compact form as

$$\mathsf{r}(\mathsf{q}, \omega) = 0 \tag{7}$$

where $\mathsf{r}(\mathsf{q}, \omega) = \left(\omega^2\, \overline{\mathsf{M}} + \omega\, \overline{\mathsf{C}}\,\right)\, \mathsf{q} + \overline{\mathsf{g}}\,(\,\mathsf{q}\,) - \overline{\mathsf{f}}$.

- Step 3: Consider an unknown solution in the neighborhood of a given solution expressed by q_0 and ω_0

$$\mathsf{q} = \mathsf{q}_0 + \Delta \mathsf{q}, \quad \omega = \omega_0 + \Delta \omega. \tag{8}$$

Substituting equation (8) into equation (7), we have

$$\mathsf{r}(\mathsf{q}_0 + \Delta \mathsf{q},\ \omega_0 + \Delta \omega) = 0 \tag{9}$$

The incremental form of the frequency-domain governing equation is

$$\mathsf{K}_q \Delta \mathsf{q} + \mathsf{K}_\omega \Delta \omega = -\mathsf{r}(\mathsf{q}_0, \omega_0) \tag{10}$$

$$\mathsf{K}_q = \left.\frac{\partial \mathsf{r}(\mathsf{q}, \omega)}{\partial \mathsf{q}}\right|_{\mathsf{q}=\mathsf{q}_0,\, \omega=\omega_0}, \quad \mathsf{K}_\omega = \left.\frac{\partial\, \mathsf{r}(\mathsf{q}, \omega)}{\partial\, \omega}\right|_{\mathsf{q}=\mathsf{q}_0,\, \omega=\omega_0}. \tag{11}$$

- The Jacobian matrices K_q and K_ω are explicitly expressed as

$$\mathsf{K}_q = \omega_0^2\,\overline{\mathsf{M}} + \omega_0\,\overline{\mathsf{C}} + \overline{\mathsf{K}}_t\,(\,\mathsf{q}_0\,)\,, \tag{12}$$

$$\mathsf{K}_\omega = 2\,\omega_0\,\overline{\mathsf{M}} + \overline{\mathsf{C}}. \tag{13}$$

where

$$\overline{\mathsf{K}}_t\,(\,\mathsf{q}_0\,) = \left.\frac{\partial \overline{\mathsf{g}}(\mathsf{q})}{\partial \mathsf{q}}\right|_{\mathsf{q}=\mathsf{q}_0} \tag{14}$$

- Consider a disturbed response at u_0 with a small disturbance $\boldsymbol{\delta}$

$$\mathsf{u} = \mathsf{u}_0 + \boldsymbol{\delta} \tag{15}$$

Substituting equation (15) into equation (4) and retaining the first-order approximation, the following equation is obtained as

$$\omega_0^2\, \mathsf{M}\, \frac{\mathrm{d}^2\boldsymbol{\delta}}{\mathrm{d}\tau^2} + \omega_0\, \mathsf{C}\, \frac{\mathrm{d}\boldsymbol{\delta}}{\mathrm{d}\tau} + \mathsf{K}_t(\mathsf{q}_0)\, \boldsymbol{\delta} = 0 \tag{16}$$

- Express the disturbance $\boldsymbol{\delta}$ as

$$\boldsymbol{\delta} = e^{\lambda\tau}\mathsf{s}\,\mathsf{p} \tag{17}$$

where λ and p denote the Floquet exponent and vector, respectively.

- The first- and second-order derivatives of $\boldsymbol{\delta}$ are

$$\frac{\mathrm{d}\boldsymbol{\delta}}{\mathrm{d}\tau} = \lambda\, e^{\lambda\tau} \mathsf{s}\,\mathsf{p} + e^{\lambda\tau}\frac{\mathrm{d}\mathsf{s}}{\mathrm{d}\tau}\mathsf{p} \tag{18}$$

$$\frac{\mathrm{d}^2\boldsymbol{\delta}}{\mathrm{d}\tau^2} = \lambda^2 e^{\lambda\tau} \mathsf{s}\,\mathsf{p} + 2\lambda e^{\lambda\tau}\frac{\mathrm{d}\mathsf{s}}{\mathrm{d}\tau}\mathsf{p} + e^{\lambda\tau}\frac{\mathrm{d}^2\mathsf{s}}{\mathrm{d}\tau^2}\mathsf{p} \tag{19}$$

- Substituting equations (17–19) into equation (16) and applying the Galerkin method, a quadratic eigenvalue problem is obtained as

$$\left(\mathsf{J}_2\lambda^2 + \mathsf{J}_1\lambda + \mathsf{J}_0\right)\mathsf{p} = 0 \tag{20}$$

$$\mathsf{J}_0 = \mathsf{K}_q(\mathsf{q}_0, \omega_0), \quad \mathsf{J}_1 = 2\,\omega_0^2\,\mathsf{M}\otimes\mathsf{h}^{(1)} + \omega_0\,\mathsf{C}\otimes\mathsf{h}^{(2)}, \quad \mathsf{J}_2 = \omega_0^2\,\mathsf{M}\otimes\mathsf{h}^{(2)} \tag{21}$$

The following frequency-domain quantities are computed as

$$\overline{\mathsf{M}} = \mathsf{M} \otimes \mathsf{h}^{(0)}, \qquad \overline{\mathsf{C}} = \mathsf{C} \otimes \mathsf{h}^{(1)},$$
$$\overline{\mathsf{g}} = \begin{bmatrix} \mathsf{h}^{(2)} \mathcal{G}_1 \\ \vdots \\ \mathsf{h}^{(2)} \mathcal{G}_{N_{dof}} \end{bmatrix}, \quad \overline{\mathsf{f}} = \begin{bmatrix} \mathsf{h}^{(2)} \mathcal{F}_1 \\ \vdots \\ \mathsf{h}^{(2)} \mathcal{F}_{N_{dof}} \end{bmatrix} \qquad (22)$$

where $\otimes$ denotes the Kronecker operator, $\mathcal{G}_i$ and $\mathcal{F}_i$ denote the Fourier coefficients of the internal and external forces for the i^{th} degree of freedom.

The key matrix $\overline{\mathrm{K}}_t$ is obtained with the assistance of a three-dimensional tensor $\mathrm{h}^{(3)}$ and the operation of tensor contraction as

$$\overline{\mathrm{K}}_t = \begin{bmatrix} \mathrm{h}^{(3)} : \mathcal{K}_{1,1} & \dots & \mathrm{h}^{(3)} : \mathcal{K}_{1,N_{dof}} \\ \vdots & \ddots & \vdots \\ \mathrm{h}^{(3)} : \mathcal{K}_{N_{dof},1} & \dots & \mathrm{h}^{(3)} : \mathcal{K}_{N_{dof},N_{dof}} \end{bmatrix} \tag{23}$$

where the symbol " : " denotes the tensor contraction operation, and $\mathcal{K}_{i,j}$ denotes a vector of Fourier coefficients corresponding to the component at the position (i, j) of $\mathrm{K}_t(\mathrm{u}(\tau))$.

$\mathsf{h}^{(0)}$, $\mathsf{h}^{(1)}$, and $\mathsf{h}^{(2)}$ are two-dimensional matrices given as

$$\mathsf{h}^{(0)} = -\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \widehat{\mathsf{I}} & 0 \\ 0 & 0 & \widehat{\mathsf{I}} \end{bmatrix}, \quad \widehat{\mathsf{I}} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & N_H^2 \end{bmatrix}_{N_H \times N_H} \tag{24}$$

$$\mathsf{h}^{(1)} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\mathsf{I}} \\ 0 & -\bar{\mathsf{I}} & 0 \end{bmatrix}, \quad \bar{\mathsf{I}} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & N_H \end{bmatrix}_{N_H \times N_H} \tag{25}$$

$$\mathsf{h}^{(2)} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & \mathsf{I} & 0 \\ 0 & 0 & \mathsf{I} \end{bmatrix}, \quad \mathsf{I} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{N_H \times N_H} \tag{26}$$

$\mathrm{h}^{(3)}$ is a three-dimensional tensor defined as

$$\mathrm{h}^{(3)}(i,j,k) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{S}_i(\tau)\,\mathcal{S}_j(\tau)\,\mathcal{S}_k(\tau)\,\mathrm{d}\tau \tag{27}$$

where $i,j,k = 1,\dots,2N_H+1$, and $\mathcal{S} = [1 \;\; \cos(\tau) \;\; \dots \;\; \cos(N_H\tau) \quad \sin(\tau) \;\; \dots \;\; \sin(N_H\tau)]$.

An improved version of the three-dimensional tensor $\mathrm{h}^{(3)}$ is given as

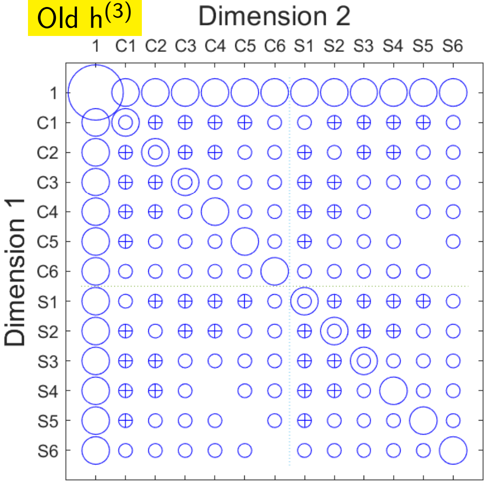
$$\mathrm{h}^{(3)}(i,j,k) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{S}_i(\tau)\,\mathcal{S}_j(\tau)\,\widehat{\mathcal{S}}_k(\tau)\,\mathrm{d}\tau \tag{28}$$

where $i,j = 1,\dots,2N_H+1$, $k = 1,\dots,4N_H+1$, and $\widehat{\mathcal{S}}$ is given as

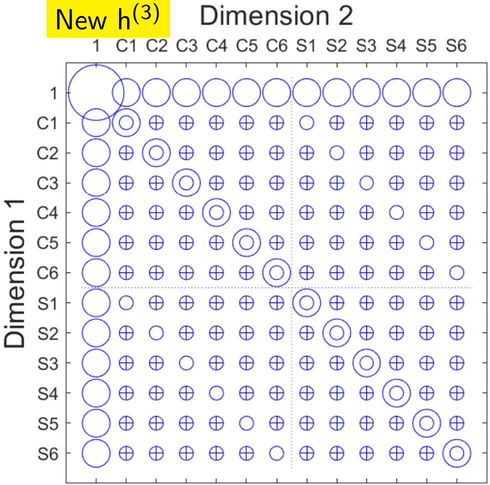
$$\widehat{\mathcal{S}} = [1 \quad \cos(\tau) \;\; \dots \;\; \cos(2N_H\tau) \quad \sin(\tau) \;\; \dots \;\; \sin(2N_H\tau)]$$

Visualization of the two versions of the three-dimensional tensor $h^{(3)}$

Old $h^{(3)}$

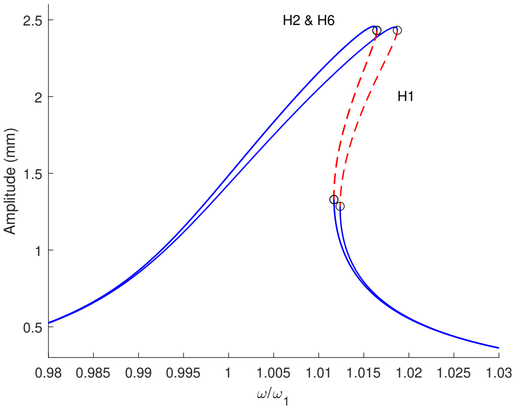


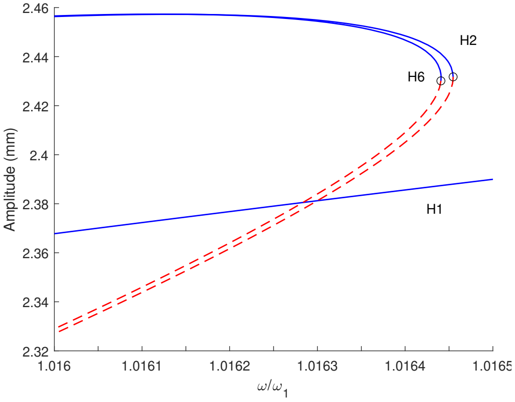
New $h^{(3)}$



- Finite element model of a clamped-clamped beam
  Young's modulus $E = 205$ Gpa, mass density $\rho = 7800$ Kg/m^3
  cross section 0.01m$\times$ 0.01m, length $L = 150\sqrt{I/A}$
  C $= \alpha$M, where $\alpha = 15.58$, damping ratio 1% for 1st flexural mode
  20 Euler-Bernoulli beam element
  Geometric nonlinearity from mid-plane stretching effect
- For the old version of h$^{(3)}$ and $N_H = 1$, response analysis fails around the resonance peak
- For the new version of h$^{(3)}$ and $N_H = 1$, response analysis and stability analysis are performed smoothly.

Results for response and stability with the new version of $h^{(3)}$ and $N_H = 1, 2, 6$





- Tensorial implementation of the incremental harmonic balance method
- An improved version of the three-dimensional tensor $h^{(3)}$
  - Accurate computation of Jacobian matrix up to cubic order
- Results for response and stability analyses can be obtained with a small value of N_H
- Future work
  - applications in sensitivity analysis and gradient based structural optimization
  - further reduction of computational cost of frequency-domain stability analysis