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Group Connectivity and Group Coloring
A Ph.D. thesis in Graph Theory

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Group Connectivity and Group Coloring

A Ph.D. thesis in Graph Theory

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Kongens Lyngby 2020

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Abstract

This Ph.D. thesis in mathematics concerns group connectivity and group coloring within the general area of graph theory. The contribution consists of the following parts:

Weak group connectivity and weak group coloring. We define the weak group connectivity number and the weak group chromatic number and show that these differ from the ordinary group connectivity number and the ordinary group chromatic number. We find examples of graphs where the group connectivity number (resp. the group chromatic number) is almost twice as big as the weak group connectivity number (resp. the weak group chromatic number). We show, however, that there is an upper bound on the group connectivity number (resp. the group chromatic number) given the weak group connectivity number (resp. the weak group chromatic number). We also present some results for groups of small order.

Many $\mathbb{Z}_6$-flows and exponentially many $\mathbb{Z}_8$-flows. Dvořák, Mohar and Šámal proved that there are exponentially many nowhere-zero flows under some conditions. Group connectivity can be seen as an extension of nowhere-zero flows and in this setting we show that there are exponentially many Γ-flows in 3-edge-connected graphs for Abelian groups Γ of order at least 8. We also show that there are at least $2^{\sqrt{\ell}/\log \ell}$ $\mathbb{Z}_6$-flows in 3-edge-connected graphs G where $\ell = |E(G)| - |V(G)| \geq 11$.

Exponentially many $\mathbb{Z}_5$-colorings and $\mathbb{Z}_5$-flows. We show that there are exponentially many $\mathbb{Z}_5$-colorings in planar simple graphs. The proof is based on an argument by Thomassen previously used to show that there are exponentially many 5-list-colorings, but adapting the proof to group coloring turns out to be highly nontrivial. The dual result is that there are exponentially many $\mathbb{Z}_5$-flows in planar 3-edge-connected graphs.

Group colorings with $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. We show that there are planar simple graphs which are $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable. These graphs are the result of the Hajos' construction on a non-simple graph with the same property (found by Hušek, Mohelníková and Šámal) and a $\mathbb{Z}_4$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-critical graph. We also show that there are planar simple graphs which are $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, but not $\mathbb{Z}_4$-colorable. The proof is not using a computer, which makes it the first of its kind. In fact, we find graphs which are $\mathbb{Z}_2^k$-colorable, but not Γ-colorable for any other Abelian group Γ of order $|\Gamma| = 2^k$. Since the graphs described above are all planar, we get dual results for group connectivity.

Resumé

Denne phd-afhandling i matematik omhandler gruppesammenhæng og gruppefarvning inden for området grafteori. Bidraget består af følgende dele:

Svag gruppesammenhæng og svag gruppefarvning. Vi definerer det svage gruppesammenhængstal og det svage gruppefarvningstal og viser, at disse er forskellige fra de sædvanlige gruppesammenhængstal og gruppefarvningstal. Vi finder eksempler på grafer, hvor gruppesammenhængstallet (hhv. gruppefarvningstallet) er næsten dobbelt så stort som det svage gruppesammenhængstal (hhv. det svage gruppesammenhængstal). Vi viser imidlertid, at der findes en øvre grænse på gruppesammenhængstallet (hhv. gruppefarvningstallet) givet det svage gruppesammenhængstal (hhv. det svage gruppefarvningstal). Vi præsenterer også nogle resultater for grupper med få elementer.

Mange $\mathbb{Z}_6$-strømninger og eksponentielt mange $\mathbb{Z}_8$-strømninger. Dvořák, Mohar og Šámal viste, at der findes eksponentielt mange intetsteds-neutrale strømninger under visse betingelser. Gruppesammenhæng kan ses som en udvidelse af intetsteds-neutrale strømninger og i denne sammenhæng viser vi, at der findes eksponentielt mange Γ-strømninger i 3-kantsammenhængende grafer for Abelske grupper Γ af orden mindst 8. Vi viser også, at der findes mindst $2^{\sqrt{\ell}/\log \ell}$ $\mathbb{Z}_6$-strømninger i 3-kantsammenhængende grafer G hvor $\ell = |E(G)| - |V(G)| \geq 11$.

Eksponentielt mange $\mathbb{Z}_5$-farvninger og $\mathbb{Z}_5$-strømninger. Vi viser, at der findes eksponentielt mange $\mathbb{Z}_5$-farvninger i plane simple grafer. Beviset er baseret på et argument af Thomassen, som tidligere er blevet brugt til at vise, at der findes eksponentielt mange 5-listefarvninger, men det viser sig at være langt fra trivielt at tilpasse til gruppefarvning. Det duale resultat viser, at der findes eksponentielt mange $\mathbb{Z}_5$-strømninger i plane 3-kantsammenhængende grafer.

Gruppefarvninger med $\mathbb{Z}_4$ og $\mathbb{Z}_2 \times \mathbb{Z}_2$. Vi viser, at der findes plane simple grafer, som er $\mathbb{Z}_4$-farvelige, men ikke $\mathbb{Z}_2 \times \mathbb{Z}_2$-farvelige. Disse grafer er resultatet af Hajos' konstruktion på en ikke-simpel graf med samme egenskab (fundet af Hušek, Mohelníková og Šámal) og en $\mathbb{Z}_4$- og $\mathbb{Z}_2 \times \mathbb{Z}_2$-kritisk graf. Vi viser også, at der findes plane simple grafer som er $\mathbb{Z}_2 \times \mathbb{Z}_2$-farvelige, men ikke $\mathbb{Z}_4$-farvelige. Beviset benytter ikke en computer, hvilket gør det til det første af sin slags. Vi finder faktisk grafer, som er $\mathbb{Z}_2^k$-farvelige, men ikke Γ-farvelige for nogen anden Abelsk gruppe Γ af orden $|\Gamma| = 2^k$. Siden graferne beskrevet ovenfor alle er plane, får vi duale resultater for gruppesammenhæng.

Preface

This thesis was prepared at the department of Applied Mathematics and Computer Science at the Technical University of Denmark in fulfillment of the requirements for acquiring a Ph.D. degree in mathematics. The Ph.D. study was conducted under the supervision of Professor Carsten Thomassen and co-supervisors Professor Inge Li Gørtz and Associate Professor Eva Rotenberg and was financed by a DTU stipend.

The thesis presents selected results within the general topic of graph theory and, more specifically, group connectivity and group coloring. All included results were obtained during my enrollment as a Ph.D. student from January 2018 to December 2020, and all results are to be submitted to, already submitted to, or already published in peer-reviewed journals:

- R. Langhede and C. Thomassen. Group connectivity and group coloring: small groups versus large groups. The Electronic Journal of Combinatorics, 2020, vol. 27.
- M. DeVos, R. Langhede, B. Mohar and R. Šámal. Many flows in the group connectivity setting. arXiv:2005.09767, 2020. In submission.
- R. Langhede and C. Thomassen. Exponentially many $\mathbb{Z}_5$-colorings in simple planar graphs. arXiv:2011.12163, 2020. In submission.

Furthermore, some of the results are based on a paper in preparation under the working title “Group coloring and group connectivity with non-isomorphic groups of the same order” (joint work with Carsten Thomassen).

Rikke Marie Langhede
Kongens Lyngby, December 11, 2020

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Contents

CHAPTER 1

Introduction

This work is within graph theory and concerns colorings and flows in graphs. Graph coloring is a major subfield in graph theory which started in the middle of the 19th century with the famous Four-Color Problem. Today, it also has many applications to optimization theory as many scheduling problems can be formulated as graph coloring problems. The Four-Color Problem was proved in 1976 by Appel and Haken [1] by a computer-assisted proof, but other long-standing coloring problems remain open, and even today researchers are working on finding a shorter proof of the Four-Color Theorem not relying on computers.

In this thesis we extend the knowledge about certain aspects of graph coloring and the related topic graph flows. We are particularly interested in three aspects: group coloring and group flows with groups of small order, group colorings and group flows with different groups, and showing that there are many group colorings and group flows.

1.1 Fundamental definitions and terminology

Throughout this thesis we will let G denote a graph, while $V(G)$ denotes to its set of vertices and $E(G)$ denotes its set of edges. All graphs in this thesis are finite and may contain multiple edges, but no loops. Below we define some concepts which are fundamental in group colouring and group connectivity. For other basic graph theory definitions and terminology we refer the reader to [4] and [7].

1.1.1 Group coloring

For the purpose of this section we let G be a graph, we let Γ be an Abelian group, and we let D be any orientation of G.

Definition 1.1. G is said to be *k-colorable* if there exists a function $c : V(G) \to \{1, \dots, k\}$ such that $c(v) \neq c(v)$ for all $uv \in E(G)$.

We say that the function c is a *k-coloring*.

Definition 1.2. G is said to be *k-list-colorable* if for any list assignment $L(v) \subseteq \mathbb{N}$ for $v \in V(G)$ with $|L(v)| = k$ there exists a function $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for all $v \in V(G)$ and $c(v) \neq c(v)$ for all $uv \in E(G)$.

We say that the function c is a *k-list-coloring* or an *L-coloring*.

Definition 1.3. G is said to be *Γ-colorable* if for any function $\varphi : E(G) \to \Gamma$ there exists a vertex coloring $c : V(G) \to \Gamma$ such that $c(w) - c(u) \neq \varphi(uw)$ for each $uw \in E(D)$.

We say that φ is a *forbidden function*.

Definition 1.4. Let $\varphi : E(G) \to \Gamma$ be a forbidden function. We say that $c : V(G) \to \Gamma$ is a *(Γ, φ)-coloring* (and G is said to be *(Γ, φ)-colorable*) if $c(w) - c(u) \neq \varphi(uw)$ for each $uw \in E(D)$.

In this thesis we will refer to a (Γ, φ)-coloring as a *Γ-coloring* or, more general, a *group coloring*. We thus distinguish ordinary vertex colorings and (Γ, φ)-colorings by saying that the latter is a group coloring, even though we may use group elements instead of integers to color the vertices in an ordinary coloring.

Definition 1.5. The *chromatic number* of G, $\chi(G)$, is the smallest number k such that G is k-colorable.

Definition 1.6. The *list chromatic number* of G, $\chi_l(G)$, is the smallest number k such that G is k-list-colorable.

Definition 1.7. The *group chromatic number* of G, $\chi_g(G)$, is the smallest number k such that G is Γ-colorable for every Abelian group Γ of order at least k.

Finally a chromatic definition which grew out of the present work.

Definition 1.8. The *weak group chromatic number* of G, $\chi_{wg}(G)$, is the smallest number k such that G is Γ-colorable for some Abelian group Γ of order k.

1.1.2 Group flows and group connectivity

For the purpose of this section we let G be a graph, we let Γ be an Abelian group, and we let D be any orientation of G.

Definition 1.9. A function $f : E(G) \to \{-k+1, \ldots, -1, 1, \ldots, k-1\}$ is said to be a *nowhere-zero k-flow* in G if $\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) = 0$ for all $v \in V(G)$.

In general, a function $f : E(G) \to \{-k+1, \ldots, k-1\}$ satisfying $\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) = 0$ for all $v \in V(G)$ is called a *flow*. A nowhere-zero flow may also be constructed using elements of $\Gamma \setminus \{0\}$, and then we will call f a *nowhere-zero Γ-flow*. A famous theorem by Tutte (see Theorems 1.34 and 1.35) states that a graph has a k-flow if and only if it has a flow with elements of some (and in fact each) Abelian group Γ with $|\Gamma| = k$ (see e.g. [7]).

Definition 1.10. G is said to be Γ-*connected* if for any function $\beta : V(G) \to \Gamma$ such that $\sum_{v \in V(G)} \beta(v) = 0$ there exists a function $f : E(G) \to \Gamma \setminus \{0\}$ which has $\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) = \beta(v)$ for all $v \in V(G)$.

This definition was introduced by Jaeger, Linial, Payan and Tarsi in [17]. We will primarily use another equivalent definition (the equivalence is proven in [17]):

Definition 1.11. G is Γ-connected if for any function $\delta : E(G) \to \Gamma$ there exists a flow $f : E(G) \to \Gamma$ which has $f(e) \neq \delta(e)$ for all $e \in E(G)$.

We say that δ is a *forbidden function*.

Definition 1.12. Let $\delta : E(G) \to \Gamma$ be a forbidden function. We say that $f : E(G) \to \Gamma$ is a (Γ, δ)-*flow* if f is a flow and has $f(e) \neq \delta(e)$ for all $e \in E(G)$.

In this thesis we will refer to a (Γ, δ)-flow as a Γ-*flow* or, more general, a *group flow*. Thus a nowhere-zero Γ-flow is a Γ-flow, but a Γ-flow is generally not a nowhere-zero Γ-flow. We will also refer to a nowhere-zero Γ-flow as a *nowhere-zero group flow*.

Definition 1.13. The *flow number* of G, $\Lambda(G)$, is the smallest number k such that G has a nowhere-zero k-flow.

Definition 1.14. The *group connectivity number* of G, $\Lambda_g(G)$, is the smallest number k such that G is Γ-connected for every Abelian group Γ of order at least k.

Finally a definition which grew out of the present work.

Definition 1.15. The *weak group connectivity number* of G, $\Lambda_{wg}(G)$, is the smallest number k such that G is Γ-connected for some Abelian group Γ of order k.

1.1.3 Orientations

It is easy to see that if a graph is Γ-connected under some orientation, then it is Γ-connected under any orientation since whenever an edge is reversed we may replace the flow value by its inverse and thus maintain the flow property. Likewise, if the graph has a (Γ, δ)-flow, is Γ-colorable, or has a (Γ, φ)-coloring, respectively, under some orientation, then the graph has a (Γ, δ)-flow, is Γ-colorable, or has a (Γ, φ)-coloring, respectively, under any orientation. Thus the specific orientation of a graph is not important.

Note on notation: Formally, $\varphi(uv)$ and $\delta(uv)$ is defined on every directed edge uv. But, we also write $\varphi(vu) = -\varphi(uv)$ and $\delta(vu) = -\delta(uv)$. This way it is redundant to mention the specific orientation of the graph.

1.1.4 Duality of group colorings and group connectivity

It is well-known that colorings and nowhere-zero flows are dual concepts:

Theorem 1.16 (Tutte [43])**.** *For every dual pair G, G^* of plane graphs, $\chi(G) = \Lambda(G^*)$.*

The authors of [17] define group colorability and prove (using matroid theory) that is is indeed the dual concept of group connectivity. We include here a direct proof for completeness.

Theorem 1.17. *Let Γ be an Abelian group. For any planar and 2-edge-connected graph G, G is Γ-connected if and only if the dual graph G^* is Γ-colorable.*

Proof. Give G and G^* orientations such that if $e \in E(G)$ is given some orientation in G then the orientation of the corresponding edge $e^* \in E(G^*)$ is obtained by turning e and its ends clockwise onto e^*. Note that any sums of values on edges in the following will be taken with respect to the orientation of the edges - if an edge is traversed along its orientation (resp. towards its orientation) the value will contribute positively (resp. negatively) to the sum.

We first prove the following statement: Let $f : E(G) \to \Gamma$ and $g : E(G^*) \to \Gamma$ such that $f(e) = g(e^*)$. Then f is a flow in G if and only g satisfies $\sum_{e^* \in C} g(e^*) = 0$ for all cycles C in G^*. This is easy to see since cuts in G becomes cycles in G^* and vice versa.

Suppose G is Γ-connected. We will prove that G^* is Γ-colorable, so let $\varphi : E(G^*) \to \Gamma$ be given. Define $\delta : E(G) \to \Gamma$ such that $\delta(e) = \varphi(e^*)$ for all $e \in E(G)$. Then there exists a Γ-flow $f : E(G) \to \Gamma$ such that $f(e) \neq \delta(e)$. We will construct a Γ-coloring $c : V(G^*) \to \Gamma$ of G^*. Let v_0 be any vertex in G^* and define $c(v_0) = 0$. For all other vertices $v \in V(G^*) \setminus \{v_0\}$ let P_v be a path from v_0 to v and define $c(v) = \sum_{e \in P_v} f(e^*)$. We first see that c is well-defined since if P, P' are different paths from v_0 to v then $P \cup P'$ forms a closed walk C in G^* hence $\sum_{e \in C} f(e^*) = 0$ as f is a flow. Thus $\sum_{e \in P} f(e^*) = \sum_{e \in P'} f(e^*)$. Furthermore, if $e_0 = uv \in E(G^*)$ (directed towards v) and P_v is a path from v_0 to v through e_0 we have

$$c(v) - c(u) = \sum_{e \in P_v} f(e^*) - \sum_{e \in P_v \setminus \{e_0\}} f(e^*) = f(e_0^*) \neq \delta(e_0^*) = \varphi(e_0). \tag{1.1}$$

Thus c is a Γ-coloring of G^*.

Now suppose G^* is Γ-colorable. We will prove that G is Γ-connected, so let $\delta : E(G) \to \Gamma$ be given. Define $\varphi : E(G^*) \to \Gamma$ such that $\varphi(e^*) = \delta(e)$ for all $e^* \in E(G^*)$. Then there exists a Γ-coloring $c : V(G^*) \to \Gamma$ such that $c(v) - c(u) \neq \varphi(e)$ for all $e = uv \in E(G^*)$ (directed towards v). We will construct a Γ-flow $f : E(G) \to \Gamma$ in G as follows: For each edge $e \in E(G)$ let u, v be the ends of the corresponding edge e^* (directed towards v) in G^*. Then we let $f(e) = c(v) - c(u)$. Clearly $f(e) \neq \varphi(e^*) = \delta(e)$. Also, if D is any minimal cut in G^* then the dual edges form a cycle, so $\sum_{e \in D} f(e^*) = 0$ by definition. Since any cut is the disjoint union of minimal cuts f is a Γ-flow in G such that $f(e) \neq \delta(e)$. □

1.2 A brief survey

In this section we will make a brief status on the progress of some of the graph problems relevant for this thesis.

1.2.1 Coloring, list coloring and group coloring

As mentioned above, the following result is fundamental in graph coloring.

Theorem 1.18 (Appell & Haaken, 1976 [1]). *Every planar graph is 4-colorable.*

Before that, a proof of the easier case of 5-coloring was found.

Theorem 1.19 (Heawood, 1890 [13]). *Every planar graph is 5-colorable.*

Another well-known result is the following.

Theorem 1.20 (Grötsch, 1959 [11]). *Every planar graph not containing a triangle is 3-colorable.*

Thomassen proved the following list-coloring versions of the 3-Color Theorem and 5-Color Theorem.

Theorem 1.21 (Thomassen, 1994 [39]). *Every planar graph is 5-list-colorable.*

Theorem 1.22 (Thomassen, 1995 [37]). *Every planar graph of girth at least 5 is 3-list-colorable.*

Later, Thomassen proved that there are an exponential number of list-colorings.

Theorem 1.23 (Thomassen, 2007 [40]). *Every planar graph with n vertices has at least $2^{n/9}$ distinct 5-list-colorings.*

Theorem 1.24 (Thomassen, 2007 [41]). *Every planar graph of girth at least 5 with n vertices has at least $2^{n/10000}$ distinct 3-list-colorings.*

Thomassen's 5-List-Color Theorem and 3-List-Color Theorem also adopts easily to group coloring as noted e.g. in [30]. Stronger versions of these theorems appear in [25] and [22]. A proof of Theorem 1.25 can be found in the beginning of Chapter 4.

Theorem 1.25. *Every simple planar graph is $\mathbb{Z}_5$-colorable.*

Theorem 1.26. *Every planar graph of girth at least 5 is $\mathbb{Z}_3$-colorable.*

Several results on group colorings can be deduced by dualizing results on group connectivity (see the following section).

Theorem 1.27 (Jaeger, Linial, Payan & Tarsi, 1992 [17]). *Every simple planar graph is Γ-colorable for any Abelian group Γ of order at least 6.*

Theorem 1.28 (Jaeger, Linial, Payan & Tarsi, 1992 [17]). *Every simple planar triangle-free graph is* Γ*-colorable for any Abelian group* Γ *of order at least 4.*

Theorem 1.29 (Lóvasz, Thomassen, Wu & Zhang, 2013 [32]). *Every planar graph of girth at least 6 is* Γ*-colorable for any Abelian group* Γ *of order at least 3.*

The last three theorems also follow from a degeneracy argument.

Note, that the group $\mathbb{Z}_2$ is uninteresting when group coloring:

Theorem 1.30 (Lai & Zhang, 2002 [26]). *A connected graph* G *is* $\mathbb{Z}_2$*-colorable if and only if* G *is a tree.*

1.2.2 Flows and group connectivity

Tutte made the following famous conjectures on nowhere-zero flows:

Conjecture 1.31 (Tutte, 1954 [43]). Every 2-edge-connected graph has a nowhere-zero 5-flow.

Conjecture 1.32 (Tutte, 1966 [45]). Every 2-edge-connected graph which does not contain the Petersen graph as a minor has a nowhere-zero 4-flow.

Conjecture 1.33 (Tutte, 1972 (see e.g. [4])). Every 4-edge-connected graph has a nowhere-zero 3-flow.

Tutte also proved the following theorems.

Theorem 1.34 (Tutte, 1954 [43]). *A graph has a nowhere-zero* k*-flow if and only if it has a nowhere-zero* $\mathbb{Z}_k$*-flow.*

Theorem 1.35 (Tutte, 1954 [43]). *If* Γ *and* Γ' *are Abelian groups such that* $|\Gamma| = |\Gamma'|$ *then a graph has a nowhere-zero* Γ*-flow if and only if it has a nowhere-zero* Γ'*-flow.*

Thus, it is not necessary to distinguish nowhere k-flows from nowhere-zero Γ-flows when $|\Gamma| = k$.

Several partial results on Tutte's Flow Conjectures have been achieved so far.

Theorem 1.36 (Seymour, 1981 [36]). *Every 2-edge-connected graph has a nowhere-zero 6-flow.*

Theorem 1.37 (Jaeger, 1979 [16]). *Every 4-edge-connected graph has a nowhere-zero 4-flow.*

Theorem 1.38 (Lóvasz, Thomassen, Wu & Zhang, 2013 [32]). *Every 6-edge-connected graph has a nowhere-zero 3-flow.*

Recently, the number of nowhere-zero flows has been investigated.

Theorem 1.39 (Dvořák, Mohar & Šámal, 2019 [8]). *Every 2-edge-connected graph with n vertices and m edges has $2^{2(m-n)/9}$ nowhere-zero $\mathbb{Z}_6$-flows.*

Theorem 1.40 (Dvořák, Mohar & Šámal, 2019 [8]). *Every 4-edge-connected graph with n vertices has $2^{n/250}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows.*

Theorem 1.41 (Dvořák, Mohar & Šámal, 2019 [8]). *Every 6-edge-connected graph with n vertices has $2^{(n-2)/12}$ nowhere-zero $\mathbb{Z}_3$-flows.*

Jaeger, Linial, Payan and Tarsi made the following conjectures on group connectivity which generalize Tutte's 5-Flow and 3-Flow Conjectures.

Conjecture 1.42 (Jaeger, Linial, Payan & Tarsi, 1992 [17]). Every 3-edge-connected graph is $\mathbb{Z}_5$-connected.

Conjecture 1.43 (Jaeger, Linial, Payan & Tarsi, 1992 [17]). Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.

The $\mathbb{Z}_5$-Connectivity Conjecture implies the 5-Flow Conjecture 1.31 by a simple reduction to 3-edge-connected graphs first made by Seymour in [36]. The $\mathbb{Z}_3$-Connectivity Conjecture implies the 3-Flow Conjecture 1.33 by a result by Kochol in [19].

Partial results have also been achieved for these two conjectures.

Theorem 1.44 (Jaeger, Linial, Payan & Tarsi, 1992 [17]). *Every 3-edge-connected graph is Γ-connected for any Abelian group Γ of order at least 6.*

Theorem 1.45 (Jaeger, Linial, Payan & Tarsi, 1992 [17]). *Every 4-edge-connected graph is Γ-connected for any Abelian group Γ of order at least 4.*

Theorem 1.46 (Lóvasz, Thomassen, Wu & Zhang, 2013 [32]). *Every 6-edge-connected graph is Γ-connected for any Abelian group Γ of order at least 3.*

By duality Theorems 1.25 and 1.26 also applies to group connectivity.

Theorem 1.47. *Every planar 3-edge-connected graph is $\mathbb{Z}_5$-connected.*

Theorem 1.48. *Every planar 5-edge-connected graph is $\mathbb{Z}_3$-connected.*

Note, that $\mathbb{Z}_2$-connectivity is uninteresting:

Theorem 1.49 (Jaeger, Linial, Payan & Tarsi, 1992 [17]). *A graph G is $\mathbb{Z}_2$-connected if and only if $|V(G)| = 1$.*

1.3 How to construct colorings and flows

In this section we will review some of the most common methods to construct group colorings and group flows in graphs. It is not at all trivial to do so. Even quite small graphs can prove difficult to group color or give group flow when none of the techniques below are directly applicable.

The most natural way to color graphs is to use what we call a *degeneracy type argument*. We say that a graph is *d-degenerate* if every subgraph contains a vertex of degree at most d. If this is the case, then it is possible to order the vertices and color them in this order such that every time we color a vertex it only has at most d colored neighbors. This works for ordinary coloring as well as list colorings. We define the *coloring number* $Col(G)$ of a graph G to be the smallest number d such G is $(d-1)$-degenerate. The argument above implies the following well-known fact:

Theorem 1.50 (See for instance Diestel [7]). *For any graph G,*

$$\chi(G) \leq \chi_l(G) \leq Col(G)$$

The argument works perfectly well for group colorings too, thus:

Theorem 1.51 (Langhede & Thomassen, 2020 [30]). *For any graph G,*

$$\chi(G) \leq \chi_{wg}(G) \leq \chi_g(G) \leq Col(G)$$

In Chapter 4 (based on [28]) we see the technique in action in several proofs when we construct $\mathbb{Z}_5$-colorings of specific families of graphs.

Degeneracy may also be used to prove the existence of a coloring without actually finding a specific coloring. In Chapter 2 we prove that all $\mathbb{Z}_3$-colorable graphs are 5-degenerate by a counting argument, thus proving that such graphs are also Γ-colorable for Abelian groups of order at least 6.

Another common technique is to reduce the problem to smaller graphs. If possible, this allows us to use induction. The strength of induction in the case of group coloring (and list coloring) is that we may sometimes prove a stronger assertion by proving it for a more flexible family of graphs. The crucial point is to find the exact right statement to prove. In Chapter 4 (based on [28]) we show that all simple planar graphs are $\mathbb{Z}_5$-colorable by only considering planar near-triangulations with two precolored vertices. For these graphs we can always reduce the coloring problem to coloring smaller subgraphs. Also in Chapter 4 the main theorems are proven by induction by considering the smallest counterexample. We deduce several properties of such a graph by reducing to smaller (colorable) graphs and conclude that no such graph can exist.

Recall, that there are two possible (equivalent) definitions of group connectivity, Definitions 1.10 and 1.11. Both definitions are used in the literature, and both definitions are valuable to keep in mind. Sometimes, it will be easy to prove one statement about group connectivity with one of them, but very tedious (or practically impossible) with the other. In Chapter 2 and Chapter 3 we use the second definition, whereas the first definition was used by Lovász et al. in [32].

The most basic way of constructing nowhere-zero flows is to send flow through cycles until all edges have been given non-zero flow. For group flows we have to be a bit more careful when sending flow trough a cycle, since we can only be sure to find a flow for which no edge has a forbidden value if the cycle has fewer edges than the number of elements in the group. Alternatively we have to know that the flow value will not be forbidden on any edge. Jaeger et al. use this method in [17] to prove Theorems 1.44 and 1.45.

As with coloring, trying to use induction to construct a group flow is a common strategy. And in order to use the induction hypothesis we need to be able to reduce the problem to a smaller one. The following observation shows one way of doing so.

Observation 1.52 (Lai, 2000 [21])**.** If H is a subgraph of G and both G/H and H are Γ-connected, then G is Γ-connected as well.

Another important technique is to use the duality between coloring and flow (Theorem 1.17). If the graph is planar and we have difficulties showing that it is Γ-colorable, we can instead show that the dual graph is Γ-connected, or vice versa. In Chapter 4 and 5 we get results for planar graphs which are valid both for group coloring and group flows by proving it just for group colorings.

For further information on group flows and group coloring techniques, see e.g. [26] or [23]. For group connectivity properties of certain graphs, see e.g. [5].

1.4 Outline

In addition to this general introduction, the thesis consists of five chapters.

- Chapter 2 contains the paper "Group connectivity and group coloring: small groups versus large groups" [30] in Sections 2.1 to 2.7, as well as a preface which describes the most important techniques used in the article. This chapter is about the weak group connectivity number and the weak group chromatic number, and how these differ from the ordinary group connectivity number and group chromatic number.

- Chapter 3 contains the paper "Many flows in the group connectivity setting" [6] in Sections 3.1 to 3.9, as well as a preface which describes the most important techniques used in the article. This chapter is about the number of group flows for groups of order at least 6.

- Chapter 4 contains the paper "Exponentially many $\mathbb{Z}_5$-colorings in simple planar graphs" [28] in Section 4.1 to 4.7, as well as a preface which describes the most important techniques used in the article. This chapter is about the number of group colorings using the group $\mathbb{Z}_5$.

- Chapter 5 contains new work on group colorings using the groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. This chapter contains examples of simple planar graphs which $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, and vice versa. The paper "Group coloring and group connectivity with non-isomorphic groups of the same order" [29] (under preparation) is based on this chapter.
- Chapter 6 is the conclusion of this thesis and contains reflections on future directions for research within group connectivity and group coloring.

CHAPTER 2

Group connectivity and weak group connectivity

This chapter is based on the paper [30] by L. and Thomassen.

Ordinary colorings, list colorings and nowhere-zero flows all have the property that if a graph is k-colorable, k-list-colorable or has a k-flow, then it is also $k+1$-colorable, $k+1$-list-colorable or has a $k+1$-flow, respectively. Group flows and group colorings do not have this property. In this chapter we introduce the weak group connectivity number and the weak group chromatic number, and we try to answer the following questions:

- How big is the difference between group connectivity and weak group connectivity? And how big is the difference between group colorability and weak group colorability?
- What can we say about the relation between Γ-connectivity and Γ'-connectivity for Abelian groups Γ, Γ' where $|\Gamma|, |\Gamma'|$ are small? And what can we say about the relation between Γ-colorability and Γ'-colorability for Abelian groups Γ, Γ' where $|\Gamma|, |\Gamma'|$ are small?

For planar graphs the questions are dual, but in general there are big differences between group connectivity and group colorability.

Techniques

Large difference between the group connectivity number and the weak group connectivity number

In the paper we investigate how big the difference between the group connectivity number and the weak group connectivity number may be.

In order investigate this we construct the graph G_{k_1,k_2} (see Figure 2.1) which consists of paths between two vertices, where k_1 is the number of paths and k_2 is the length of each path. For $k_1 = p$ and $k_2 = 2^{k-1}$, where k is any natural number and p is the smallest prime such that $p > 2^{k-1}$, G_{k_1,k_2} is $\mathbb{Z}_p$-connected, but not $\mathbb{Z}_2^k$-connected. Its dual is $\mathbb{Z}_p$-colorable, but not $\mathbb{Z}_2^k$-colorable.

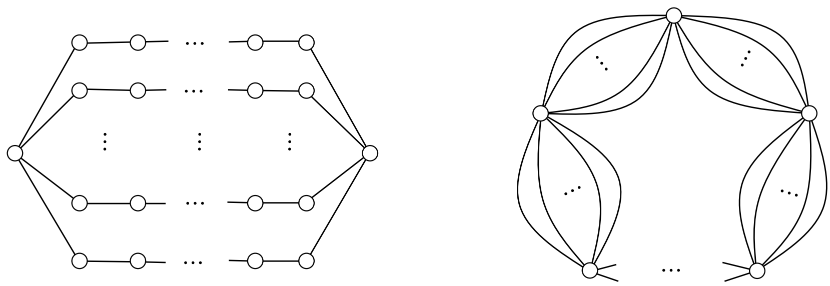

Figure 2.1: The graph G_{k_1,k_2} (on the left) and its dual (on the right).

The cyclicity of a graph

One of contributions of the paper is the definition of the *cyclicity* of a graph. The cyclicity of a graph is the number of edges in a largest set of edges such that each pair in the set is a 2-edge-cut in the graph. The first important property is that a graph cannot be Γ-connected if $|\Gamma|$ is smaller than or equal to the cyclicity of the graph.

Another aspect of the cyclicity is the construction of a new graph whose group flows are equivalent to those in the original graph. This is done by grouping all edges of a cycle-equivalence class together to form a path (see Figure 2.2). The resulting graph will then be a subdivision of a 3-edge-connected graph. This construction is central to the proof of Theorem 2.17.

Modifying an argument by Jaeger et al.

The proof of the main theorem, Theorem 2.15, is based on the technique used in the proof of Theorem 3.2 in the paper [17] by Jaeger et al. They prove that the edges of a cubic 3-edge-connected graph (with one vertex removed) can be divided into two parts, a spanning tree T and the remaining edges R such that G/R is 2-constructible, i.e. can be constructed recursively from an edgeless graph by adding two new edges between disjoint components. First they create a flow which on T avoid the forbidden flow value, as well as this value plus/minus a constant x (for the group $\mathbb{Z}_6$, x is 3

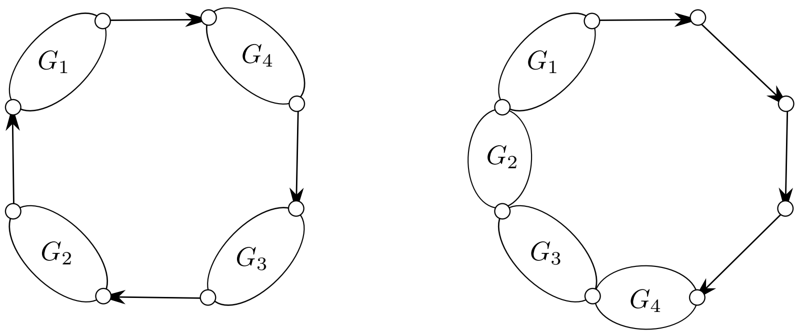

Figure 2.2: A graph of cyclicity 4 (on the left) and the resulting graph (on the right) after using the technique in the proof of Theorem 2.17.

since $-3 = 3$). Then they create a flow which send the flow x through edges of R with forbidden flow. The sum of the two flows is the wanted flow.

In our paper we allow each edge to have up to k forbidden values, where Jaeger et al. allow only one forbidden value. Therefore the first flow has to avoid a much larger set of values in order to be able to construct the second flow such that the edges of T does not get new forbidden flow-values. In particular, this set is created in a specific way to contain the sums of elements of a smaller set, which then can be send through edges of R with forbidden flow. This set is not necessary for Jaeger et al. who only need a single element to correct forbidden flow in edges of R. As a result, the size of the group needed to create a satisfying flow is dependent on k.

For $k = 1$ our result says that any 3-edge-connected graph is Γ-connected for $\Gamma > 8$, whereas Jaeger et al. get the result for $\Gamma \geq 6$. This is because the set of flow values to be avoided by Jaeger et al. consists of only two elements for the group $\mathbb{Z}_6$, whereas in our more general approach we need many more.

Group flows and group colorings for small groups

Lastly, the results for small groups should be mentioned. By a simple argument stating that $\mathbb{Z}_3$-connected graphs must have a certain number of edges and therefore two edge-disjoint spanning trees, we get that $\mathbb{Z}_3$-connected graphs are also Γ-connected for all Abelian groups Γ of order at least 4. By an argument about the number of vertices compared to edges, we get that a $\mathbb{Z}_3$-colorable graph is 5-degenerate and hence Γ-colorable for all groups Γ of order at least 6. And by a transformation of a $\mathbb{Z}_3$-coloring to a $\mathbb{Z}_5$-coloring, we see that $\mathbb{Z}_3$-colorable graphs are also $\mathbb{Z}_5$-colorable. All these result are proven with relatively simple, but surprisingly different arguments.

Group connectivity and group coloring: small groups versus large groups[1]

Abstract

A well-known result of Tutte says that if Γ is an Abelian group and G is a graph having a nowhere-zero Γ-flow, then G has a nowhere-zero Γ'-flow for each Abelian group Γ' whose order is at least the order of Γ. Jaeger, Linial, Payan, and Tarsi observed that this does not extend to their more general concept of group connectivity. Motivated by this we define $g(k)$ as the least number such that, if G is Γ-connected for some Abelian group Γ of order k, then G is also Γ'-connected for every Abelian group Γ' of order $|\Gamma'| \geq g(k)$. We prove that $g(k)$ exists and satisfies for infinitely many k,

$$(2 - o(1))k < g(k) \leq 8k^3 + 1.$$

The upper bound holds for all k. Analogously, we define $h(k)$ as the least number such that, if G is Γ-colorable for some Abelian group Γ of order k, then G is also Γ'-colorable for every Abelian group Γ' of order $|\Gamma'| \geq h(k)$. Then $h(k)$ exists and satisfies for infinitely many k,

$$(2 - o(1))k < h(k) < (2 + o(1))k \ln(k).$$

The upper bound (for all k) follows from a result of Král', Pangrác, and Voss. The lower bound follows by duality from our lower bound on $g(k)$ as that bound is demonstrated by planar graphs.

2.1 Introduction

Tutte's 5-Flow Conjecture states that any 2-edge-connected graph has a nowhere-zero 5-flow (see e.g. [4, 15, 18, 49]). Seymour [36] proved that every 2-edge-connected graph has a nowhere-zero 6-flow.

Jaeger et al. [17] introduced the concept of group-connectivity and proved that every 3-edge-connected graph is Γ-connected for any Abelian group Γ of size $|\Gamma| \geq 6$. This extends the 6-flow theorem since that theorem is easily reduced to the 3-connected case as pointed out by Seymour [36].

Tutte [46] (see also [49]) proved that if Γ is an Abelian group and G is a graph having a nowhere-zero Γ-flow, then G has a nowhere-zero Γ'-flow for each Abelian group Γ' whose order is at least the order of Γ. Jaeger et al. [17] observed that this does not extend to their more general concept of group connectivity. We prove that

[1] The content of the remaining part of this chapter is the paper [30] published in Electronic Journal of Combinatorics in 2020. This is joint work with Carsten Thomassen.

the statement becomes true if Γ' is sufficiently large compared to Γ, as explained in the Abstract.

If Γ is an Abelian group and G is a planar 2-edge-connected graph, then G is Γ-connected if and only if the dual graph G is Γ-colorable. This duality was part of the motivation for these concepts [17]. This suggests the definition of $g(k)$, $h(k)$ in the Abstract, and we prove:

$$(2 - o(1))k < g(k) \leq 8k^3 + 1, \tag{2.1}$$

and

$$(2 - o(1))k < h(k) < (2 + o(1))k \ln(k). \tag{2.2}$$

The lower bounds hold for infinitely many k, and we do not know if g, h are monotone. The upper bounds hold for all k. The upper bound on $h(k)$ follows from a result of Král' et al. [20]. The lower bound on $h(k)$ follows from the lower bound of $g(k)$ as that bound is demonstrated by planar graphs.

It was proved in [31] that $g(3) = 3$. We conjecture that $h(3) = 3$ and prove that $h(3) \leq 5$.

The *group chromatic number* (see e.g. [20, 26]) $\chi_g(G)$ is the smallest number k such that G is Γ-colorable for every Abelian group Γ of order at least k. We point out that there is another possible definition, namely the *weak group chromatic number* $\chi_{wg}(G)$ which is the smallest number k such that G is Γ-colorable for *some* Abelian group Γ of order k. Clearly

$$\chi(G) \leq \chi_{wg}(G) \leq \chi_g(G). \tag{2.3}$$

Thus our lower bound on $h(k)$ shows that $\chi_g(G)$ may be almost twice as big as $\chi_{wg}(G)$.

The *group connectivity number* (see e.g. [24]) $\Lambda_g(G)$ is the smallest number k such that G is Γ-connected for every Abelian group Γ of order at least k. Similarly, we define the *weak group connectivity number* $\Lambda_{wg}(G)$ which is the smallest number k such that G is Γ-connected for *some* Abelian group Γ of order k.

Our lower bound on $g(k)$ shows that $\Lambda_g(G)$ may be almost twice as big as $\Lambda_{wg}(G)$.

2.2 Group connectivity

We use essentially the terminology and notation in [4]. We allow a graph to have multiple edges but no loops. If v is a vertex in a directed graph, then $E^+(v)$ (respectively $E^-(v)$) denotes the set of edges going out from v (respectively going into v). Jaeger et al. [17] introduced the concept of group connectivity as follows.

Definition 2.1. Let Γ be an Abelian group. The graph G is said to be Γ-*connected* if the following holds: Given some orientation D of G and any function $\beta : V(G) \to \Gamma$ satisfying $\sum_{v\in V(G)} \beta(v) = 0$, there exists a function $f : E(G) \to \Gamma$ such that $\sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e) = \beta(v)$ for all $v \in V(G)$ and such that $f(e) \neq 0$ for all $e \in E(G)$.

Note that the direction of an edge is not important. Indeed, we may replace "some orientation" by "any orientation" in the definition because we may replace $f(e)$ by $-f(e)$, if necessary.

Jaeger et al. [17] gave the following criterion for group connectivity in terms of forbidden flow values.

Theorem 2.2. *The graph G is Γ-connected if and only if the following holds: Given any orientation D of G and any function $\varphi : E(G) \to \Gamma$, there exists a function $f : E(G) \to \Gamma$ which has $f(e) \neq \varphi(e)$ for all $e \in E(G)$ and $\sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e) = 0$ for all $v \in V(G)$.*

The function f in Theorem 2.2 is called a *flow*, and $f(e)$ is called a *flow value*. $\phi(e)$ is called a *forbidden flow value*.

2.3 A lower bound for g

We shall use the following lemma which is an easy exercise. A proof can be found in [17].

Lemma 2.3. *Let P be a cyclic group of prime order, let S be a non-empty proper subset of P, and let T be a subset of P which contains at least two elements. Then $|S + T| > |S|$.*

Let q, k be natural numbers. We define $G_{q,k}$ as the graph consisting of two vertices s, t connected by q internally disjoint paths of length k.

Theorem 2.4. *Let k be a natural number, and let p be the smallest prime $> 2^{k-1}+1$. Then $G_{q,2^{k-1}}$ is not $\mathbb{Z}_2^k$-connected for any odd q. $G_{q,2^{k-1}}$ is $\mathbb{Z}_p$-connected when $q \geq p$.*

Proof. We first prove that $G = G_{q,2^{k-1}}$ is not $\mathbb{Z}_2^k$-connected. By reversing directions, if necessary, we may assume that all edges are directed towards t. Let $\varphi : E(G) \to \mathbb{Z}_2^k$ be a function such that each path forbids all the 2^{k-1} elements having an even number of 1's. As the flow values on a directed path has to be the same on all edges of the path, the forbidden values imply that the flow value of any of the paths must have an odd number of 1's. Since G consists of an odd number of paths which all have a flow with an odd number of 1's, the in-flow in t (or, similarly, out-flow in s) can never sum to 0. Thus, G is not $\mathbb{Z}_2^k$-connected.

To prove that G is $\mathbb{Z}_p$-connected we consider any function $\varphi : E(G) \to \mathbb{Z}_p$, and again, we let all edges be directed towards t. The goal is to find a flow $f : E(G) \to \mathbb{Z}_p$

such that $f(e) \neq \varphi(e)$ for all $e \in E(G)$. As noted above f must have the same value on all edges of a path between s and t, and as each path has length 2^{k-1}, it has at least $p - 2^{k-1} \geq 3$ possible flow values of $\mathbb{Z}_p$. It remains to check that we can choose the values of f on each path such that the sum is 0 in s and t. Given $q \geq p$ subsets of $\mathbb{Z}_p$ of size at least 2, it follows from Lemma 2.3 that the sum of these contains all elements of $\mathbb{Z}_p$, in particular the sum contains 0. Thus we can choose f such that $f(e) \neq \varphi(e)$ for all $e \in E(G)$ and the sum in s and t is 0. □

Corollary 2.5. *Given any $\epsilon > 0$ there exists an infinite number of graphs which are Γ-connected for some group Γ of prime order, but not Γ'-connected for some group Γ' satisfying $|\Gamma'| = (2-\epsilon)|\Gamma|$. Hence $g(k) > (2 - o(1))k$ for infinitely many primes k.*

Proof. Let p_n denote the n'th prime, and let $g_n = p_{n+1} - p_n$. The Prime Number Theorem implies that for any $\epsilon > 0$ there exists a natural number N such that for any $n \geq N$, $g_n < \epsilon p_n$. Now choose n, k such that $N < p_n < 2^k$, and furthermore, p_n is the largest prime $< 2^k$. Then

$$p_{n+1} < 2^k + g_n < (1+\epsilon)2^k.$$

Put $p = p_{n+1}$. If q is any odd number $\geq p$ then, by Theorem 2.4, $G_{q,2^k}$ is $\mathbb{Z}_p$-connected, but not $\mathbb{Z}_2^{k+1}$-connected. It follows that $g(p) > 2(1-\epsilon)p$. □

There is a slight inaccuracy in the proof above, namely when $p = p_{n+1} = 2^k + 1$, that is, p is a Fermat prime. But this can happen only if k is a power of 2 which does not affect the correctness of Corollary 2.5.

2.4 The cyclicity of a graph

Definition 2.6. Let G be a 2-edge-connected graph. We say that two edges e_1, e_2 are *cycle-equivalent* if every cycle that contains one of e_1, e_2 also contains the other. It is easy to see that this defines an equivalence relation on $E(G)$ and that two distinct edges e_1, e_2 are cycle-equivalent if and only if the two edges form a *2-edge-cut*, that is, $G - e_1 - e_2$ is disconnected. We define the *cyclicity* of G, denoted $q(G)$, to be the size of a largest equivalence class. In particular, if G has no 2-edge-cuts (i.e. G is 3-edge-connected), then $q(G) = 1$.

The following result follows from Proposition 3.2 and Lemma 3.3 in [21]. For the sake of completeness we include a proof.

Proposition 2.7. *If G is Γ-connected, then $|\Gamma| > q(G)$.*

Proof. Suppose G is Γ-connected. Let $e_1, \ldots, e_{q(G)}$ be the edges in a largest cycle-equivalence class. Let C be a cycle which contains one, and hence all of $e_1, \ldots, e_{q(G)}$. By reversing directions, if necessary, we may assume that all edges in $e_1, \ldots, e_{q(G)}$ have the same direction when we traverse C. For any flow, all edges in $e_1, \ldots, e_{q(G)}$

have the same flow value. We now use Theorem 2.2. If $|\Gamma| \leq q(G)$ we can define $\varphi : E(G) \to \Gamma$ such that it is surjective on $e_1, \ldots, e_{q(G)}$, that is, all elements in Γ are forbidden on the edges $e_1, \ldots, e_{q(G)}$. Hence $|\Gamma| > q(G)$. □

We shall prove the following (see Theorem 2.17).

Theorem 2.8. *If G is 2-edge-connected, then G is Γ-connected for any Abelian group Γ of order $|\Gamma| > 8q(G)^3$.*

We can now combine Proposition 2.7 and Theorem 2.8 to get:

Corollary 2.9. *If G is Γ-connected for some Abelian group Γ, then G is Γ'-connected for any Abelian group Γ' of order $|\Gamma'| > 8|\Gamma|^3$.*

2.5 Flows in 3-edge-connected graphs with multiple forbidden flow-values

We shall use the following definition and theorems by Jaeger et al. [17]:

Definition 2.10. A *2-constructible* graph G is defined recursively as follows.
(i) The graph with one vertex (and no edges) is 2-constructible.
(ii) If G_1, G_2 are 2-constructible, then the disjoint union of G_1 and G_2 together with two new edges joining them is 2-constructible.

Jaeger et al. [17] proved the following.

Theorem 2.11. *Let G be a cubic 3-edge-connected graph and let v be a vertex in G. Define $H = G - v$. Then H has a spanning tree T such that the contraction of the edges of H which are not in T yields a 2-constructible graph.*

They used it to prove the following.

Theorem 2.12. *Let G be a 3-edge-connected graph and let v be a vertex of degree 3 in G. Then $G - v$ is Γ-connected for any Abelian group Γ of order $|\Gamma| \geq 6$.*

We shall also use the following definition and lemma.

Definition 2.13. Given a finite subset $\Pi = \{a_1, a_2, \ldots, a_k\}$ of an Abelian group Γ, we define the *simple sum* Π' of Π to be the set of all elements on the form

$$\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_k a_k$$

where $\alpha_i \in \{0, \pm 1\}$ for $1 \leq i \leq k$. In particular, Π' contains 0.

Lemma 2.14. *Given a natural number k and an Abelian group Γ of order $|\Gamma| > k$, there exists a subset $\Pi \subseteq \Gamma$ which is closed under taking inverses and has $|\Pi| = k$ such that the simple sum Π' of Π satisfies $|\Pi'| \leq k^2$.*

Proof. For $k = 1$ we let Π consist of 0, so let $k > 1$. If some element a in Γ has order at least k, then we put $\Pi = \{\pm a, \pm 2a, \ldots, \pm \frac{k}{2} a\}$ if k is even, and $\Pi = \{0, \pm a, \pm 2a, \ldots, \pm \frac{k-1}{2} a\}$ if k is odd. So assume all elements have order $< k$.

Let Γ_0 be the largest subgroup of order $< k$. Let $a \in \Gamma \setminus \Gamma_0$. Then the subgroup generated by $\Gamma_0 \cup \{a\}$ has order greater than k but less than k^2 and any k-subset closed under taking inverses can play the role of Π. □

We use these results to prove the following:

Theorem 2.15. *Let G be a 3-edge-connected graph and let v be a vertex of degree 3 in G. Define $H = G - v$. Assume each edge of H has a direction. Let $k \in \mathbb{N}$, and let Γ be any Abelian group of order $|\Gamma| > 8k^3$. Assume that for each edge e, F_e is a set of at most k elements in Γ. Then there exists a flow $f : E(H) \to \Gamma$ such that $f(e) \notin F_e$ for all $e \in E(G)$.*

We say that F_e is the set of *forbidden flow values* for e. If it becomes convenient to reverse the orientation of an edge e we replace F_e by $-F_e = \{-\gamma \mid \gamma \in F_e\}$.

Proof. By Lemma 2.14 there exist subsets $\Pi, \Pi' \subseteq \Gamma$ such that Π is closed under inverses and has $|\Pi| = 2k$ and Π' is the simple sum of Π and has $|\Pi'| \leq 4k^2$.

It suffices to prove Theorem 2.15 in the case where G is cubic. For if u is a vertex of degree $d > 3$, then we replace u by a cycle of length d such that all vertices in that cycle have degree 3 in the resulting graph. (If u is a cutvertex, we can make sure that no two edges of the cycle form a 2-edge-cut.) So assume that G is cubic and 3-edge-connected. The case $k = 1$ follows from Theorem 2.2 and Theorem 2.12, so we may assume that $k > 1$.

By Theorem 2.11, $H = G - v$ has a spanning tree T in H such that the contraction of the edges of H which are not in T yields a 2-constructible graph. We colour the edges of T red and colour the edges in $H - T$ blue. Let E_b be the set of blue edges.

As H/E_b is 2-constructible there exists a sequence $H_0, H_1, \ldots, H_t$ of graphs such that H_0 is the empty graph on $|V(H/E_b)|$ vertices, $H_t = H/E_b$, and each H_i is obtained from H_{i-1} by adding two red edges (which we denote by e_i and e'_i) between two disjoint connected components of H_{i-1}.

We first describe informally the method, which is similar to the method in [17]. We give H flow in two steps. In the first step we give all red edges flow values which are non-forbidden and which remain to be non-forbidden after Step 2. Step 1 also affects the blue edges but this is not important. In Step 2 we send flow through the blue edges such that the flow values of the blue edges are non-forbidden. There is already flow through the blue edges after Step 1. The additional flow values (added in Step 2) in the blue edges will be in Π. This will then affect the red edges in such a way that the additional flow through a red edge will be in Π'. In Step 1, the flow is chosen such that an additional flow value in Π' will keep the flow value in the red edge non-forbidden.

We now argue formally.

Step 1: Consider a cycle C'_t in $H_t = H/E_b$ through e_t and e'_t and let C_t be a cycle in H containg the edges of C'_t and no other red edges. If the orientations of e_t and e'_t do not agree on C_t then reverse the orientation of e_t and replace F_e by $-F_e$. Let $F = F_{e_t} \cup F_{e'_t}$. We pick flow γ_t to send through C_t such that:

$$(\gamma_t + \Pi') \cap F = \emptyset. \tag{2.4}$$

Since Π' has size at most $4k^2$, F has size at most $2k$, and γ_t can be chosen in more than $8k^3$ ways, this is indeed possible. Modify the set of forbidden flow values in each red edge e of $C_t - e_t - e'_t$ such that the new set of forbidden flow values is $F_e - \gamma_t = \{\alpha - \gamma_t \mid \alpha \in F_t\}$ if the orientation of e agrees with that of e_t, and $F_e + \gamma_t = \{\alpha + \gamma_t \mid \alpha \in F_t\}$ if not. We call this the *first iteration* of Step 1.

Next, we consider a cycle C'_{t-1} in $H/E_b - e_t - e'_t$ through e_{t-1} and e'_{t-1} and let C_{t-1} be a cycle in $H - e_t - e'_t$ containing the edges of C'_{t-1} and no other red edges. As above we find an appropriate flow γ_{t-1} to send though C_{t-1}. Note that e_t and e'_t are not in C_{t-1} (or any other C_i for $1 \le i \le t-1$) by the construction of H/E_b, so the flow in e_t, e'_t will not be changed in Step 1. We call this the *second iteration* of Step 1. We repeat this argument for e_{t-2} and e'_{t-2} and then e_{t-3} and e'_{t-3}, etc., until all red edges have received a flow value. Once e_i, e'_i have received a flow value in the $(t+1-i)$'th iteration of Step 1, that flow value will not be further changed in Step 1.

Step 2: If all blue edges have a non-forbidden flow after Step 1, the proof is complete and there will be no Step 2. So consider a blue edge e which has a forbidden flow value $f'(e)$, say, after Step 1. Since $|\Pi| = 2k$ and F_e has at most k elements, there exists a $\gamma \in \Pi$ such that $f'(e) + \gamma$ and $f'(e) - \gamma$ are both non-forbidden, that is, they are both in $\Gamma \setminus F_e$. We say that γ is *good for e*. Let E_γ be the set of all edges which currently have forbidden flow values and for which γ is good. For each edge in E_γ, let C_e be the unique cycle in $T + e$. Form the symmetric difference H_γ of C_e taken over all edges e in E_γ. Then H_γ is an even graph (that is, a graph where each component is Eulerian), and hence H_γ has a flow using only γ and $-\gamma$. We add this flow to the flow obtained after Step 1, and now all edges in E_γ have non-forbidden flow values.

Repeat this step as long as there are blue edges with forbidden flow values.

Consider now a red edge e after Step 2. Let $f'(e)$ be its flow value after Step 1. In Step 2 we add some elements in Π to $f'(e)$. Thus the final flow of e is of the form $f'(e) + \gamma'$ where γ' is a simple sum of elements from Π, that is $\gamma' \in \Pi'$. By the choice of $f'(e)$ it follows that the final flow value $f'(e) + \gamma' \notin F_e$, as required. (In Step 1 we modified the forbidden flow values in the red edges. Here in Step 2, F_e denotes the original forbidden flow values.) □

Theorem 2.16 below follows from Theorem 2.15 by adding a vertex of degree 3 to G which may be removed again.

Corollary 2.16. *Let G be a 3-edge-connected graph. Let $k \in \mathbb{N}$, and let Γ be any Abelian group of order $|\Gamma| > 8k^3$. Given any orientation of G, if G has at most k forbidden flow values from Γ in each edge, then there exists a flow $f : E(G) \to \Gamma$ such that $f(e)$ is not forbidden for any $e \in E(G)$.*

2.6 Flows in 2-edge-connected graphs with forbidden flow-values

Now we can use Theorem 2.16 to prove a similar statement about 2-edge-connected graphs.

Theorem 2.17. *Let G be a 2-edge-connected graph. Let $k \in \mathbb{N}$, and let Γ be an Abelian group of order $|\Gamma| > 8(kq(G))^3$. Given any orientation of G, if G has at most k forbidden flow values from Γ in each edge, then there exists a flow $f : E(G) \to \Gamma$ such that $f(e)$ is not forbidden for any $e \in E(G)$.*

Proof. We may assume that the orientation of G is strongly connected by reversing the edges necessary and adjusting the forbidden sets in the edges accordingly. Then all edges in a cycle-equivalence class have the same direction in each cycle containing them. We form a new directed graph G' on the same edge set as G (but possibly with a different vertex set) such that

(i): the edge set of any cycle in G is also the edge set of a cycle in G' and vice versa (in particular G and G' have the same cycle-equivalence classes).

(ii): the orientations of all edges agree in any two cycles with the same edge set in G and G', respectively. More precisely: If E is the edge set of a cycle in G and hence also a cycle in G', then we can choose a clockwise orientation of C, C' such that an edge e is directed clockwise in C if and only if e is directed clockwise in C'.

(iii): every Γ-flow in G is also a Γ-flow in G' and vice versa.

(iv): the edges of each cycle-equivalence class in G' form a directed path such that each intermediate vertex has indegree 1 and outdegree 1.

Note that (iv) is equivalent with the following:

(v): If we delete the edges of a cycle-equivalence class in G', then the resulting graph has precisely one component with edges.

If G satisfies (iv), we put $G' = G$. Otherwise, there exists a cycle-equivalence class such that the deletion of its edges results in a graph with more than one component. Let G_1 be one component. Let G_2 consist of all other components containing edges together with those paths in the cycle-equivalence class that connect them. Then the edges in the cycle-equivalence class that are not in $G_1 \cup G_2$ form two directed paths P_1 with vertices $v_1, v_2, \ldots, v_s$ and directed edges $e_1, e_2, \ldots, e_{s-1}$ and P_2 with vertices $u_1, u_2, \ldots, u_t$ and directed edges $e'_1, e'_2, \ldots, e'_{t-1}$ where $v_1, u_t \in V(G_1)$ and $u_1, v_s \in V(G_2)$. The two paths P_1, P_2 are disjoint except that possibly $v_1 = u_t$ and possibly $u_1 = v_s$. Only their endvertices are in $G_1 \cup G_2$. Now we form a new graph H from $G_1 \cup G_2$ by first identifying v_1, v_s and then adding a directed path with edges $e_1, e_2, \ldots, e_{s-1}, e'_1, e'_2, \ldots, e'_{t-1}$ from u_1 to u_t. Then H also satisfies (i),(ii),(iii). Moreover, H has more vertices of indegree 1 and outdegree 1 than G. So, in a finite number of steps we obtain a graph satisfying (i),(ii),(iii),(iv).

It follows that G' is a subdivision of a 3-edge-connected graph G''. An edge in G'' corresponds to a cycle-equivalence class in G and is therefore subdivided into at most $q(G)$ edges, by the definition of $q(G)$. Now we complete the proof by applying Theorem 2.16 to G''. As each edge in G has at most k forbidden flow values, each edge in G'' has at most $kq(G)$ forbidden flow values. □

Theorem 2.17 for $k = 1$ combined with Theorem 2.2 implies Theorem 2.8 as well as the upper bound $g(k) \leq 8k^3 + 1$.

2.7 Group coloring

Jaeger et al. [17] define group colourability as follows.

Definition 2.18. Let Γ be an Abelian group. The graph G is said to be Γ-*colorable* if the following holds: Given some orientation D of G and any function $\varphi : E(G) \to \Gamma$ there exists a vertex coloring $c : V(G) \to \Gamma$ such that $c(w) - c(u) \neq \varphi(uw)$ for each $uw \in E(D)$.

We say that φ *allows* c.

A graph is d-*degenerate* if every subgraph contains a vertex of degree at most d. The *coloring number* $Col(G)$ is the largest number such that G is $(d-1)$-degenerate. Equivalently, $Col(G) - 1$ is the maximum minimum degree where the maximum is taken over all subgraphs of G. Clearly

$$\chi(G) \leq \chi_{wg}(G) \leq \chi_g(G) \leq Col(G). \tag{2.5}$$

Also

$$\chi_l(G) \leq Col(G), \tag{2.6}$$

where $\chi_l(G)$ is the list-chromatic number.

Now let G be a graph with $\chi_{wg}(G) = k$, and let Γ be an Abelian group such that G is Γ-colorable. Král' et al. [20] proved that $k > \delta/2\ln(\delta)$ where δ is the minimum degree of G. (Král' et al. formulated their result as one about the group chromatic number but the proof works for the weak group chromatic number as well.) As this holds for every subgraph of G it also holds for a subgraph of minimum degree $\delta = Col(G) - 1$. Hence

$$\chi_{wg}(G) = k > \frac{\delta}{2\ln\delta} = \frac{Col(G)-1}{2\ln(Col(G)-1)} \geq \frac{\chi_g(G)-1}{2\ln(\chi_g(G)-1)}. \quad (2.7)$$

This implies, for each natural number k, the upper bound in the following:

$$(2-o(1))k < h(k) < (2+o(1))k\ln(k). \quad (2.8)$$

Since the graphs $G_{q,k}$ are planar, their dual graphs establish the lower bound (for infinitely many primes k) by the proof of Theorem 2.4.

The group chromatic number and weak group chromatic number have some similarity to the list-chromatic number. Indeed, the proof of the 5-list-color theorem for planar graphs [39] translates, word for word, to the result that every planar graph has group chromatic number at most 5. And the proof of the 3-list-color theorem for planar graphs of girth at least 5 [38] translates to the result that every planar graph has group chromatic number at most 3. Only a minor detail in the proof needs additional explanation, see [35].

The following conjecture was made by Král' et al. in [20] and, according to [20], independently by Margit Voigt.

Conjecture 2.19. For every graph G, $\chi_l(G) \leq \chi_g(G)$.

We propose the analogous conjecture for the weak group chromatic number.

Conjecture 2.20. For every graph G, $\chi_l(G) \leq \chi_{wg}(G)$.

In [17] it is shown that every graph with two edge-disjoint spanning trees is Γ-connected for every Abelian group Γ of order at least 4. This implies that $g(3) = 3$ as pointed out in [31]. For, if G is $\mathbb{Z}_3$-connected we let $\beta(v) = 1 - d(v)$ (where $d(v)$ denotes the degree of v) for every vertex v, except possibly one, in Definition 2.1. We may choose the resulting f such that $f(e) = 1$ for every edge e, by reversing the direction of those edges having flow value 2. This gives an orientation of the edges such that all vertices, except possibly one, have outdegree $2 \pmod 3$ and hence outdegree at least 2. Thus G has at least $2|V(G)| - 2$ edges. As this also holds for every graph obtained from G by identifying vertices (and removing the loops that may arise), G has two edge-disjoint spanning trees, by a fundamental result of Edmonds [10], Nash-Williams [34] and Tutte [47].

Note that the dual statement does not hold: A $\mathbb{Z}_3$-colorable graph is not necessarily the union of two spanning trees. For example is $K_{3,5}$ $\mathbb{Z}_3$-colorable (see e.g. [26]) but contains 8 vertices and 15 edges, thus it has too many edges to be the union of two spanning trees.

Conjecture 2.21. $h(3) = 3$.

Conjecture 2.21 clearly implies Conjecture 2.22 below.

Conjecture 2.22. If G is $\mathbb{Z}_3$-colorable, then G is $\mathbb{Z}_4$-colorable and also $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable.

Since $g(3) = 3$, the answer is affirmative for planar graphs. Jaeger et al. [17] asked if a graph is $\mathbb{Z}_4$-connected if and only if it is $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected. Hušek et al. [14] answered this in the negative for both implications. As some of the counterexamples are planar, their dual graphs show that $\mathbb{Z}_4$-colorability does not imply $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorability. These graphs have multiple edges. Examples without multiple edges can be obtained using Hajos' construction. Such examples can even be made planar. Their dual graphs are therefore 3-edge-connected planar graphs that are $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected. For details, see [27]. We do not know if $\mathbb{Z}_4$-colorability is implied by $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorability.

We conclude the paper by showing that Conjecture 2.22 in fact is equivalent to Conjecture 2.21. This follows from the two propositions below.

Proposition 2.23. *If G is $\mathbb{Z}_3$-colorable, then G is 5-degenerate, that is, $\chi_g(G) \leq Col(G) \leq 6$.*

Proof. Consider the bipartite graph with vertices $A \cup B$ in which the vertices in A corresponds to all possible $\mathbb{Z}_3$-colorings of the vertices in G, and the vertices in B correspond to all possible $\mathbb{Z}_3$-colorings of the edges in G. We join two vertices by an edge if the edge-coloring allows the vertex-coloring.

Let $n = |V(G)|$ and $m = |E(G)|$. Note that there 3^n vertices in A and 3^m vertices in B. Furthermore, each vertex in A has degree 2^m. As each edge-coloring will have at least one allowed vertex-coloring, all vertices in B has degree at least 1. Thus $3^n \cdot 2^m \geq 3^m$. We conclude

$$m \leq n \cdot \frac{\log(3)}{\log(\frac{3}{2})} < 2.8n < 3n, \tag{2.9}$$

so G is 5-degenerate. Thus, G is Γ-colorable for every Abelian group Γ of order $|\Gamma| \geq 6$. □

We do not know if every $\mathbb{Z}_3$-colorable graph is even 3-degenerate.

Proposition 2.24. *If G is $\mathbb{Z}_3$-colorable, then G is $\mathbb{Z}_5$-colorable.*

Proof. Let G be a $\mathbb{Z}_3$-colorable graph, let D be an orientation of G, and let $\varphi : E(D) \to \mathbb{Z}_5$ be given. We think of $\mathbb{Z}_5$ as the numbers $0, 1, -1, 2, -2$. We define $\varphi' : E(D) \to \mathbb{Z}_3$ by reducing these numbers modulo 3, where we think of $\mathbb{Z}_3$ as the numbers $0, 1, -1$. More precisely, for each edge $e \in E(D)$,

$$\varphi'(e) = \begin{cases} 0 & \text{if } \varphi(e) = 0, \\ 1 & \text{if } \varphi(e) = 1 \text{ or } \varphi(e) = -2, \\ -1 & \text{if } \varphi(e) = 2 \text{ or } \varphi(e) = -1. \end{cases} \tag{2.10}$$

As G is $\mathbb{Z}_3$-colorable, there exists a vertex-coloring $c' : V(G) \to \mathbb{Z}_3$ such that $c'(v) - c'(u) \neq \varphi'(uv)$ for each $uv \in E(D)$. Consider the vertex-coloring $c : V(G) \to \mathbb{Z}_5$ defined by $c(v) = c'(v) \pmod 5$ for each vertex v. We argue that c is a proper coloring. Let $e = uv$ be a directed edge. If $c'(v) - c'(u) = 0$, then $c(v) - c(u) = 0$. As $\varphi'(e) \neq 0$ we get $\varphi(e) \neq 0$ so $c(v) - c(u) \neq \varphi(e)$. If $c'(v) - c'(u) = 1$, then $c(v) - c(u) = 1$ or $c(v) - c(u) = -2$. As $\varphi'(e) \neq 1$ we get $\varphi(e) \neq 1$ and $\varphi(e) \neq -2$ so $c(v) - c(u) \neq \varphi(e)$. If $c'(v) - c'(u) = -1$, then $c(v) - c(u) = 2$ or $c(v) - c(u) = -1$. As $\varphi'(e) \neq -1$ we get $\varphi(e) \neq 2$ and $\varphi(e) \neq -1$ so $c(v) - c(u) \neq \varphi(e)$. $\square$

CHAPTER 3

The number of flows

This chapter is based on the paper [6] by DeVos, L., Mohar and Šámal.

It is a natural question to ask whether, given the existence of a certain coloring or flow in a graph, there are exponentially many. This has been asked and answered in the positive for e.g. 5-list-colorings ([40]), 3-list-colorings ([41]) and nowhere-zero $\mathbb{Z}_3$-, $\mathbb{Z}_4$-, and $\mathbb{Z}_6$-flows ([8]).

In this chapter we will consider the same question for group flows.

Techniques

A lemma on nowhere-zero flows

We first prove a lemma which states that if a graph contains a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_k$-flow (ϕ, ψ), then it contains at least $2^{m-n-t/k}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_k$-flows, where t is the size of the support of ϕ. The idea is to modify ϕ on edges where ψ is non-zero. That way the flow remains nowhere-zero. The lemma is shown in two steps: First we modify ψ to cover as much as of $\text{supp}(\phi)$ as possible. And then we may change ϕ on any cycle in $\text{supp}(\psi)$. As the dimension of the cycle space of a graph with n vertices and $m - t/k$ edges is $m - n - t/k$, we get at least $2^{m-n-t/k}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_k$-flows.

This lemma is then used to improve the bounds on the number of nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows in 4-edge-connected graphs and the number of nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flows in 2-edge-connected graphs.

Modifying the technique by Jaeger et al.

For Abelian groups of order $k \geq 8$, the exponential number of group flows is based on an argument by Jaeger, Linial, Payan and Tarsi in [17] who show the existence of group flows in 3-edge-connected graphs when the order of the group is at least 6. Our proof exploits that there is room for choice when constructing the flow in [17], and then we find a large subset of flows which are not counted twice.

This proof uses the notion of a *k-base* which is a subset of the edges of the graph such that we can construct the entire edge set by successively adding cycles that have at most 2 edges not in the current edge set.

We obtain a cubic graph G from the graph using a lemma, and then we choose a 2-base B in G such that $E(G) \setminus B$ is a 1-base. In the first step we construct many flows which on each edge of $E(G) \setminus B$ is neither the forbidden element nor the forbidden element plus/minus a special element x. Since this flow is chosen recursively such that two edges in $E(G) \setminus B$ is given flow in each round, there are at least $k-6$ ways to choose the flow. In total this gives $(k-6)^\ell$ flows.

In the second step we fix the edges in B by sending flow through cycles C_e. As there are $2^{\ell+1}$ possible subsets of B, at least $(k-6)^\ell/2^{\ell+1}$ of the flows use the same subset in B, hence these flows are distinct. Thus we get the desired result.

When k is even we may improve the result slightly: choose x to have order 2 in the group, then there are $k-4$ ways to choose the flow in the first step. This results in $(k-4)^\ell/2^{\ell+1}$ distinct flows.

Peripheral cycles and bases

It is central to the special case $\mathbb{Z}_6$ to be able to show the existence of some peripheral (that is, induced non-separating) cycles and bases with specific properties, but the topic is also of independent interest. We first show a couple of lemmas on the existence of peripheral paths in certain subgraphs, and then we show some lemmas on the existence of (many) peripheral 2-bases. Many peripheral 2-bases will then give many flows.

In the first part we use a technique where we choose a path for which the lexicographic ordering of the sizes of the components appearing when the path is removed is maximum. If there is more than one component, that is, the path is not peripheral, this allows us to deduce some properties of the path and in the end reach a contradiction, e.g. by concluding that the 3-edge-connected graph has a 2-edge-cut or that there is a path inducing a better lexicographic ordering of the components.

In the second part the crucial technique used to construct peripheral 2-bases is to maintain two sequences of nested subsets. The first one are nested edge-sets, and the other one are vertex-sets. To form the next set in the sequence we add a peripheral path using the lemmas. The final edge-set will be a peripheral 2-base in the graph.

The special case $\mathbb{Z}_6$

In the case $\mathbb{Z}_6$ we cannot prove that there are exponentially many group flows, so instead we prove that there are at least $2^{\sqrt{\ell}/\log \ell}$ $\mathbb{Z}_2 \times \mathbb{Z}_3$-flows, where $\ell = m - n$.

Either there is a long peripheral cycle (of length at least $\frac{3}{2}\sqrt{\ell}/\log \ell$) in the graph, in which case we use a lemma which gives the result directly by sending flow through fundamental cycles. This strategy is similar to that in the lemma showing there are many nowhere-zero group flows. Otherwise all peripheral cycles have length at most $\frac{3}{2}\sqrt{\ell}/\log \ell$. Then we can choose $\frac{3}{2}\sqrt{\ell}/\log \ell$ distinct peripheral 2-bases and we can choose $\mathbb{Z}_3$-flows which have non-forbidden flow on all edges not in the 2-base. We may assume that all these $\mathbb{Z}_3$-flows satisfy that at most $\sqrt{\ell}/\log \ell$ edges in the 2-base have non-forbidden flow, since otherwise we may again send flow through fundamental cycles to construct the wanted number of flows. By bounding (from below) the number of ways to select a 2-base in which all but at most $\sqrt{\ell}/\log \ell$ edges are fixed we can now show that the number of distinct $\mathbb{Z}_3$-flows is at least $2^{\sqrt{\ell}/\log \ell}$.

Many flows in the group connectivity setting[2]

Abstract

Two well-known results in the world of nowhere-zero flows are Jaeger's 4-flow theorem asserting that every 4-edge-connected graph has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow and Seymour's 6-flow theorem asserting that every 2-edge-connected graph has a nowhere-zero $\mathbb{Z}_6$-flow. Dvořák and the last two authors of this paper extended these results by proving the existence of exponentially many nowhere-zero flows under the same assumptions. We revisit this setting and provide extensions and simpler proofs of these results.

The concept of a nowhere-zero flow was extended in a significant paper of Jaeger, Linial, Payan, and Tarsi to a choosability-type setting. For a fixed abelian group Γ, an oriented graph $G = (V, E)$ is called Γ-*connected* if for every function $f : E \to \Gamma$ there is a flow $\phi : E \to \Gamma$ with $\phi(e) \neq f(e)$ for every $e \in E$ (note that taking $f = 0$ forces ϕ to be nowhere-zero). Jaeger et al. proved that every oriented 3-edge-connected graph is Γ-connected whenever $|\Gamma| \geq 6$. We prove that there are exponentially many solutions whenever $|\Gamma| \geq 8$. For the group $\mathbb{Z}_6$ we prove that for every oriented 3-edge-connected $G = (V, E)$ with $\ell = |E| - |V| \geq 11$ and every $f : E \to \mathbb{Z}_6$, there are at least $2^{\sqrt{\ell}/\log \ell}$ flows ϕ with $\phi(e) \neq f(e)$ for every $e \in E$.

3.1 Introduction

Throughout this paper we permit graphs to have loops and parallel edges. We use standard graph theory terminology and notation as in [4] and [7]. In particular, if G is a graph and $X \subseteq V(G)$ and $S \subseteq E(G)$, we let $G - X$ and $G - S$ denote the subgraph of G obtained by removing all vertices in X (and their incident edges), and removing all edges in S (but keeping all vertices), respectively. We also write $G[X]$ to denote the subgraph induced on X.

For a graph $G = (V, E)$, we define a k-*coloring* to be a function $f : V \to \{1, 2, \ldots, k\}$ with the property that $f(u) \neq f(v)$ for every $uv \in E$. If G is equipped with an orientation and $v \in V$, we let $\delta^-(v)$ denote the set of edges that have v as terminal vertex and $\delta^+(v)$ the set of edges with v as initial vertex; we also put $\delta(v) = \delta^+(v) \cup \delta^-(v)$. If Γ is an additive abelian group, a function $\phi : E \to \Gamma$ is called a *flow* or a Γ-*flow* if the following rule is satisfied at every $v \in V$:

$$\sum_{e \in \delta^+(v)} \phi(e) = \sum_{e \in \delta^-(v)} \phi(e).$$

We say that ϕ is *nowhere-zero* if $0 \notin \phi(E)$. If $\Gamma = \mathbb{Z}$ and $|\phi(e)| < k$ for every $e \in E$ we call ϕ a k-*flow*. Note that if ϕ is a flow and we reverse the direction of an edge e, we

[2]The content of the remaining part of this chapter is the paper [6] which is in submission. This is joint work with Matt DeVos, Bojan Mohar and Robert Šámal.

may replace $\phi(e)$ with $-\phi(e)$ and this gives us a flow relative to this new orientation. Since this operation preserves the properties of nowhere-zero and k-flow, the presence of a nowhere-zero Γ-flow or a nowhere-zero k-flow depends only on the underlying graph and not on the particular orientation.

The study of nowhere-zero flows was initiated by Tutte [43] who observed that these are dual to colorings in planar graphs. Namely, he proved the following.

Theorem 3.1 (Tutte [43]). *Let G and G^* be dual planar graphs and orient the edges of G arbitrarily. If Γ is an abelian group with $|\Gamma| = k$, the number of k-colorings of G^* is equal to k times the number of nowhere-zero Γ-flows of G.*

The above theorem has the curious corollary that for planar graphs, the number of nowhere-zero flows in an abelian group Γ depends only on $|\Gamma|$. Tutte proved that this holds more generally for arbitrary graphs. That is, for any abelian groups Γ_1 and Γ_2 with $|\Gamma_1| = |\Gamma_2|$, the number of nowhere-zero Γ_1-flows is equal to the number of nowhere-zero Γ_2-flows in every oriented graph. Furthermore, the inherent monotonicity in coloring (every graph with a k-coloring has a k'-coloring for every $k' \geq k$) is also present in flows. This follows from another theorem of Tutte asserting that for every $k \geq 2$, an oriented graph G has a nowhere-zero k-flow if and only if it has a nowhere-zero $\mathbb{Z}_k$-flow. In addition to establishing these fundamental properties, Tutte made three fascinating conjectures concerning nowhere-zero flows that have directed the field since then.

Conjecture 3.2 (Tutte [43, 45, 4]). Let G be an oriented 2-edge-connected graph.

1. G has a nowhere-zero 5-flow.
2. If G does not have a Petersen graph minor, it has a nowhere-zero 4-flow.
3. If G is 4-edge-connected, it has a nowhere-zero 3-flow.

Despite a wealth of research, all three of Tutte's conjectures remain open. Below we have summarized some of the most significant results to date on the presence of nowhere-zero flows.

Theorem 3.3. *Let G be an oriented graph.*

1. (Seymour [36]) *If G is 2-edge-connected, it has a nowhere-zero 6-flow.*
2. (Jaeger [16]) *If G is 4-edge-connected, it has a nowhere-zero 4-flow.*
3. (Lovász, Thomassen, Wu, Zhang [32]) *If G is 6-edge-connected, it has a nowhere-zero 3-flow.*

Beyond showing that a graph G has a k-coloring, one may look to find lower bounds on the number of k-colorings. Although there are infinite families of planar graphs where any two 4-colorings differ by a permutation of the colors (so there are just $4! = 24$ in total), for 5-colorings the following theorem provides an exponential lower bound.

Theorem 3.4 (Birkhoff, Lewis [3]). *Every simple planar graph on n vertices has at least 2^n 5-colorings.*

By Theorem 3.1 the above result implies that 2-edge-connected planar graphs have exponentially many nowhere-zero $\mathbb{Z}_5$-flows. More recently, Thomassen [41] showed a similar exponential bound for the number of 3-colorings of triangle-free planar graphs, which yields exponentially many nowhere-zero $\mathbb{Z}_3$-flows in any 4-edge-connected planar graph. In a recent work, Dvořák and the latter two authors of this paper investigated the problem of finding exponentially many nowhere-zero flows in general graphs and established the following results.

Theorem 3.5 (Dvořák, Mohar, and Šámal [8]). *Let G be an oriented graph with n vertices and m edges.*

1. *If G is 2-edge-connected, it has $2^{2(m-n)/9}$ nowhere-zero $\mathbb{Z}_6$-flows.*
2. *If G is 4-edge-connected, it has $2^{n/250}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows.*
3. *If G is 6-edge-connected, it has $2^{(n-2)/12}$ nowhere-zero $\mathbb{Z}_3$-flows.*

We have revisited this topic and found shorter proofs and improved exponential bounds in the first two instances appearing in the following theorem.

Theorem 3.6. *Let G be an oriented graph with n vertices and m edges.*

1. *If G is 3-edge-connected, it has $2^{(m-n)/3}$ nowhere-zero $\mathbb{Z}_6$-flows.*
2. *If G is 4-edge-connected, it has $2^{n/3}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows.*

Theorem 3.6 will be proved in Section 3.3, see Theorems 3.15 and 3.16.

List-coloring provides a more general setting than that of standard coloring. Let $G = (V, E)$ be a graph and let $\mathcal{L} = \{L_v\}_{v \in V}$ be a family of sets indexed by the vertices of G. An *$\mathcal{L}$-coloring* of G is a function f so that $f(v) \in L_v$ holds for every $v \in V$ and $f(u) \neq f(v)$ whenever $uv \in E$. We say that a graph G is *k-choosable* if an $\mathcal{L}$-coloring exists whenever $|L_v| \geq k$ holds for every $v \in V$. Note that every k-choosable graph is necessarily k-colorable since we may take $L_v = \{1, 2, \ldots, k\}$ for every $v \in V$. For planar graphs, Thomassen [40] proved the following theorem showing the existence of exponentially many $\mathcal{L}$-colorings when every set has size five.

Theorem 3.7 (Thomassen [40]). *Let $G = (V, E)$ be a simple planar graph of order n. If $\mathcal{L} = \{L_v\}_{v \in V}$ is a family of sets with $|L_v| = 5$ for every $v \in V$, then there are at least $2^{n/9}$ $\mathcal{L}$-colorings of G.*

A recent concept due to Dvořák and Postle [9] provides an even more challenging setting than list-coloring. Let $G = (V, E)$ be a graph with an arbitrary orientation, fix a positive integer k and for every edge $e \in E$ let σ_e be a permutation of $\{1, 2, \ldots, k\}$. We define a *DP-coloring* of this system to be a function $f : V \to \{1, 2, \ldots, k\}$ with the property that for every directed edge $e = (u, v)$ we have $f(u) \neq \sigma_e(f(v))$. The

DP-coloring number of G is the smallest k so that such a coloring exists for every assignment of permutations. Every graph with DP-coloring number k is also k-choosable. To see this, let $\{L_v\}_{v\in V}$ be a family of sets with $|L_v| = k$ for every $v \in V$ and for every oriented edge $e = (u, v)$ of G choose a bijection $\sigma_e : L_u \to L_v$ such that σ_e is the identity when restricted to $L_u \cap L_v$. A DP-coloring for the family $\{\sigma_e\}_{e\in E}$ then gives a list-coloring.

There is a natural group-valued specialization of DP-coloring. Again, let $G = (V, E)$ be an oriented graph and let Γ be an additive abelian group. For every $e \in E$, let $\gamma_e \in \Gamma$. Then a function $c : V \to \Gamma$ may be considered a coloring if for every $e = (u, v) \in E$ we have $c(u) \neq \gamma_e + c(v)$. So here we are operating in the realm of DP-coloring, but using a group as the ground set, and taking permutations associated with group elements on the edges. This group-valued DP-coloring is usually called group coloring. Its dual was investigated many years ago by Jaeger, Linial, Payan, and Tarsi [17], who gave the following definition.

Definition 3.8. Let $G = (V, E)$ be an oriented graph and let Γ be an abelian group. We say that G is Γ-*connected* if it satisfies the following property: For every $f : E \to \Gamma$ there is a flow $\phi : E \to \Gamma$ satisfying $\phi(e) \neq f(e)$ for every $e \in E$.

In fact, the authors showed in [17] that the above concept has several equivalent formulations. Our interest here will be in counting the number of functions ϕ satisfying the above property, and for this purpose all of the equivalent forms of group-connectivity operate the same. So for simplicity we shall stay with the above definition. Using the basic method of Jaeger et al. we will prove the following theorem in Section 3.5.

Theorem 3.9. *Let $G = (V, E)$ be an oriented 3-edge-connected graph with $\ell = |E| - |V|$ and let Γ be an abelian group with $|\Gamma| = k \geq 6$. For every $f : E \to \Gamma$ we have*

$$|\{\phi : E \to \Gamma \mid \phi \text{ is a flow and } \phi(e) \neq f(e) \text{ for every } e \in E\}| \geq \begin{cases} \frac{1}{2}(\frac{k-6}{2})^\ell & \text{if } k \text{ is odd,} \\ \frac{1}{2}(\frac{k-4}{2})^\ell & \text{if } k \text{ is even.} \end{cases}$$

For $k \geq 8$ the above theorem gives us an exponential number of flows, but for $k = 6, 7$ it only implies the existence of one. We believe that this represents a shortcoming of our techniques and suspect the result holds as well when $k = 6$, 7.

Conjecture 3.10. There exists a fixed constant $c > 1$ so that the following holds. For every 3-edge-connected oriented graph $G = (V, E)$ of order n, every abelian group Γ with $|\Gamma| \geq 6$, and every $f : E \to \Gamma$, there exist at least c^n flows $\phi : E \to \Gamma$ with $\phi(e) \neq f(e)$ for every $e \in E$.

Theorem 3.9 shows the above conjecture is true for all abelian groups except for $\mathbb{Z}_6$ and $\mathbb{Z}_7$. In the case of $\mathbb{Z}_7$ we have no interesting lower bound. In the case of $\mathbb{Z}_6$, we have the following result showing a super-polynomial lower bound, which has the flavor of a similar coloring result by Asadi et al. [2]. Throughout we use log to denote the logarithm base 2.

Theorem 3.11. *Let $G = (V, E)$ be an oriented 3-edge-connected graph with $\ell = |E| - |V| \geq 11$ and let $f : E \to \mathbb{Z}_6$. There exist at least $2^{\sqrt{\ell}/\log \ell}$ flows $\phi : E \to \mathbb{Z}_6$ with the property that $\phi(e) \neq f(e)$ for every $e \in E$.*

The rest of this paper is organized as follows. In the next section we show that many of our counting problems can be reduced to cubic graphs. In the third section we establish Theorem 3.6 on the existence of many nowhere-zero flows. Section 3.4 introduces Seymour's concept of a k-base and then this is used in Section 5 to prove Theorem 3.9. Sections 3.6 and 3.7 develop the techniques required to find many decompositions of a 3-connected cubic graph into a 1-base and a 2-base, and then in Section 3.8 this is exploited to prove Theorem 3.11. The results about 1-base / 2-base decompositions (Lemmas 3.26 and 3.27) may be of independent interest.

3.2 Reduction to cubic graphs

As is common in the world of nowhere-zero flows, certain basic operations will reduce some of our problems to the setting of cubic graphs. If $G = (V, E)$ is a graph and $X, Y \subseteq V$ are disjoint we let $E(X, Y) = \{xy \in E \mid x \in X \text{ and } y \in Y\}$. For $z \in V$ with $z \notin Y$ we let $E(z, Y) = E(\{z\}, Y)$.

Lemma 3.12. *Let $G = (V, E)$ be a 3-edge-connected graph with $|E| - |V| = \ell$. Then there exists a 3-edge-connected cubic graph G' with $|V(G')| = 2\ell$ so that G can be obtained from G' by contracting the edges of a forest.*

Proof. We proceed by induction on $S = \sum_{v \in V} (\deg(v) - 3)$. In the base case this parameter is 0 and the result holds trivially by setting $G' = G$. For the inductive step we may choose $v \in V$ with $\deg(v) > 3$. For $e, e' \in \delta(v)$ with $e \neq e'$ we may form a new graph G' from G by adding a new vertex v', changing e and e' to have v' as one end instead of v (in case of a loop at v, change just one end) and then adding a new edge f with ends v, v'. We say that G' is obtained from G by *expanding relative to* e, e'. Note that we may return from G' to G by contracting the newly added edge f. If we can construct a new graph G' by expanding so that G' is still 3-edge-connected, then the result follows by applying induction to G', so it suffices to prove this. (Note that we have added one vertex and one edge, so ℓ stays the same, while the sum S decreases.)

If $G - v$ is disconnected, there must be at least 3 edges between v and each component of $G - v$. Now choosing $e, e' \in \delta(v)$ so that e and e' have ends in different components of $G - v$ and expanding relative to e, e' preserves 3-edge-connectivity. So we may assume $G - v$ is connected. If $G - v$ is 2-edge-connected, then any expansion suffices, so we may assume otherwise and choose a minimal nonempty set $X \subseteq V(G - v)$ so that there is just one edge between X and $Y = V(G - v) \setminus X$. Since

G is 3-edge-connected, there must be at least two edges between v and X and at least two between v and Y. Choose $e \in E(v, X)$ and $e' \in E(v, Y)$ and then expanding relative to e, e' gives us a 3-edge-connected graph G' and this completes the proof. □

Observation 3.13. Let $G = (V, E)$ be an oriented graph, let $F \subseteq E$ and assume that (V, F) is an (oriented) forest. Let Γ be an abelian group, and let $\phi_1, \phi_2 : E(G) \to \Gamma$ be flows. If $\phi_1(e) = \phi_2(e)$ holds for every $e \in E \setminus F$, then $\phi_1 = \phi_2$.

Proof. The function $\phi_1 - \phi_2$ is also a flow and must be identically 0 since the support of every flow is a union of (edge-sets of) cycles. □

3.3 Many nowhere-zero flows

In this section we prove Theorem 3.6 concerning the existence of many nowhere-zero flows.

Lemma 3.14. *Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. Let $\phi : E \to \mathbb{Z}_2$ and $\psi : E \to \mathbb{Z}_k$ be flows with $\operatorname{supp}(\phi) \cup \operatorname{supp}(\psi) = E$ and let $t = |\operatorname{supp}(\phi)|$. Then G has at least $2^{m-n-t/k}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_k$-flows.*

Proof. The support of ϕ may be expressed as a disjoint union $\sqcup_{i=1}^{k} C_i$ where each C_i is the edge-set of a cycle. Let us fix an arbitrary orientation of G. (In the sequel we will no longer be mentioning that we have an implicit chosen orientation of the graph whenever we speak of flows in the graph.) For every $1 \leq i \leq k$ there is a $\mathbb{Z}_k$-flow ρ_i of G with support C_i that has values ± 1 on every edge in C_i. By adding a suitable multiple of ρ_i to ψ we may assume that at most $\frac{1}{k}|E(C_i)|$ edges of C_i are not in the support of ψ. After applying this operation to each C_i, the resulting $\mathbb{Z}_k$-flow ψ' will still have $\operatorname{supp}(\phi) \cup \operatorname{supp}(\psi') = E$ but will additionally satisfy $|\operatorname{supp}(\psi')| \geq m - t/k$.

Let $E' = \operatorname{supp}(\psi')$, let $G' = (V, E')$ and note that the dimension of the cycle space of G' is at least $|E'| - |V| \geq m - n - t/k$. It follows that the number of $\mathbb{Z}_2$-flows of G supported on a subset of E' is at least $2^{m-n-t/k}$. If we take any such $\mathbb{Z}_2$-flow, say η, the mapping from E to $\mathbb{Z}_2 \times \mathbb{Z}_k$ given by $e \mapsto (\phi(e) + \eta(e), \psi'(e))$ will be a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_k$-flow of G and this gives the desired count. □

With this lemma in hand we are ready to prove Theorem 3.6. The first part of this is given by Theorem 3.15 and the second by Theorem 3.16.

Theorem 3.15. *Every oriented 2-edge-connected graph with n vertices and m edges has at least $2^{(m-n)/3}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flows.*

Proof. We proceed by induction on n with the base case $n = 1$ holding trivially (each of the m loop edges may be assigned any non-zero value in $\mathbb{Z}_2 \times \mathbb{Z}_3$ to get a nowhere-zero flow and this can be done in $5^m \geq 2^{(m-1)/3} = 2^{(m-n)/3}$ ways). Next suppose that there is an edge e so that e is in a 2-edge-cut with another edge e', and apply the theorem inductively to G/e. Every nowhere-zero flow ϕ of G/e extends uniquely to one in G (if e and e' are consistently oriented in the edge-cut $\{e, e'\}$, then we extend by setting $\phi(e) = -\phi(e')$). This gives the desired count of nowhere-zero flows, and we may therefore assume that G is 3-edge-connected.

Now apply Lemma 3.12 to choose a 3-edge-connected cubic graph $G' = (V', E')$ with 2ℓ vertices where $\ell = m - n$ so that G can be obtained from G' by contracting the edges of a forest. In light of Observation 3.13, it suffices to prove that G' has at least $2^{(m-n)/3} = 2^{\ell/3}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flows. By Seymour's 6-flow theorem (part 1 of Theorem 3.3) we may choose flows $\phi : E' \to \mathbb{Z}_2$ and $\psi : E' \to \mathbb{Z}_3$ with $\mathrm{supp}(\phi) \cup \mathrm{supp}(\psi) = E'$. Since G' is cubic, the support of ϕ is a disjoint union of (edge-sets of) cycles and thus $|\mathrm{supp}(\phi)| \leq |V'| = 2\ell$. Now applying Lemma 3.14 gives us at least $2^{\ell-2\ell/3} = 2^{\ell/3}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flows, as claimed. □

Theorem 3.16. *Every oriented 4-edge-connected graph with n vertices has at least $2^{n/3}$ nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows.*

Proof. Let $G = (V, E)$ be a 4-edge-connected graph with $|V| = n$ and $|E| = m$ and apply Jaeger's 4-flow theorem to choose a nowhere-zero flow $\phi : E \to \mathbb{Z}_2 \times \mathbb{Z}_2$. Thanks to the symmetry of the factors of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ we may permute these flow values $(1, 0), (0, 1), (1, 1)$ while maintaining a flow (for instance, we can change all edges with flow value $(0, 1)$ to $(1, 1)$ and all those with flow value $(1, 1)$ to $(0, 1)$ and we still have a flow). By way of this operation, we may assume that at least $\frac{1}{3}m$ edges e satisfy $\phi(e) = (0, 1)$. Now letting $\phi_1 : E \to \mathbb{Z}_2$ denote the first coordinate of ϕ and $\phi_2 : E \to \mathbb{Z}_2$ the second, we have $t = |supp(\phi_1)| \leq \frac{2}{3}m$. Since every vertex has degree at least 4 we have $m \geq 2n$ and thus

$$m - n - \tfrac{1}{2}t \geq \tfrac{2}{3}m - n \geq \tfrac{1}{3}n.$$

The result follows from this inequality and Lemma 3.14 applied to ϕ_1 and ϕ_2. □

3.4 Seymour's bases

Following Seymour [36], we introduce a closure operator for subsets of edges of a graph. Let $G = (V, E)$ be a graph and let k be a positive integer. For every $S \subseteq E$, the *k-closure* of S, denoted $\langle S \rangle_k$, is defined to be the minimal edge-set R such that $S \subseteq R \subseteq E$ and every cycle $C \subseteq G$ with $E(C) \setminus R \neq \emptyset$, has at least $k+1$ edges that are not in R. Clearly, $\langle S \rangle_k$ can be constructed starting with $R = S$ and then consecutively add the edges of any cycle C violating the condition, i.e., if $0 < |E(C) \setminus R| \leq k$, then we add $E(C)$ into R and repeat. This is a closure operator as evidenced by the following properties, all of which are easy to verify:

$$S \subseteq \langle S \rangle_k, \qquad \langle \langle S \rangle_k \rangle_k = \langle S \rangle_k, \qquad S' \subseteq S \Rightarrow \langle S' \rangle_k \subseteq \langle S \rangle_k.$$

A subset $S \subseteq E$ is called a k-base if $\langle S \rangle_k = E$. It is easy to see that a set of edges of a connected graph is a 1-base if and only if it contains the edge-set of a spanning tree. A key feature of the definition of k-bases is that they can be extended to the whole edge-set by a sequence of steps, each of which adds at most k new edges along a cycle. This property can be used to find Γ-flows whose support contains $E \setminus S$ for all groups Γ with $|\Gamma| > k$. In particular, we have the following well-known statement:

Lemma 3.17. *Let k and q be integers with $0 < k < q$, let $G = (V, E)$ be a graph, and let $S \subseteq E$ be a k-base. For every $f : E \setminus S \to \mathbb{Z}_q$ there exists a flow $\phi : E \to \mathbb{Z}_q$ satisfying $\phi(e) \neq f(e)$ for every $e \in E \setminus S$.*

Proof. Since every k-base is a $(q-1)$-base, we may assume that $q = k+1$. We proceed by induction on $|E \setminus S|$. As a base case, when $|E \setminus S| = 0$ the result holds trivially (with ϕ zero everywhere). For the inductive step $S \neq E$ so we may choose a cycle $C \subseteq G$ so that $0 < |E(C) \setminus S| \leq k$ and we let $S' = S \cup E(C)$. By applying the induction hypothesis to S' we may choose a flow $\phi' : E \to \mathbb{Z}_{k+1}$ so that $\phi'(e) \neq f(e)$ holds for every $e \in E \setminus S'$. Now let $\nu : E \to \mathbb{Z}_{k+1}$ be a $\mathbb{Z}_{k+1}$-flow taking the values ± 1 on $E(C)$ and the value 0 elsewhere. Let $x \in \mathbb{Z}_{k+1}$ and consider the flow $\phi = \phi' + x\nu$. For every $e \in E(C) \setminus S$ there is precisely one value of x for which we will have $\phi(e) = f(e)$. Since there are at most k edges in $E(C) \setminus S$ but $k+1$ possible values for x, we may assign x a value so that $\phi(e) \neq f(e)$ holds for every $e \in E(C) \setminus S$, and then ϕ satisfies the lemma. □

In Seymour's original paper [36] on nowhere-zero 6-flows, he proves two structure theorems giving the existence of a 2-base (these give two different proofs of the 6-flow theorem). We summarize these results below.

Theorem 3.18 (Seymour [36])**.** *Let G be a 3-edge-connected cubic graph.*

1. *There exists a collection of edge-disjoint cycles $C_1, \ldots, C_t$ so that $\cup_{i=1}^t E(C_i)$ is a 2-base.*

2. *There exists a partition of $E(G)$ into $\{B_1, B_2\}$ so that B_i is an i-base for $i = 1, 2$.*

Both parts of this theorem give the existence of a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow quite immediately. For the first, we may choose a $\mathbb{Z}_2$-flow with support equal to $\cup_{i=1}^t E(C_i)$ and Lemma 3.17 (applied with $f = 0$) gives a $\mathbb{Z}_3$-flow whose support contains the complement. For the second part, we may apply the above lemma twice (both times with $f = 0$) to choose a $\mathbb{Z}_2$-flow with support containing $E \setminus B_1$ and a $\mathbb{Z}_3$-flow with support containing $E \setminus B_2$.

The second decomposition in Theorem 3.18 will be more useful to us, and in fact we will require some slightly stronger variants of this, so we will develop a proof of this later in the paper. To see the utility of this second decomposition in the setting of group-connectivity, we follow Jaeger et al. [17] to prove the following result.

Theorem 3.19 (Jaeger, Linial, Payan, and Tarsi [17])**.** *Every 3-edge-connected graph is $\mathbb{Z}_6$-connected.*

Proof. Let $G = (V, E)$ be an oriented 3-edge-connected graph, let $f_1 : E \to \mathbb{Z}_2$, and let $f_2 : E \to \mathbb{Z}_3$. Apply the second part of Theorem 3.18 to choose a partition of E into $\{B_1, B_2\}$ so that B_i is an i-base for $i = 1, 2$. Now for $i = 1, 2$ we apply Lemma 3.17 to choose a flow $\phi_i : E \to \mathbb{Z}_i$ so that $\phi_i(e) \neq f_i(e)$ holds for every $e \in E \setminus B_i$. Now $(f_1(e), f_2(e)) \neq (\phi_1(e), \phi_2(e))$ holds for every $e \in E$ and this completes the proof. □

3.5 Large groups

In this section we prove Theorem 3.9, our result for groups of order 8 or more. For such large groups all that we need is Seymour's decomposition theorem. We have restated this theorem below for convenience.

Theorem. Let $G = (V, E)$ be an oriented 3-edge-connected graph with $\ell = |E| - |V|$ and let Γ be an abelian group with $|\Gamma| = k \geq 6$. For every $f : E \to \Gamma$ we have

$$|\{\phi : E \to \Gamma \mid \phi \text{ is a flow and } \phi(e) \neq f(e) \text{ for every } e \in E\}| \geq \begin{cases} \frac{1}{2}(\frac{k-6}{2})^\ell & \text{if } k \text{ is odd,} \\ \frac{1}{2}(\frac{k-4}{2})^\ell & \text{if } k \text{ is even.} \end{cases}$$

Proof. Apply Lemma 3.12 to choose a cubic graph $G' = (V', E')$ with $|V'| = 2\ell$ and note that by Observation 3.13 it suffices to prove the above result with G' in place of G. If k is even, choose $x \in \Gamma$ to be an element of order 2; otherwise choose $x \in \Gamma \setminus \{0\}$.

Apply the second part of Theorem 3.18 to choose a 2-base B of G' such that $E' \setminus B$ is a 1-base. This implies that $G' - B$ is connected. The first stage in our proof will be to construct many flows $\phi : E' \to \Gamma$ with the following property for every $e \in E' \setminus B$:

$$(\star) \quad \phi(e) \notin \{f(e), f(e) + x, f(e) - x\}.$$

Let us observe that $f(e)+x = f(e)-x$ when k is even. Let us now invoke the definition of 2-base to choose a sequence of nested sets $B = B_0 \subset B_1 \subset \ldots \subset B_t = E'$ satisfying:

- $|B_i \setminus B_{i-1}| \leq 2$ for every $1 \leq i \leq t$, and
- there exists a cycle C_i with $(B_i \setminus B_{i-1}) \subseteq E(C) \subseteq B_i$ for every $1 \leq i \leq t$.

We will construct our flows recursively using elementary flows on the cycles C_i (working backwards). Initially start with $\phi : E' \to \Gamma$ to be the zero flow. Let ν_t be either the zero flow or an elementary flow supported on $E(C_t)$ (so there are k choices for ν_t) and modify ϕ by adding ν_t to it. If k is odd (even), there are at most 3 (2) possible choice of ν_t so that $(\star)$ fails on an edge in $B_t \setminus B_{t-1}$. Thus, there at least $k-6$ $(k-4)$ ways to choose ν_t so that condition $(\star)$ is satisfied on every edge in $B_t \setminus B_{t-1}$. Next choose ν_{t-1} to be either the zero flow or an elementary flow supported on $E(C_{t-1})$ and modify ϕ by adding ν_{t-1} so that $(\star)$ is satisfied on every edge in $B_{t-1} \setminus B_{t-2}$, and continue in this manner. Since the edges in $B_i \setminus B_{i-1}$ satisfy $(\star)$ at the point when we add the flow ν_i and these edges do not appear in the support of $\nu_{i-1}, \nu_{i-2}, \ldots, \nu_1$, at the end of this process we have a flow ϕ that satisfies ($\star$) on every edge in $E' \setminus B$. Since $G' - B$ is connected we have $|E' \setminus B| \geq 2\ell - 1$ and this means that $t \geq \ell$. Therefore, the number of flows ϕ satisfying ($\star$) on every edge in $E' \setminus B$ is at least $(k-6)^\ell$ when k is odd and at least $(k-4)^\ell$ when k is even.

Choose a spanning tree T with $E(T) \subseteq E' \setminus B$. For every edge $e \in B$ let C_e be the edge-set of the fundamental cycle of e with respect to T. For every $S \subseteq B$ define $\hat{S} = \bigoplus_{e \in S} C_e$ where $\bigoplus$ denotes the symmetric difference. The set $\hat{S}$ may be expressed as a disjoint union of (edge-sets of) cycles so we may choose a flow $\mu_S : E' \to \Gamma$ supported on $\hat{S}$ so that $\mu_S(e) = \pm x$ for every $e \in S$. Now for every flow ϕ satisfying ($\star$) on every edge in $E' \setminus B$ we let $S = \{e \in B \mid \phi(e) = f(e)\}$ and we define $\phi' = \phi + \mu_S$. It follows from this construction that the resulting flow ϕ' will satisfy $\phi'(e) \neq f(e)$ for every $e \in E'$. Since $|B| \leq \ell + 1$, the number of subsets $S \subseteq B$ is at most $2^{\ell+1}$, so if k is odd we have at least $\frac{1}{2}\left(\frac{k-6}{2}\right)^\ell$ flows ϕ with the same set S and for k even this count will be $\frac{1}{2}\left(\frac{k-4}{2}\right)^\ell$. Since each of these flows is modified by adding the same flow, μ_S, this gives us the desired number of flows ϕ'. □

3.6 Peripheral paths and cycles

Seymour's proof of the second part of Theorem 3.18 is based on an iterative procedure during which the edge partition is formed, and we will require a strong form of this. A key concept in this process is that of a path or cycle who's removal leaves the graph connected. The purpose of this section is to prove three lemmas that provide the tools we need to find such paths and cycles.

Tutte [44] called a cycle C of a graph G *peripheral* if C is induced and $G - V(C)$ is connected. A key feature of peripheral cycles is that for a graph G embedded in the plane, every peripheral cycle must bound a face. Tutte proved that for a 3-connected graph, every edge is contained in at least two peripheral cycles, thus giving an abstract characterization of the faces of a 3-connected planar graph (they are precisely the peripheral cycles). For our purpose, we will be interested only in subcubic graphs and in this setting we can use the following definition of peripheral edge-sets.

Definition 3.20. If $G = (V, E)$ is a subcubic graph, an edge-set $S \subseteq E$ is *peripheral* if $G - S$ is connected. We call a subgraph $H \subseteq G$ *peripheral* if $E(H)$ is peripheral.

Note that with this definition, Seymour's second decomposition theorem asserts the existence of a peripheral 2-base in every 3-edge-connected cubic graph. Below we state a restricted form of Tutte's theorem (for cubic graphs) of use to us. We will provide a proof of this below.

Theorem 3.21 (Tutte [44])**.** *Let G be a 3-edge-connected cubic graph. For any two edges of G incident with the same vertex, there exists a peripheral cycle containing both of them.*

Now we are ready for the first of the lemmas from this section.

Lemma 3.22. *Let $G = (V, E)$ be a 3-edge-connected cubic graph, let $X \subset V$ be nonempty, let H be a component of $G - X$, and let $f \in E(X, V(H))$. Then there exists a (possibly trivial) path $P \subseteq H$ with ends y_0, y_1 satisfying:*

- *P is peripheral in H;*
- *there exist distinct edges e_0, e_1, where $e_i \in E(y_i, X) \setminus \{f\}$ for $i = 0, 1$;*
- *no internal vertex on the path P has a neighbor in X.*

Moreover, if H is 2-edge-connected and we prescribe any $y_0 \in V(H)$ such that $E(y_0, X) \setminus \{f\}$ is nonempty, we can still guarantee a peripheral path as above.

Proof. If H is a single vertex, then the result is obviously true. If H is not 2-edge-connected, then consider its block structure. If one of the leaf blocks of H is just a cut-edge vy, where y has degree 1 in H and is not incident with f, then we take $y_0 = y_1 = y$ and $P = y$, and the result follows. Otherwise, we let H' be a leaf block that is not incident with f. Further, we let f' be the only cut-edge of H incident with H'. We extend X to $X' = X \cup (V(H) \setminus V(H'))$. If we find a path P for X', H' and f', then the same path works for X, H and f because f' is the only edge in $E(H', X') \setminus E(H, X)$ and the path P does not contain the vertex z' (the end of f' in H'). We define $Y = \{y \in V(H') \mid E(y, X') \setminus \{f'\} \neq \emptyset\}$ and pick arbitrary $y_0 \in Y$.

If, on the other hand, H was 2-edge-connected (and y_0 was specified), then we put $H' = H$, $X' = X$ and $f' = f$; we write $f' = x'z'$ with $x' \in X'$. In both cases we now have H' 2-edge-connected and we want to find a peripheral path P in it with one end specified. Note that since H' is 2-connected, the end z' of f' in H' is not in Y, and by the third property it should not be on the path.

As we have dealt with the case of H' being a single vertex, we now have the useful property that each vertex of H' has at most one incident edge going to X'. Choose a path $P \subseteq H' - z'$ starting at y_0 with the other end in $Y \setminus \{y_0\}$, subject to the following conditions:

(i) The component of $H' - E(P)$ containing z' has maximum size.

(ii) Subject to (i), the lexicographic ordering of the sizes of the components of $H' - E(P)$ not containing z' is maximum (i.e., the largest component not containing z' has maximum size, and subject to this the second largest has maximum size, and so on).

The assumption that G is 3-edge-connected implies that $|Y| \geq 2$, and as H' is 2-edge-connected, there exists some path P. Let y_1 be the other end of P. Note that our path P chosen according to the above criteria has no interior vertices in Y, otherwise we may take a subpath. We claim that P is peripheral in H'. Suppose (for a contradiction) that this is not the case and let F be a component of $H' - E(P)$

such that $z' \notin V(F)$ and F is of minimum possible size. Define P' to be the minimal subpath of P that contains all vertices of $V(F) \cap V(P)$.

Suppose there exists a vertex $q \in V(P')$ that is not contained in $V(F)$. In this case we may choose a path $P^* \subseteq F$ with the same ends as P'. As P^* avoids q, modifying our original path P by replacing the subpath P' with P^* gives us a path that contradicts the choice of P: the component containing q increases, all others except F stay the same or increase.

Thus we must have $V(P') \subseteq V(F)$. If P' shares an end with P then F is connected to the rest of H' by a single edge, a contradiction with 2-edge-connectivity of H'.

Suppose next, there exists a vertex $w \in Y \cap V(F)$. By now we know, that w is not an end of P. We choose a path $Q \subseteq F$ with one end w and the other one the first vertex of P'; note that $z' \notin V(Q)$ by the choice of P and F. Now $P \cup Q$ contains a path $\tilde{P}$ that contradicts the choice of P relative to (i) or (ii): components of $H' - E(\tilde{P})$ are larger than or equal to the corresponding components of $H' - E(P)$, except for F, which was the least significant in our selection process and the component containing y_1 becomes strictly larger. Therefore, no such vertex w can exist.

It follows that F is connected to the rest of G by only two edges, a contradiction with 3-edge-connectivity of G. We deduce that P is peripheral in H', and this completes the proof. □

Before we prove our strong form of the above result let us pause to prove Tutte's peripheral cycles theorem for cubic graphs using Lemma 3.22.

Proof of Theorem 3.21. Let x be the vertex incident with both prescribed edges e_0, e_1. Let $X = \{x\}$, let $f \in \delta(x) \setminus \{e_0, e_1\}$ and apply Lemma 3.22 for the set X and the edge f. This gives us a peripheral path P of $G - x$. The cycle C formed by adding the vertex x and the edges $\{e_0, e_1\}$ is a peripheral cycle in G. □

Next we establish a stronger version of the above lemma that will provide us with some choice in our basic process. This is a key ingredient for us in proving the existence of many flows in the group $\mathbb{Z}_6$. The proof has similar basic structure as the proof of Lemma 3.22; with a few more subtleties – including using Lemma 3.22 in one of the steps. We recall that if P is a path containing vertices a, b, then aPb denotes the subpath of P from a to b.

Lemma 3.23. *Let $G = (V, E)$ be a cyclically 4-edge-connected cubic graph, let $X \subset V$ have $|X| \geq 2$, let H be a component of $G - X$, and let $f = xz \in E$ have $x \in X$ and $z \in V(H)$. If $|E(v, X) \setminus \{f\}| \leq 1$ for every $v \in V(H)$, then there exist distinct vertices $y, y_1, y_2 \in V(H) \setminus \{z\}$ such that $E(y, X) \neq \emptyset$ and $E(y_i, X) \neq \emptyset$ for $i = 1, 2$, and for $i = 1, 2$ there exists a path $P_i \subseteq H - z$ with ends y and y_i that is peripheral in H and contains no internal vertices with a neighbor in X. Moreover, the edge of P_1 incident with y is distinct from the edge of P_2 incident with y.*

Proof. As in the proof of Lemma 3.22, we first suppose that the graph H has a cut-edge, and therefore a nontrivial block structure. In this case, choose H' to be a leaf block of H that does not contain z. The condition that $|E(v,X) \setminus \{f\}| \leq 1$ for every $v \in V(H)$ implies that H' is nonempty, is not just a vertex, and has no vertices of degree 1 except possibly z. This implies that H' is 2-connected. Let $z' \in V(H')$ be the unique vertex of H' incident with a cut-edge of H and let $X' = X \cup (V(H) \setminus V(H'))$. In the case that our graph H has no cut-edge, then we set $H' = H$, set $z' = z$, and set $X' = X$. Observe that to complete the proof of the lemma, it suffices to solve the problem with H', z', X' in place of H, z, and X. This adjustment has granted us the useful property that H' is 2-connected.

Set $Y = \{y \in V(H') \mid E(y, X') \neq \emptyset\}$ and note that $|Y| \geq 4$ as neither X' nor H' can be a single vertex and G is cyclically 4-edge-connected. Also, let $Y' = Y \setminus \{z'\}$. Declare a nontrivial path $P \subseteq H' - z'$ to be *good* if P is peripheral in H', both ends of P are in Y, and no interior vertex of P is in Y. Let $S \subseteq E(H')$ be the set of edges incident with a vertex in Y' and contained in a good path. Lemma 3.22 gives us a good path starting at any vertex of Y', thus any such vertex is incident with at least one edge in S. To complete the proof it suffices to prove that there is a vertex in Y' incident with two such edges. Accordingly, we now assume (for a contradiction) that every vertex in Y' is incident with precisely one edge in S.

Claim: There exists a path $Q \subseteq H'$ with ends z', y' and interior vertex y such that $y, y' \in Y'$ and the edge of yQy' that is incident with y is not in S.

Proof of the claim: Call a cycle $C \subseteq H'$ *obliging* if it contains distinct vertices $y, y' \in Y'$ with the property that one of the two paths in C with ends y, y' contains the edge of S incident with y, and the other path contains the edge in S incident with y'. If C is obliging, we may choose a (possibly trivial) path from z' to $V(C)$ and this path together with C will contain a path satisfying the claim. Thus we may assume no cycle is obliging. Note that this implies that every cycle contains at most two vertices of Y'. Choose a cycle C containing two distinct vertices, say $y_1, y_2 \in Y'$ and then choose $y_3 \in Y' \setminus V(C)$. Since H' is 2-connected, we may choose a path P_3 internally disjoint from C so that both ends of P_3 are in $V(C)$ and y_3 is an internal vertex of P_3. Let w, w' be the ends of P_3 and for $i = 1, 2$ let P_i be the path of C with ends w, w' that contains y_i. Now we must have $C \cup P_3 = P_1 \cup P_2 \cup P_3$ or cycle $P_1 \cup P_3$ contains three vertices y_1, y_2, y_3 of Y'. Moreover, since there is no obliging cycle, by possibly interchanging w and w' we may assume that wP_iy_i avoids S for $i = 1, 2, 3$. Finally, choose a (possibly trivial) path of H' from z' to $V(P_1 \cup P_2 \cup P_3)$ and observe that this path together with $P_1 \cup P_2 \cup P_3$ contains a path satisfying the claim. $\square$

Now apply the claim to choose a path Q and vertices y, y'. Let $Q' = z'Qy$ and note that the unique edge of S incident with y is contained in Q'. Now we will take advantage of Q' to construct another good path. Thanks to the presence of the path Q we may choose a path $P \subseteq H' - E(Q')$ so that P has y as one end and the other end in $Y' \setminus \{y\}$ and subject to this we choose P so that:

(i) The component of $H' - E(P)$ containing Q' has maximum size.

(ii) Subject to (i), the lexicographic ordering of the sizes of the components of $H' - E(P)$ not containing Q' is maximum (i.e., the largest component not containing Q' has maximum size, subject to this the second largest has maximum size, and so on).

We claim that the resulting path P will be good. Suppose otherwise and let F be the smallest component of $H' - E(P)$ not containing Q'. Note that F cannot be an isolated vertex, since that would be an interior vertex of P in Y' – and we could shorten P to end at this vertex, improving our criteria.

Let P' be the minimal subpath of P containing all vertices of $F \cap V(P)$. If there is another component of $H' - E(P)$ containing a vertex in $V(P')$ then we may choose a path $P^* \subseteq F$ with the same ends as P' and modify P by replacing the subpath P' by P^* to obtain a path superior to P thus contradicting our choice. (Note that P^* is disjoint from Q' by the choice of F.) Thus, all vertices in P' belong to F. There must exist a vertex $u \in Y \cap V(F)$. Otherwise, the two edges incident with the ends of P' that are not in $P' \cup F$ would form a 2-edge-cut in G. Now we may choose a path from $V(P')$ to u and reroute P using this path. This will result in a path superior to P thus contradicting our choice. This proves that $H' - E(P)$ is connected, so P is a good path. This gives us a contradiction, since now both edges of H' incident with y are contained in S. This completes the proof. □

Our last lemma provides a technical property that we will use to control the behaviour of our process.

Lemma 3.24. *Let $G = (V, E)$ be a 3-edge-connected cubic graph, and let $X \subseteq V$ have $G[X]$ connected. Let H be a component of $G - X$ and let $P \subseteq H$ be a nontrivial path with ends y_1, y_2. If P is peripheral in H, $E(y_i, X) \neq \emptyset$ for $i = 1, 2$, and $E(y, X) = \emptyset$ for all other vertices $y \in V(P)$, then there exists a peripheral cycle C of G with $C \cap H = P$.*

Proof. For $i = 1, 2$ let $x_i y_i \in E(y_i, X)$. By assumption there exists a path in G from x_1 to x_2, say Q, so that $E(Q) \cap E(H) = \emptyset$. Among all such paths, choose one so that:

(i) The component of $G - E(Q)$ containing $E(H)$ has maximum size.

(ii) Subject to (i), the lexicographic ordering of the sizes of the components of $G - E(Q)$ not containing $E(H)$ is maximum.

Suppose (for a contradiction) that Q is not peripheral and let F be a minimum size component of $G - E(Q)$ not containing H. If Q' is the minimum subpath of Q containing all vertices in F, then $V(Q') \not\subseteq V(F)$ as otherwise G would have just two edges between $V(F)$ and the other vertices. However we may then choose a path $Q^* \subseteq F$ with the same ends as Q' and then modifying Q by replacing the subpath Q' with Q^* gives us an improvement to Q. Therefore, our chosen path Q is peripheral. Moreover, the path P is peripheral in H and G is 3-edge-connected. It follows that the cycle C consisting of $P \cup Q$ together with the edges $x_1 y_1$ and $x_2 y_2$ is peripheral in G and this completes the proof. □

3.7 Peripheral 2-bases

Jaeger, Linial, Payan, and Tarsi [17] found an alternative proof of Seymour's 1-base and 2-base decomposition theorem. Their theorem is slightly sharper than Seymour's in that it saves a vertex (a feature we will need). Recall that an edge-set $F \subseteq E(G)$ being peripheral in the graph G means that $G - F$ is connected, in other words the set $E - F$ contains edge-set of some spanning tree of G.

Theorem 3.25 (Jaeger, Linial, Payan, and Tarsi [17])**.** *If G is a graph obtained from a 3-connected cubic graph by deleting a single vertex then G has a peripheral 2-base.*

The proof of the above theorem in [17] is based on an inductive approach applied to the class of graphs which are a single-vertex deletion from a cubic 3-connected graph. For our purpose we will adopt a blend of these ideas. We will operate iteratively following Seymour, but we will save a vertex like Jaeger et al. For any graph $G = (V, E)$ and $E' \subseteq E$ we let $V(E')$ denote the set of vertices of G incident to some edge in E'.

Lemma 3.26. *For every 3-edge-connected cubic graph G, the following holds:*

1. *If $C \subseteq G$ is a peripheral cycle, there exists a peripheral 2-base $B \subseteq E(G)$ with $E(C) \subseteq B$.*
2. *For every $r \in V(G)$, the graph $G - r$ has at least three peripheral 2-bases.*

Proof. Although the two parts to the lemma have slightly different inputs, we will prove both simultaneously. In the first case, choose $f \in E$ to be an edge with exactly one endpoint in $V(C)$. For the second case, let f be an edge incident with r and apply Theorem 3.21 to choose a peripheral cycle C so that $r \in V(C)$ but $f \notin E(C)$. Now for both parts of the proof we will use the cycle C and the edge f and construct two sequences of nested subsets. The first one are nested edge-sets $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_t \subseteq E$, the second one are nested vertex-sets $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t = V$, where $B_0 = E(C)$ and $X_0 = V(C)$. For every $0 \leq i \leq t$ we will maintain the following properties:

(i) $B_i \subseteq E(X_i)$.

(ii) B_i is peripheral in G.

(iii) The graphs $G[X_i]$ and $G - X_i$ are connected (or empty).

(iv) $\langle B_i \rangle_2$ contains every edge with both ends in X_i.

Note that the initial sets B_0 and X_0 satisfy (i)–(iv) for $i = 0$. Assuming $V(B_i) \neq V$ we form the next sets as follows: Apply Lemma 3.22 to G with the set X_i, the edge f, and H the unique component of $G - X_i$. If P is the path selected by this lemma, we define $B_{i+1} = B_i \cup E(P)$ and $X_{i+1} = X_i \cup V(P)$. Observe that all four of the above properties are still satisfied. We continue this process until $X_t = V$ at which point the set B_t is a peripheral 2-base. This finishes the proof of the first part of the lemma. To complete the proof of the second part we will investigate the behaviour of the edge f in our process. Let $f = rz$ where $r \in V(C)$ and observe that the vertex z cannot appear in the path P selected by Lemma 3.22 (because this path is peripheral in H and no intermediate vertex on the path is adjacent to X_i) until H is just the single isolated vertex z. It follows that $B_t \setminus \delta(r)$ is a peripheral 2-base in the graph $G - r$. If z, z', z'' are the vertices adjacent to r in G, then $B_t \setminus \delta(r)$ will contain an edge incident with z' and one incident with z'' but none incident with z. Since $f \in \delta(r)$ may be chosen arbitrarily, we have found three 2-bases in $G - r$ as desired. □

The above lemma gives us peripheral 2-bases with a couple of useful properties. However, in order to prove our main theorem about $\mathbb{Z}_6$-flows we require the existence of many peripheral 2-bases. This is achieved by the following lemma.

Lemma 3.27. *Let G be a 3-connected cubic graph on n vertices with a distinguished root $r \in V(G)$. If every peripheral cycle of G has length at most q, then the graph $G - r$ has at least $2^{n/(2q)}$ decompositions into a spanning tree and a 2-base.*

Proof. We proceed by induction on n. As a base case, observe that when $n \leq 2q$, the result follows immediately from the previous lemma. For the inductive step we begin by considering the case that there exists a partition $\{X_1, X_2\}$ of V with $|X_i| \geq 2$ for $i = 1, 2$ and $|E(X_1, X_2)| = 3$. We may assume that $r \in X_1$ and for $i = 1, 2$ form a graph G_i from G by identifying X_i to a single vertex called x_i and deleting any loops formed in this process. It is easy to see that G_1 and G_2 are 3-connected cubic graphs. Let $e, e' \in E(X_1, X_2)$ be distinct and for $i = 1, 2$ let C_i be a peripheral cycle of G_i that contains e, e'. Now the cycle of G formed from the union of $C_1 - x_1$ and $C_2 - x_2$ by adding the edges e, e' is a peripheral cycle of G. It follows from this and Theorem 3.21 that neither G_1 nor G_2 has a peripheral cycle with length greater than q. So, by the induction hypothesis, there are at least $2^{|V(G_1)|/(2q)}$ peripheral 2-bases of $G_1 - x_1$ and at least $2^{|V(G_2)|/(2q)}$ peripheral 2-bases of $G_2 - r$. The union of a peripheral 2-base of $G_1 - x_1$ with a peripheral 2-base of $G_2 - r$ is a peripheral 2-base of $G - r$ and this gives the desired count.

So we may now assume that G is cyclically 4-edge-connected. Now we will show that we have many degrees of freedom in selecting a peripheral 2-base using a procedure similar to that used in the proof of Lemma 3.26. We construct two sequences of nested subsets, edge-sets $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_t \subseteq E$ and vertex-sets $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t = V$. For every $0 \leq i \leq t$ we will maintain the same properties (i)–(iv) as in the proof of the previous lemma.

We begin by choosing a peripheral cycle C containing r (note that we have three ways to do this). Let $B_0 = E(C)$, let $X_0 = V(C)$ and we let $\{f\} = \delta(r) \setminus B_0$. Now at each step assuming $X_i \neq V$ we operate as follows: If there is a vertex $y \in V \setminus X_i$ so that $|E(y, X_i) \setminus \{f\}| \geq 2$ then we let $B_{i+1} = B_i$ and let $X_{i+1} = X_i \cup \{y\}$ (we have added a trivial path of one new vertex and no new edge to the 2-base). If no such vertex exists, then we apply Lemma 3.23 to choose a vertex $y \in V \setminus X_i$ and peripheral paths P_1, P_2. Now we can choose to either set $X_{i+1} = X_i \cup V(P_1)$ and

$B_{i+1} = B_i \cup E(P_1)$ or we may set $X_{i+1} = X_i \cup V(P_2)$ and $B_{i+1} = B_i \cup E(P_2)$. We continue the process until we have $X_t = V$.

In order to see that this operation gives us the desired flexibility, it is helpful to introduce another nested sequence of edges $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_t$ defined by the rule $T_i = E(X_i) \setminus B_i$. The key feature of these sets (verified by a straightforward induction) is that for every $1 \leq i \leq t-1$, the set $T_i \cup E(X_i, V \setminus X_i)$ is a spanning tree in the graph obtained from G by identifying $V \setminus X_i$ to a single vertex. For every $1 \leq i \leq t-1$ we have $|T_i \setminus T_{i-1}| = 2$. At the last step we have $|T_t \setminus T_{t-1}| = 3$ and the set T_t forms the edge-set of a spanning tree in T. Therefore, $|V| - 1 = |T_t| = 2(t-1) + 3$ and we have $t = \frac{1}{2}|V| - 1$. It follows from this that $|B_t| = \frac{1}{2}|V| + 1$. It follows from Lemma 3.24 that every path P we select using Lemma 3.23 has length at most q. So the total number of nontrivial paths selected in our process must be at least $\frac{|V|}{2q}$.

It remains to show that different choices of paths during our process yield different peripheral 2-bases. From our construction follows immediately that $E(X_i, V \setminus X_i) \subseteq T_t$ for every $0 \leq i \leq t-1$. Suppose that when we have B_i and X_i and apply Lemma 3.23 we select the vertex $y \in V \setminus X_i$ and the paths P_1, P_2 (both ending at y). Let $\{e_0\} = E(y, X_i)$ and for $j = 1, 2$ let e_j be the edge of P_j incident with y (the lemma gives $e_1 \neq e_2$). If we choose the path P_j and set $B_{i+1} = B_i \cup E(P_j)$ and $X_{i+1} = X_i \cup V(P_j)$ then upon completion of our process we will have $\delta(y) \cap B_t = \{e_j\}$. So the 2-bases constructed by making a different choice of P_1 or P_2 will always be distinct. This gives us at least $2^{n/(2q)}$ peripheral 2-bases of G, as desired. □

3.8 Flows in $\mathbb{Z}_6$

In this section we will first prove a lemma that provides the existence of many peripheral 2-bases in a 3-edge-connected cubic graph with a long peripheral cycle. We will then use this to prove our main theorem showing the existence of many $\mathbb{Z}_6$-flows in the setting of group connectivity for 3-edge-connected graphs.

Lemma 3.28. *Let $G = (V, E)$ be an oriented 3-edge-connected cubic graph with a peripheral cycle C with $|V(C)| = q$. For every $f : E \to \mathbb{Z}_2 \times \mathbb{Z}_3$ there exist at least $2^{2q/3}$ flows $\phi : E \to \mathbb{Z}_2 \times \mathbb{Z}_3$ with $\phi(e) \neq f(e)$ for every $e \in E$.*

Proof. Put $f = (f_1, f_2)$. Now choose a partition of E into $\{B, T\}$ so that B is a 2-base with $E(C) \subseteq B$ and T is the edge-set of a spanning tree. Apply Lemma 3.17 to choose a flow $\phi_2 : E \to \mathbb{Z}_3$ satisfying $\phi_2(e) \neq f_2(e)$ for every $e \in T$. By possibly modifying ϕ_2 by adding an elementary flow around C, we may further assume that $A = \{e \in E(C) \mid \phi_2(e) \neq f_2(e)\}$ satisfies $|A| \geq \frac{2}{3}|E(C)| \geq \frac{2}{3}q$. Let $B' = \{e \in B \setminus A \mid f_1(e) = 0\}$, and for every $e \in B$, let C_e be the edge-set of the fundamental cycle of e with respect to the spanning tree (V, T). Now for every set S with $B' \subseteq S \subseteq B' \cup A$, there is a $\mathbb{Z}_2$-flow ϕ_1 with support $\bigoplus_{e \in S} C_e$ and the $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow $\phi = (\phi_1, \phi_2)$ satisfies $\phi(e) \neq f(e)$ for every $e \in E$. There are $2^{|A|}$ choices for S, each of which gives a different flow. Since $|A| \geq \frac{2}{3}q$, this gives the desired bound. □

With this last lemma in place we are ready to prove Theorem 3.11, our main theorem concerning flows in $\mathbb{Z}_6$. We have restated it for convenience.

Theorem. Let $G = (V, E)$ be an oriented 3-edge-connected graph with $\ell = |E| - |V| \geq 11$, and let $f : E \to \mathbb{Z}_2 \times \mathbb{Z}_3$. There exist at least $2^{\sqrt{\ell}/\log \ell}$ flows $\phi : E \to \mathbb{Z}_2 \times \mathbb{Z}_3$ with the property that $\phi(e) \neq f(e)$ for every $e \in E$.

Proof. Apply Lemma 3.12 to choose a cubic graph $G' = (V', E')$ with $|V'| = 2\ell$ so that G can be obtained from G' by contracting the edges of a forest. Extend the function f to have domain E' arbitrarily, and let $f = (f_1, f_2)$ where $f_1 : E' \to \mathbb{Z}_2$ and $f_2 : E' \to \mathbb{Z}_3$. Observe that by Theorem 3.13 it suffices to prove the result for G' in place of G.

If G' has a peripheral cycle of length at least $q = \frac{3}{2}\sqrt{\ell}/\log \ell$, then the result follows immediately from the previous lemma. Otherwise, by Lemma 3.27, we can choose $N = \lceil 2^{2\ell/(2q)} \rceil = \lceil 2^{\frac{2}{3}\sqrt{\ell}\log \ell} \rceil$ pairwise distinct partitions of E', say $\{T_1, B_1\}, \ldots, \{T_N, B_N\}$ where for $1 \leq i \leq N$ the set T_i is the edge-set of a spanning tree and B_i is a 2-base. (Note that $|B_i| = \ell + 1$.) For each such partition $\{T_i, B_i\}$, $1 \leq i \leq N$, we apply Lemma 3.17 to choose a $\mathbb{Z}_3$-flow $\phi_i : E' \to \mathbb{Z}_3$ satisfying $\phi_i(e) \neq f_2(e)$ for every $e \in E' \setminus B_i$.

First suppose there exists $1 \leq i \leq N$ for which the flow ϕ_i has the property that $\phi_i(e) \neq f_2(e)$ holds for at least $\sqrt{\ell}/\log \ell$ edges $e \in B_i$. In this case, we may proceed as in the proof of the previous lemma to construct $2^{\sqrt{\ell}/\log \ell}$ flows: Set $A = \{e \in B_i \mid \phi_i(e) \neq f_2(e)\}$, set $B' = \{e \in B_i \setminus A \mid f_1(e) = 0\}$, and then for every $B' \subseteq S \subseteq B' \cup A$ form a $\mathbb{Z}_2$-flow with support $\bigoplus_{e \in S} C_e$ (where C_e is the edge-set of the fundamental cycle of e with respect to the tree T_i) and combine this with ϕ_i to get at least $2^{\sqrt{\ell}/\log \ell}$ valid flows.

Thus we may assume that every ϕ_i has at most $\sqrt{\ell}/\log \ell$ edges $e \in B_i$ for which $\phi_i(e) \neq f_2(e)$. This means that each flow ϕ_i will agree with the function f_2 on all but at most $\sqrt{\ell}/\log \ell$ elements of B_i and on no elements in T_i. In particular, we have

$$\ell + 1 - \sqrt{\ell}/\log \ell \leq |\{e \in E' \mid \phi_i(e) = f_2(e)\}| \leq \ell + 1. \tag{3.1}$$

Now let $\nu : E' \to \mathbb{Z}_3$ be a flow, let $A = \{e \in E' \mid \nu(e) = f_2(e)\}$ and let $r := \ell + 1 - |A|$. We will find an upper bound on the number of indices $1 \leq i \leq N$ for which $\phi_i = \nu$. If $r < 0$ or $r > \sqrt{\ell}/\log \ell$, then (3.1) shows that $\nu \neq \phi_i$ for every $1 \leq i \leq N$. Otherwise, in order for $\nu = \phi_i$ it must be that the 2-base B_i consists of all of the edges in A plus r edges from $E' \setminus A$. The number of ways to select such a set is equal to $\binom{|E \setminus A|}{r}$, which we further estimate using the bound $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ and the fact that the function $k \mapsto (\frac{en}{k})^k$ is increasing for $k < n$:

$$\binom{|E \setminus A|}{r} = \binom{2\ell - 1 + r}{r} \leq \left(\frac{3e\ell}{r}\right)^r \leq \left(\frac{3e\ell}{\sqrt{\ell}/\log \ell}\right)^{\sqrt{\ell}/\log \ell} = \left(3e\sqrt{\ell}\log \ell\right)^{\sqrt{\ell}/\log \ell}.$$

It follows that the number of distinct flows in our list $\phi_1, \dots, \phi_N$ is at least

$$\frac{2^{\frac{2}{3}\sqrt{\ell}\log\ell}}{2^{\left(\sqrt{\ell}/\log\ell\right)\log\left(3e\sqrt{\ell}\log\ell\right)}} = 2^{(\sqrt{\ell}/\log\ell)\left(\frac{2}{3}(\log\ell)^2-\log(3e\sqrt{\ell}\log\ell)\right)}.$$

Since $\ell \geq 11$, we have $\frac{2}{3}(\log\ell)^2 - \log(3e\sqrt{\ell}\log\ell) \geq 1$, and our list $\phi_1, \phi_2, \dots, \phi_N$ contains at least $2^{\sqrt{\ell}/\log\ell}$ distinct flows. For every $1 \leq i \leq N$ we may apply Lemma 3.17 to choose a flow $\psi_i : E' \to \mathbb{Z}_2$ so that $\psi_i(e) \neq f_1(e)$ holds for every $e \in B_i$. So, every (ψ_i, ϕ_i) is a $\mathbb{Z}_2 \times \mathbb{Z}_3$ flow for which $(\psi_i(e), \phi_i(e)) \neq (f_1(e), f_2(e))$ holds for every $e \in E'$ and we have at least $2^{\sqrt{\ell}/\log\ell}$ such flows, thus completing our proof. □

3.9 Open problems

As our main question, we would like to know what is that status of Conjecture 3.10 for $\mathbb{Z}_6$ and $\mathbb{Z}_7$. However, we also want to list here some further questions that came up during our work on this paper.

We conjecture that a 3-edge-connected, nonplanar graph with representativity at least 5 has exponentially many peripheral cycles. Note that for 3-edge-connected planar graphs, peripheral cycles are exactly facial walks, so there is at most $2n-4$ of them.

As mentioned before, a result of Jaeger et al. [17] (Theorem 3.25) gives a decomposition of a graph obtained from a 3-connected cubic graph by deleting a single vertex into a 1-base and a 2-base. We conjecture that such graph can also be decomposed into three 2-bases.

CHAPTER 4

The number of group colorings in simple planar graphs

This chapter is based on the paper [28] by L. and Thomassen.

For planar graphs it is well-known that all planar graphs are 5-list-colorable [39], and not all planar graphs are 4-list-colorable [48]. Furthermore, in [40] it is shown that there are even exponentially many 5-list-colorings of all planar graphs. In this chapter we extend this result to group colorings.

Before we discuss the techniques used in the paper we include a proof of the $\mathbb{Z}_5$-Color Theorem. The proof is an easy adaption of the proof of the 5-List-Color Theorem [39]. As far as we know this proof has not appeared anywhere, but in [25] Lai and Zhang prove a stronger version of this result.

We also include a discussion on the τ-function and how this function can be used to view group colorings from a certain perspective. This can be very useful when reading proofs in which specific graphs are colored, for example the graphs of Lemma 4.17 a) and b).

$\mathbb{Z}_5$-coloring planar graphs

We prove the group coloring version of the 5-list-coloring theorem. As pointed out in [35] the proof follows by an adaption of the proof in [39]. Note that we use the τ-function as defined in Definition 4.6 later in this chapter.

Theorem 4.1. *Let G be a simple planar graph. Then G is $\mathbb{Z}_5$-colorable.*

Proof. We prove the more general statement: If G is an oriented near-triangulation in which two neighboring vertices on the outer cycle are precolored, the rest of the vertices on the outer cycle have at most two forbidden colors each, and any vertex in the interior has no forbidden colors, then given any $\varphi : E(G) \to \mathbb{Z}_5$, G can be $(\mathbb{Z}_5, \varphi)$-colored.

Let the outer cycle be denoted by C and consist of vertices $v_1, v_2, \dots, v_k$ where v_1, v_2 are precolored. We denote by F_v the set of forbidden colors at vertex v. We prove the statement by induction on the number of vertices in G. If G has three vertices there is nothing to prove. So suppose G has at least four vertices. If G has a chord v_iv_j where $2 \leq i \leq j-2 \leq k-1$ then we apply the induction hypothesis first to the graph G_1 consisting of the cycle $v_1v_2 \dots v_iv_j \dots v_1$ and its interior and then to the graph G_2 consisting of the cycle $v_iv_{i+1} \dots v_jv_i$ and its interior. So we can assume that G has no chord.

Let $v_1, u_1, u_2, \dots, u_p, v_{k-1}$ be the neighbors of v_k. As the interior of C is triangulated $v_1u_1u_2 \cdots u_pv_{k-1}$ forms a path P, and as C is chordless, $(C - v_k) \cup P$ is a cycle. Let a, b be distinct colors in $\mathbb{Z}_5 \setminus (F_{v_k} \cup \{\tau_{v_1}(v_k)\})$. Now define $F_{u_i} = \{\tau_{v_k}(a, u_i), \tau_{v_k}(b, u_i)\}$ for $1 \leq i \leq p$. Then the induction hypothesis applied to the graph $G' := G - v_k$ gives a coloring of all vertices except v_k. We complete the coloring by assigning either a or b to v_k such that $c(v_k) \neq \tau_{v_{k-1}}(v_k)$. □

Techniques

The τ-function and the matching graph

In the paper we introduce a new notation called the τ-function. Let v be some vertex. Given a forbidden function φ, this function takes as input (if v is not precolored) a color and a neighbor of v, and determines the color which is forbidden at the neighbor by the color of v and the value of φ on their common edge. In this paper the colors are elements of $\mathbb{Z}_5$, but in general it can be elements of any given Abelian group Γ.

This notation is useful since we can forget the specific orientation of the graph. Given a forbidden function $\varphi : E(G) \to \Gamma$, the τ-function can also be used to construct the so-called *matching graph* of a graph G with n vertices:

Given $\alpha, \beta \in \Gamma$ and $uv \in E(G)$, we will say that α and β *match* on uv if $\tau_v(\alpha, u) = \beta$. Observe, that this is equivalent to $\varphi(uv) = \alpha - \beta$ if uv is directed towards v, and $\varphi(uv) = \beta - \alpha$ if uv is directed towards u. Given lists L_v of available colors on v for every $v \in V(G)$, we will define the *matching graph*, M_G, of G to be the n-partite graph consisting of $\sum_{v \in V(G)} |L_v|$ vertices, one for each color in the lists, and edges between matching colors. Now, G is (Γ, φ)-colorable if and only if it is possible to find an independent set consisting of n vertices in the matching graph, one in each part.

Proposition 4.8 in the paper gives an important result on the $\mathbb{Z}_5$-coloring of triangles, namely that if one of the vertices is given a color such that the two colors forbidden by this color on the other two vertices also forbid each other, then this happens no matter what color the first vertex has. This implies that the matching graph of the triangle has a special structure, as seen in the following observation.

Observation 4.2. Given $\varphi : E(G) \to \mathbb{Z}_5$, $u, v, w \in V(G)$ such that $uv, vw, uw \in E(G)$, and lists L_u, L_v, L_w of available colors on u, v, w, the structure of the matching graph M of uvw is as follows.

a) If $L_u = L_v = L_w = \mathbb{Z}_5$ then the elements of L_u, L_v, L_w either form a 15-cycle or five 3-cycles.

b) If at least one of L_u, L_v or L_w are different from $\mathbb{Z}_5$, then the matching graph is a subgraph of the matching graph above, hence it consists of a collection of paths and 3-cycles.

Proof. Proof of a): The edges between two of L_u, L_v, L_w is a matching with five edges. Hence M is a collection of cycles. If $\varphi(uv) + \varphi(vw) + \varphi(wu) \equiv 0 \pmod 5$, they are all 3-cycles. Otherwise, M is a 15-cycle.

b) follows immediately from a). □

A minimal counterexample

The proofs of the main Theorems 4.19 and 4.23 use a common technique, namely that of a minimal counterexample. That is, assume that the statement of the theorem is false, and consider a graph for which it does not hold such that the number of vertices is as small as possible. Now we may prove several properties of this graph using the fact that all smaller graphs, and in particular all subgraphs, satisfy the statement of the theorem.

In Theorem 4.19 we prove that all near-triangulations except a special class of graphs are $\mathbb{Z}_5$-colorable when three vertices on the outer cycle are precolored and the remaining vertices on the outer cycle have at least three available colors. We prove that a minimal counterexample, among other properties, has no separating triangles or 4-cycles, has no chords, and that certain vertices must have degree at least 4. These properties imply that the graph has a very special structure which enables us to $\mathbb{Z}_5$-color after all.

In Theorem 4.23 we prove that all triangulated graphs have at least $2^{n/9-r/3}$ $\mathbb{Z}_5$-colorings when three vertices on the outer cycle are precolored and the remaining vertices on the outer cycle have at least three available colors. (Here, n is the number of non-precolored vertices, and r is the number of vertices with precisely three available colors.) Again, we prove that a minimal counterexample, among other properties, has no separating triangles, has no chords, and that the outer cycle has at least 5 vertices. In the end we show that the graph has many $\mathbb{Z}_5$-colorings after all.

Coloring special triangulations

In order to complete the proofs mentioned above it is necessary to color several different classes of graphs. One strategy is to use what we refer to as a degeneracy argument in Section 1.3 of the introduction. We will explain how: If we can find an ordering $v_1, v_2, \ldots, v_n$ of the vertices, such that a vertex v_i has less neighbors among $v_1, \ldots, v_{i-1}$ than the number of available colors at v_i, then it is easy to color $v_1, v_2, \ldots, v_n$ in this order since at each vertex there is always at least one available color left. This technique is used in several proofs in the paper, most notably Lemma 4.17 and Lemma 4.22.

Exponentially many $\mathbb{Z}_5$-colorings in simple planar graphs [3]

Abstract

Every planar simple graph with n vertices has at least $2^{n/9}$ $\mathbb{Z}_5$-colorings.

4.1 Introduction

List coloring and group coloring are generalizations of (ordinary) graph coloring. While the two generalizations are formally unrelated, it is believed that group coloring is more difficult than list coloring. Specifically, it is conjectured in [20] that, for every graph G, the list chromatic number is less than or equal to the group chromatic number which is defined as the smallest k such that G is Γ-colorable (defined below) for every group Γ of order at least k. In [30] it is conjectured that the list chromatic number is even less than or equal to the weak group chromatic number which is the smallest k such that G is Γ-colorable for **some** group Γ of order k. (In [30] it is proved that the two group chromatic numbers are bounded by each other, but may differ by a factor close to 2.)

Group connectivity and group coloring are introduced by Jaeger et al. in [17]. For planar graphs they are dual concepts. It was shown in [6] that graphs with an edge-connectivity condition imposed have exponentially many group flows for groups of order at least 8. [42] proved the weak 3-flow conjecture, specifically, every 8-edge-connected graph has a nowhere zero 3-flow. In [32] the proof was refined to 6-edge-connected graphs, and in [8] that refinement was used to prove that every 8-edge-connected graph has exponentially many nowhere-zero 3-flows.

The groups of order 3, 4, 5 are particularly interesting because they relate to the 4-color theorem and Tutte's flow conjectures. Jaeger et al. [17] conjectured that every 3-edge-connected graph is $\mathbb{Z}_5$-connected, which is a strengthening of Tutte's 5-Flow Conjecture. In [40] it was proven that planar graphs have exponentially many 5-list-colorings (for every list assignment to the vertices), and in [41] it was proven that planar graphs of girth at least 5 have exponentially many 3-list-colorings (for every list assignment to the vertices). Perhaps somewhat surprising, the list-color proof in [39] carries over, word for word, to a proof of Theorem A below.

Theorem A. Let G be a simple planar graph. Then G is $\mathbb{Z}_5$-colorable.

Also, the proof in [41] needs only minor modifications to give the analogous result for group coloring (saying that planar graphs of girth at least 5 have exponentially many $\mathbb{Z}_3$-colorings). However, the proof in [40] does not immediately extend to group coloring. In this paper we prove

[3] The content of the remaining part of this chapter is the paper [28] which is in submission. This is joint work with Carsten Thomassen.

Theorem B. Every planar simple graph with n vertices has at least $2^{n/9}$ $\mathbb{Z}_5$-colorings.

Note, that Theorem A proves the conjecture of Jaeger et al. when restricted to planar graphs. Theorem B even shows that there are many solutions for this family of graphs.

Although both the bound in Theorem B as well as the strategy of proof are identical to those in [40], the details are significantly different. The main idea in [40] (as well as in the present paper) is an application of the 5-list-color theorem in [39]. In [39] two neighboring vertices on the outer cycle are allowed to be precolored. In [40] (and in the present paper) we need the extension where a path with three vertices on the outer cycle is precolored. Such a coloring cannot always be extended, but the exceptions (called *generalized wheels* in [40]) are easily characterized and studied. Their nice behaviour allows exponentially many list colorings. For group colorings, however, there are more exceptions (which we call *generalized multi-wheels*) and, more important, their group coloring properties are far more subtle. Here, the group structure is essential, and the proof does not extend to e.g. DP-colorings [9]. We do conjecture, though, that the planar graphs have exponentially many DP-colorings. That would be a common generalization of [40] and the present paper.

Also, we conjecture that if every graph in a graph family has exponentially many $\mathbb{Z}_k$-colorings, then it has exponentially many k-list-colorings. As mentioned earlier, a graph may be Γ-colorable and non-Γ'-colorable for some Abelian groups Γ, Γ' where $|\Gamma| < |\Gamma'|$. But maybe the existence of many Γ-colorings implies some (or even many) Γ'-colorings. Jaeger et al. [17] proved that $\mathbb{Z}_5$-colorability does not imply $\mathbb{Z}_6$-colorability. A planar graph has at least $(k-5)^k$ $\mathbb{Z}_k$-colorings. For $\mathbb{Z}_6$ we can do better: By repeating the present proof we obtain at least $2^{n/9}$ $\mathbb{Z}_6$-colorings. By using the proof in [39] we obtain even $(3/2)^n$ $\mathbb{Z}_6$-colorings.

In this paper, we will maintain the same structure as in [40]. Thus the theorems and lemmas, etc., will be given the same numbers, and those proofs from [40] which carry over will stand as in [40].

4.2 Definitions

In this paper we consider simple planar graphs. We follow the notation of Mohar and Thomassen [33]. Each edge in the graph will be given an orientation. The orientation will be fixed, but the specific orientation of a graph will not be important.

We will introduce an additional constraint on the group colorings of graphs by letting $F_v \subseteq \mathbb{Z}_5$ denote a set of *forbidden* colors at the vertex v. We thus require a group coloring $c : V(G) \to \mathbb{Z}_5$ to satisfy $c(v) \notin F_v$. Furthermore, we will use the notation $L_v = \mathbb{Z}_5 \setminus F_v$ to denote the set of *available* colors at v.

In general we define group colorability as follows.

Definition 4.3. Let Γ be an Abelian group. The graph G is said to be Γ-*colorable* if the following holds: Given some orientation of G and any function $\varphi : E(G) \to \Gamma$

there exists a vertex coloring $c : V(G) \to \Gamma$ such that $c(w) - c(u) \neq \varphi(uw)$ for each $uw \in E(G)$ where uw is directed towards w.

If this holds we say that c is *proper* with respect to φ. If the function $\varphi : E(G) \to \Gamma$ is given, we define (Γ, φ)-colorability as follows:

Definition 4.4. G is said to be (Γ, φ)-*colorable* if there exists a vertex coloring $c : V(G) \to \Gamma$ such that $c(w) - c(u) \neq \varphi(uw)$ for each $uw \in E(G)$ where uw is directed towards w.

Note on notation: Formally, $\varphi(uw)$ is defined on every directed edge uw. But, we also write $\varphi(wu) = -\varphi(uw)$.

The following property of group coloring will prove useful later.

Proposition 4.5. *Let* $\varphi : E(G) \to \Gamma$. *Given* $v_0 \in V(G)$ *and* $\alpha \in \Gamma$, *we define* $\varphi' : E(G) \to \Gamma$ *as follows:*

$$\varphi'(e) = \begin{cases} \varphi(e) + \alpha & \text{if } e \text{ is incident to } v_0 \text{ and directed towards } v_0, \\ \varphi(e) - \alpha & \text{if } e \text{ is incident to } v_0 \text{ and directed away from } v_0, \\ \varphi(e) & \text{otherwise.} \end{cases} \tag{4.1}$$

Then G *is* (Γ, φ)*-colorable if and only if* G *is* (Γ, φ')*-colorable.*

4.2.1 The function τ

In addition to the standard definitions in Section 2 above we introduce a collection of functions τ which, given a coloring of some vertex $v \in V(G)$, determines the colors at the neighboring vertices of v that are not allowed by the coloring of v:

Definition 4.6. Given a function $\varphi : E(G) \to \mathbb{Z}_5$ and a vertex v with prescribed color $c(v)$ we will define the function $\tau_v : N(v) \to \mathbb{Z}_5$ to be:

$$\tau_v(u) = \begin{cases} c(v) + \varphi(uv) & \text{if } uv \text{ is directed towards } u, \\ c(v) - \varphi(uv) & \text{if } uv \text{ is directed towards } v. \end{cases} \tag{4.2}$$

In case the coloring of v is not prescribed we will define $\tau_v : \mathbb{Z}_5 \times N(v) \to \mathbb{Z}_5$ to be:

$$\tau_v(\alpha, u) = \begin{cases} \alpha + \varphi(uv) & \text{if } uv \text{ is directed towards } u, \\ \alpha - \varphi(uv) & \text{if } uv \text{ is directed towards } v. \end{cases} \tag{4.3}$$

Furthermore, given $S \subseteq \mathbb{Z}_5$ we will define $\tau_v(S, u) := \{\tau_v(s, u) \mid s \in S\}$.

Note, that τ is well-defined since G is simple. Now, a $\mathbb{Z}_5$-coloring $c : V(G) \to \mathbb{Z}_5$ of G is proper with respect to φ if and only if $c(u) \neq \tau_v(u)$ for all pairs of neighbors $v, u \in V(G)$. Informally, if we give v the color α, then we cannot give u the color $\tau_v(\alpha, u)$.

Note, that when using the τ-function we will no longer need to specify the orientation of G.

Observation 4.7. If $c(v) = \alpha$, then $\tau_u(\tau_v(u), v) = \alpha$ for $\alpha \in \mathbb{Z}_5$ (regardless of the value of $\varphi(uv)$). Similarly, $\tau_u(\tau_v(S, u), v) = S$ for $S \subseteq \mathbb{Z}_5$. This can also be expressed as $\tau_v(\alpha, u) = \beta$ if and only if $\tau_u(\beta, v) = \alpha$ for $\alpha, \beta \in \mathbb{Z}_5$, and $\tau_v(S_1, u) = S_2$ if and only if $\tau_u(S_2, v) = S_1$ for $S_1, S_2 \subseteq \mathbb{Z}_5$.

The function τ can also be defined for the more general DP-colorings introduced by Dvořák and Postle in [9], and Observation 4.7 also holds in this more general setup whereas the following Proposition 4.8 which is an important feature of group colorings does not.

Proposition 4.8. *Given vertices $u, v, w \in V(G)$ such that $uv, vw, uw \in E(G)$. If $\tau_v(\tau_u(\alpha, v), w) = \tau_u(\alpha, w)$ for some $\alpha \in \mathbb{Z}_5$, then it holds for any $\alpha \in \mathbb{Z}_5$.*

Proof. We can assume without loss of generality that uv is directed towards v, vw is directed towards w, and wu is directed towards u. If there exists an $\alpha \in \mathbb{Z}_5$, such that $\tau_v(\tau_u(\alpha, v), w) = \tau_u(\alpha, w)$, then

$$(\alpha + \varphi(uv)) + \varphi(vw) = \alpha - \varphi(wu). \tag{4.4}$$

Hence $\varphi(uv) + \varphi(vw) + \varphi(wu) = 0 \pmod 5$. Thus $\tau_v(\tau_u(\alpha, v), w) = \tau_u(\alpha, w)$ for any $\alpha \in \mathbb{Z}_5$. □

4.3 $\mathbb{Z}_5$-colorings with precolored vertices

In the rest of this paper we assume G is an oriented plane near-triangulation with outer cycle $C : v_1 v_2 \dots v_k v_1$.

Definition 4.9. Given $\varphi : E(G) \to \mathbb{Z}_5$ we say that G is $(\mathbb{Z}_5, 3)$-*extendable* with respect to φ and the vertices v_1, v_2, v_k if the following holds: Assume that the vertices v_k, v_1 and v_2 are precolored $c(v_k), c(v_1), c(v_2)$, respectively, such that $c(v_k) \neq \tau_{v_1}(v_k)$ and $c(v_2) \neq \tau_{v_1}(v_2)$, and for each $v \in C \setminus \{v_1, v_2, v_k\}$, F_v is a set containing at most two forbidden colors. For all other vertices v, F_v is empty. Then c can be extended to a $(\mathbb{Z}_5, \varphi)$-coloring of G which we also call c and which satisfies $c(v) \notin F_v$ for any $v \in C \setminus \{v_1, v_2, v_k\}$.

Note, that the analogous definition of $(\mathbb{Z}_5, 2)$-*extendability* is used to prove Theorem A above which can be phrased as follows:

Theorem 4.10. *Any oriented near-triangulation is $(\mathbb{Z}_5, 2)$-extendable with respect to any φ-function and any path on two vertices on the outer cycle.*

This implies the following:

Theorem 4.11. *Let $\varphi : E(G) \to \mathbb{Z}_5$ be given where G is a near-triangulation with precolored outer cycle C of length $k \leq 5$. Then G has a $(\mathbb{Z}_5, \varphi)$-coloring unless C has length precisely 5, and int(C) has a vertex v joined to all vertices of C such that $\{\tau_{v_1}(v), \dots, \tau_{v_5}(v)\} = \mathbb{Z}_5$.*

Proof. The proof is by induction on the number of vertices of G. If no vertex of $int(C)$ is joined to more than two vertices of C, then we consider the subgraph H induced by the vertices in $int(C)$. We let the set of forbidden colors of a vertex be the colors forbidden by its neighbors in C. By Theorem 4.10, H is $(\mathbb{Z}_5, 2)$-extendable with these sets of forbidden colors. (If H is not 2-connected, then we color the blocks of H successively.) So we may assume that some vertex u has at least three neighbors in C. If it is not possible to color u, then G satisfies the conclusion of Theorem 4.11. On the other hand, if it is possible to color u, then we color it and complete the proof by induction by coloring the interior of each precolored cycle on the form $v_i \cdots v_{i+j} u v_i$ (where $j = 1, 2, 3$). The only case where this might not work is if there is some vertex v in the interior of one of the colored cycles which is joined to all vertices in a colored 5-cycle, but then u must have precisely three consecutive neighbors in C, and we therefore have two possibilities for coloring u. So, the exceptional case in Theorem 4.11 can be avoided. □

4.4 Generalized wheels and generalized multi-wheels

We define *wheels*, *broken wheels* and *generalized wheels* as in [40]: The outer cycle C is of the form $v_1 v_2 \cdots v_k v_1$ where v_1 is the *major vertex*, v_k, v_2 are *principal neighbours*, $v_k v_1, v_1 v_2$ are *principal edges*, and $v_k v_1 v_2$ is the *principal path*. If the interior of C consists of the edges $v_1 v_3, v_1 v_4, \ldots, v_1 v_{k-1}$, then we call G a *broken wheel*. If the interior of C consists of a vertex v and the edges $vv_1, vv_2, \ldots, vv_k$, then we call G a *wheel*. We define *generalized wheels* to be the class of graphs containing all broken wheels and wheels, as well as the graphs obtained from two generalized wheels by identifying a principal edge in one of them with a principal edge in the other such that their major vertices are identified.

Note, that it is easy to see that a broken wheel with at least four vertices is not $(\mathbb{Z}_5, 3)$-extendable with respect to v_k, v_1, v_2, and a wheel with an even number of (at least six) vertices is not $(\mathbb{Z}_5, 3)$-extendable with respect to v_k, v_1, v_2.

In addition to these graphs we need a class of graphs which extends the wheels, as well as a class which extends the generalized wheels.

We define an operation as follows: Let G be a generalized wheel and assume that $v_i, u, v_{i+1} \in V(G) \setminus \{v_1\}$ form a facial triangle where $v_i v_{i+1}$ is an edge on the outer cycle C and $v_i u, v_{i+1} u$ are edges in $int(C)$. We obtain a new graph G' from G by adding a new vertex w and the edges $uw, v_i w, v_{i+1} w$, as well as replacing the edge $v_i v_{i+1}$ by a path $v_i w_1 \cdots w_j v_{i+1}$ with $j \geq 0$ and adding the edges $w_1 w, \ldots, w_j w$. We say that we *insert a wheel* into the triangle $v_i u v_{i+1}$.

Definition 4.12. We define *multi-wheels* to be the class of graphs containing all wheels, as well as the graphs obtained from a multi-wheel by inserting a wheel into a triangle as above.

Definition 4.13. We define *generalized multi-wheels* to be the class of graphs containing all generalized wheels, as well as the graphs obtained from a generalized multi-wheel by inserting a wheel into a triangle as above.

Note, that a broken wheel is also a generalized wheel (and therefore also a generalized multi-wheel), but a broken wheel is not a multi-wheel.

Observe, that if we replace the operation in Definition 4.13 by inserting generalized multi-wheels into triangles instead of inserting wheels, then we get the exact same class of graphs.

Proposition 4.14. *Let G be a generalized multi-wheel with outer cycle C. If $uvwu$ is a facial triangle with $u, v, w \in V(G)$, then at least one of u, v, w is on $C - v_1$.*

Proof. The statement is clearly true for all facial triangles in wheels, broken wheels and generalized wheels. As the statement remains true whenever a wheel is inserted into a triangle, it is also true for multi-wheels and generalized multi-wheels. ☐

Lemma 4.15. *Let $\varphi : E(G) \to \mathbb{Z}_5$ be given where G is a multi-wheel. Assume that for each $v \in \{v_3, v_4, \ldots, v_{k-1}\}$, F_v is a forbidden set containing at most two colors in $\mathbb{Z}_5$. For all other vertices v, F_v is empty. Then there exists $\alpha \in \mathbb{Z}_5$ such that the $(\mathbb{Z}_5, \varphi)$-colorings of v_k, v_1, v_2 which cannot be extended to G satisfy that $c(v_k) - c(v_2) = \alpha$.*

It is easy to see such an α does not exist if G is a broken wheel on 4 (and hence any larger number of) vertices. This may explain why the proof of Lemma 4.15 is not trivial.

Proof of Lemma 1. We prove Lemma 4.15 by induction on the number of vertices n. Assume $n \geq 5$ since otherwise there is nothing to prove. Also, by Theorem 4.11 we can assume that $k \geq 5$. Consider first the case where G is a wheel. Let v be the vertex not in C. Suppose v_k, v_1, v_2 are colored $c(v_k), c(v_1), c(v_2)$, respectively, and that this coloring cannot be extended to G. Construct $\varphi' : E(G) \to \mathbb{Z}_5$ from φ using Proposition 4.5 successively with $v_{k-1}, v_k, v_1, v_2, v_3$ respectively playing the role of v_0 such that $\varphi'(v_{k-1}v) = \varphi'(v_kv) = \varphi'(v_1v) = \varphi'(v_2v) = \varphi'(v_3v) = 0$ with corresponding precoloring $c'(v_k), c'(v_1), c'(v_2)$ and τ-function τ'. It suffices to prove Lemma 4.15 with this φ' instead of φ. Now $\tau'_{v_i}(v) = c'(v_i)$ for $i \in \{k, 1, 2\}$, and similarly $\tau'_{v_i}(\alpha, v) = \alpha$ for $i \in \{3, k-1\}, \alpha \in \mathbb{Z}_5$. Then $L_{v_3} \setminus \tau'_{v_2}(v_3)$ consists of precisely two colors of $\mathbb{Z}_5$, say α, β, since otherwise we can color v (with at least two color options) and extend that coloring to G by applying Theorem 4.10 to $G - v_1 - v_2$. Similarly, $L_{v_{k-1}} \setminus \tau'_{v_k}(v_{k-1})$ consists of precisely two colors, say γ, δ. If $L_v \setminus \{c'(v_k), c'(v_1), c'(v_2)\}$ contains a color ϵ distinct from α, β, then we can give v that color, put $L_{v_3} = \{\alpha, \beta, \epsilon\}$, and then extend the resulting coloring to G by applying Theorem 4.10 to $G - v_1 - v_2$, a contradiction. So we may assume that $L_v \setminus \{c'(v_k), c'(v_1), c'(v_2)\} = \{\alpha, \beta\}$. In particular, $c'(v_k), c'(v_1), c'(v_2)$ are distinct. Similarly, $L_v \setminus \{c'(v_k), c'(v_1), c'(v_2)\} = \{\gamma, \delta\}$. Thus $L_{v_3}, L_{v_{k-1}}$ have at least two colors in common, namely α, β. Consider first the case where $L_{v_3}, L_{v_{k-1}}$ have precisely two colors in common. In this case we argue as in [40]: $c'(v_2)$ is the unique color of $L_{v_3} \setminus L_{v_{k-1}}$, $c'(v_k)$ is the unique color of

$L_{v_{k-1}} \setminus L_{v_3}$, and $c'(v_1)$ is the unique color of $L_v \setminus (L_{v_3} \cup L_{v_{k-1}})$. This shows that the coloring of v_1, v_2, v_k is unique. Consider next the case where $L_{v_3}, L_{v_{k-1}}$ have more than two colors in common, that is, they are equal. In this case, assume without loss of generality that v_2v_3 is directed towards v_3 and v_kv_{k-1} is directed towards v_{k-1}. We observe that $\tau'_{v_2}(v_3) = \tau'_{v_k}(v_{k-1})$, i.e. $c'(v_2) + \varphi'(v_2v_3) = c'(v_k) + \varphi'(v_kv_{k-1})$. Thus $c'(v_k) - c'(v_2) = \varphi'(v_2v_3) - \varphi'(v_kv_{k-1})$ can play the role of α in Lemma 4.15.

Consider now the case where G is a multi-wheel, but not a wheel. Recall, that G is obtained from a multi-wheel by inserting a wheel into a triangle. More precisely, G has a vertex u in $int(C)$ joined to $v_i, v_{i+1}, \dots, v_j$ and also joined to a vertex v in $int(C)$ such that v is joined to v_i, v_j. We may assume that $j > i+1$ since otherwise, we delete u and complete the proof by induction. Let G' be the subgraph of G which has outer cycle $C' = vv_i \cdots v_jv$. Note that G' is a wheel. By the induction hypothesis there exists $\alpha' \in \mathbb{Z}_5$ such that all colorings of v_i, v, v_j which cannot be extended to G' satisfy $c(v_j) - c(v_i) = \alpha'$. Now use the induction hypothesis on the graph G'' obtained from G by replacing G' by the triangle vv_iv_jv (with v_iv_j directed towards v_j) where we define $\varphi(v_iv_j) = \alpha'$. All colorings of v_k, v_1, v_2 that can be extended to G'' clearly also extends to G. Thus the colorings of v_k, v_1, v_2 that cannot be extended to G satisfy the conclusion of Lemma 4.15 with the same α as the one we found for G'' using the induction hypothesis. □

Note, that α does not depend on $\varphi(v_kv_1), \varphi(v_1v_2)$. More precisely, if we let $\varphi' : E(G) \to \mathbb{Z}_5$ be a function that agrees with φ on all edges except v_kv_1, v_1v_2, then the α that works for φ also works for φ'.

Lemma 4.16. *Let $\varphi : E(G) \to \mathbb{Z}_5$ be given where G is a generalized multi-wheel with no separating triangles. Assume that each for each $v \in \{v_3, v_4, \dots, v_{k-1}\}$, F_v is a forbidden set containing at most two colors of $\mathbb{Z}_5$, and assume that v_k, v_1, v_2 are precolored. For all other vertices v, F_v is empty. Let e be any edge in $E(G) \setminus \{v_kv_1, v_1v_2\}$. Then $G - e$ has a $(\mathbb{Z}_5, \varphi)$-coloring $c : V(G) \to \mathbb{Z}_5$ that extends the precoloring and satisfies $c(v) \notin F_v$ for any $v \in C \setminus \{v_1, v_2, v_k\}$.*

Proof. By induction on the number of vertices in G. The statement is easy to verify if G is a broken wheel. Consider now the case where G is a wheel. Consider the subcase $e = vv_i$ where v is the vertex in $int(C)$. If $3 \le i \le k-1$ then we color $v, v_3, v_4, \dots, v_{i-1}, v_{k-1}, v_{k-2}, \dots, v_i$ in that order. If $i \in \{k, 1, 2\}$ then we choose the color of v such that $\tau_v(v_{k-1}) \notin L_{v_{k-1}} \setminus \{\tau_{v_k}(v_{k-1})\}$ (in case that set has precisely two colors) where $L_{v_{k-1}}$ denotes the list of available colors at v_{k-1}. Then we color $v_3, v_4, \dots, v_{k-1}$ in that order. If $e = v_iv_{i+1}$ is on C then we color $v, v_3, v_4, \dots, v_i, v_{k-1}$, $v_{k-2}, \dots, v_{i+1}$ in that order.

Consider next the case where G is a multi-wheel, but not a wheel. Recall, that G is obtained from a multi-wheel by inserting a wheel into a triangle. More precisely, G has a vertex u in $int(C)$ joined to $v_i, v_{i+1}, \dots, v_j$ and also joined to a vertex v in $int(C)$ such that v is joined to v_i, v_j. Now $j \ge i+2$ as G has no separating triangles. Let G' be the subgraph of G which has outer cycle $C' = vv_i \cdots v_jv$. Note that G' is a wheel. Let G'' be the graph obtained from G by replacing G' by the triangle vv_iv_jv with v_iv_j

directed towards v_j. If e is in $E(G'')$, then use Lemma 4.15 to obtain $\alpha \in \mathbb{Z}_5$ such that all colorings of v_i, v, v_j which cannot be extended to G' satisfy $c(v_j) - c(v_i) = \alpha$. The induction hypothesis implies that there exists a $(\mathbb{Z}_5, \varphi)$-coloring of $G'' - e$ where we define $\varphi(v_iv_j) = \alpha$. By Lemma 4.15 this coloring can be extended to G'. (If e is one of the two edges vv_i, vv_j, then we use the remark following the proof of Lemma 4.15.) Thus we get a $(\mathbb{Z}_5, \varphi)$-coloring of $G - e$.

If e is not in $E(G'')$, then the induction hypothesis implies that $G'' - e'$ is $(\mathbb{Z}_5, \varphi)$-colorable where $e' = v_iv_j$. This coloring can be extended to $G' - e$, again using the induction hypothesis. Thus $G - e$ is $(\mathbb{Z}_5, \varphi)$-colorable.

Assume now that G contains a chord v_1v_i and that e is not v_1v_i. Then v_1v_i divides G into near-triangulations G_1, G_2 where G_1 has outer cycle $v_1v_2 \cdots v_iv_1$ and G_2 has outer cycle $v_1v_iv_{i+1} \cdots v_1$. Assume without loss of generality that $e \in G_1$. Then G_2 can be colored by Theorem 4.10, and the induction hypothesis implies that the coloring can be extended to $G_1 - e$.

Assume finally that G is the union of two multi-wheels G_1, G_2 and $e = v_1v_i$ is their common edge. We may assume that v_i has precisely three available colors since otherwise we delete one or two available colors. By Theorem 4.10 each of G_1, G_2 can be $(\mathbb{Z}_5, \varphi)$-colored, and we get another coloring of each graph by using Theorem 4.10 on G_1, G_2 where we define $\varphi(v_1v_i)$ to be the color of v_i minus the color of v_1 in the first coloring. Thus we have two colorings of each of G_1, G_2 where v_i has distinct colors. Combining the colorings of G_1, G_2 in which the color of v_i is the same gives a $(\mathbb{Z}_5, \varphi)$-coloring of $G - e$. □

Lemma 4.17. *Let $\varphi : E(G) \to \mathbb{Z}_5$ be given where G is a near-triangulation.*

a) Assume that the interior of the outer cycle C has precisely two vertices u, v, and that there exists a natural number i, $3 \le i \le k-1$, such that u is joined to $v, v_1, v_2, \ldots, v_i$, and v is joined to $u, v_i, v_{i+1}, \ldots, v_k, v_1$. Then G is $(\mathbb{Z}_5, 3)$-extendable with respect to φ and the path $v_kv_1v_2$.

b) Assume next that the interior of the outer cycle C has precisely two vertices u, v, and that there exists a natural number i, $4 \le i \le k-1$, such that u is joined to $v, v_2, \ldots, v_i$, and v is joined to $u, v_1, v_2, v_i, v_{i+1}, \ldots, v_k$. Then G is $(\mathbb{Z}_5, 3)$-extendable with respect to φ and the path $v_kv_1v_2$.

Proof of a). Assume that v_k, v_1, v_2 are precolored. Let $S_u = \mathbb{Z}_5 \setminus \{\tau_{v_1}(u), \tau_{v_2}(u)\}$, $S_v = \mathbb{Z}_5 \setminus \{\tau_{v_k}(v), \tau_{v_1}(v)\}$, and $S_i = L_{v_i}$.

We give u a color from S_u, say α_u, such that $L_{v_3} \setminus \{\tau_{v_2}(v_3), \tau_u(\alpha_u, v_3)\}$ contains at least two colors. If $i = k-1$ then we color $v_{k-1}, v_{k-2}, \ldots, v_3, v$ in that order. So assume that $i \le k-2$, and, similarly, $i \ge 4$.

If it is now possible to color v such that $L_{v_i} \setminus \{\tau_v(v_i), \tau_u(v_i)\}$ has at least two colors, then it is easy to complete the coloring by coloring $v_{k-1}, v_{k-2}, \ldots, v_3$ in that order. So we may assume that such colorings of u and v, respectively, are not possible. Then we must have $|S_v| = |S_i| = 3$, so we let $S_v = \{\alpha_v, \beta_v, \gamma_v\}$, $S_i = \{\alpha_i, \beta_i, \gamma_i\}$. In particular, after u has received color α_u, the colors $\tau_u(\alpha_u, v) =: \alpha_v$ and $\tau_u(\alpha_u, v_i) =: \alpha_i$ are no longer available at v and v_i, respectively, and furthermore $\tau_v(\beta_v, v_i) =: \beta_i$ and $\tau_v(\gamma_v, v_i) =: \gamma_i$ are the remaining available colors at v_i.

Similarly, we can choose a color δ from S_v, such that $L_{v_{k-1}} \setminus \{\tau_{v_k}(v_{k-1}), \tau_v(\delta, v_{k-1})\}$ contains at least two colors. If δ is not α_v, then we may color G by letting u have color α_u, v have color δ, and completing the coloring by coloring $v_i, v_{i-1}, \ldots, v_3, v_{i+1}, \ldots, v_{k-1}$ in that order. So we may assume that $\delta = \alpha_v$. As above, we conclude $|S_u| = 3$, so we let $S_u = \{\alpha_u, \beta_u, \gamma_u\}$. In particular, after v has received color α_v, the colors $\tau_v(\alpha_v, u) = \alpha_u$ and $\tau_v(\alpha_v, v_i) = \alpha_i$ (the latter equality holds since we know $\tau_v(\beta_v, v_i) = \beta_i$ and $\tau_v(\gamma_v, v_i) = \gamma_i$) are no longer available at u and v_i, respectively. Choose the notation for β_u, γ_u such that $\tau_u(\beta_u, v_i) = \beta_i$ and $\tau_u(\gamma_u, v_i) = \gamma_i$ are the remaining available colors at v_i. Using Proposition 4.8, we get that $\tau_u(\beta_u, v) = \beta_v$ and $\tau_u(\gamma_u, v) = \gamma_v$. Thus, on the triangle uvv_iu any $(\mathbb{Z}_5, \varphi)$-coloring must consist of one α, one β and one γ.

Now, we give u the color α_u. If $\tau_v(\beta_v, v_{k-1})$ is not in $L_{v_{k-1}} \setminus \tau_{v_k}(v_{k-1})$ then we give v color β_v and we color $v_i, v_{i+1}, \ldots, v_{k-1}, v_{i-1}, \ldots, v_3$. So we may assume that $\tau_v(\beta_v, v_{k-1})$, $\tau_v(\gamma_v, v_{k-1})$ are the only colors in $L_{v_{k-1}} \setminus \tau_{v_k}(v_{k-1})$.

We give u the color α_u, we give v the color β_v, and we color $v_{k-1}, v_{k-2}, \ldots, v_{i+1}$. If this coloring can be extended to v_i, it is easy to complete the coloring by coloring $v_{i-1}, \ldots, v_3$. So we may assume that v_{i+1} has color $\tau_{v_i}(\gamma_i, v_{i+1})$.

We now try another coloring. We give u the color γ_u, we give v the color α_v, and we color $v_3, v_4, \ldots, v_{i-1}$. We may assume that this coloring cannot be extended to v_i, that is, v_{i-1} has color $\tau_{v_i}(\beta_i, v_{i-1})$.

Now we keep the colors of $v_3, v_4, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k-1}$ given above. And we give u, v, v_i the colors $\gamma_u, \beta_v, \alpha_i$, respectively. This gives a $(\mathbb{Z}_5, \varphi)$-coloring of G. □

Proof of b). Assume again that v_k, v_1, v_2 are precolored. We delete the precolored vertices and call the resulting graph H. Note that v_3, v_{k-1}, v each has at least two available colors, u has at least four available colors, and each other vertex has at least three available colors. We complete the proof by induction on the number of vertices of H.

Consider first the case where $i = k-1$, that is, v has degree 2 in H. It is easy to see that we can give u a color such that two of v_3, v_{k-1}, v still has at least two available colors. We then color the third of v_3, v_{k-1}, v, and thereafter it is easy to color the remaining vertices one by one.

Consider next the case where $i = k-2$, that is, v has degree 3 in H. If possible, we give v_{k-2} a color such that each of v, v_{k-1} still has two available colors. Then we delete v_{k-2}, v_{k-1} and can easily color the rest of H. So assume that such a coloring of v_{k-2} is not possible. If it is possible to color one of v, v_{k-1} such that v_{k-2} still has three available colors, then we color both of v, v_{k-1} such that v_{k-2} still has two available colors, we delete these two vertices, and then it is again easy to color the rest. So, we can assume that no such coloring of v or v_{k-1} is possible. Then $L_{v_{k-2}}$ contains a color α such that if we give v_{k-2} the color α, then each of v, v_{k-1} has precisely one available color left. Now let β, γ be two other colors in $L_{v_{k-2}}$. We choose the notation such that $\tau_{v_{k-2}}(\alpha, v) = \alpha_v$ and $\tau_{v_{k-2}}(\beta, v) = \beta_v$ where $\alpha_v, \beta_v \in L_v$. And we choose the notation such that $\tau_{v_{k-2}}(\alpha, v_{k-1}) = \alpha'$ and $\tau_{v_{k-2}}(\gamma, v_{k-1}) = \gamma'$ where $\alpha', \gamma' \in L_{v_{k-1}}$. If we can give v_{k-1} a color such that α_v, β_v are still available at v, then the proof reduces to the previous case where v has degree 2. So, such a coloring of v_{k-1} is not possible. Now we use Proposition 4.8 to conclude that $\tau_{v_{k-1}}(\alpha', v) = \beta_v$ and $\tau_{v_{k-1}}(\gamma', v) = \alpha_v$. Now we delete v_{k-1} and repeat the proof in the case where v has degree 2. We let v_{k-2} have the available colors α, β. It is easy to see that the coloring of $H - v_{k-1}$ extends to H.

Consider finally the case where $i < k-2$, that is, v_{k-2} has degree 3 in H. We repeat the proof above with the exception that where we above after deleting vertices color the rest of the graph, we will in this case use induction on the remaining graph. □

Corollary 4.18. *Assume G is a multi-wheel with no separating triangle and with at least two inner vertices such that all inner vertices are joined to v_2. Then G is $(\mathbb{Z}_5, 3)$-extendable.*

Proof. G has a unique path $v_1 u_1 u_2 \cdots u_q v_2$, such that all of $u_1, u_2, \dots, u_q$ are joined to v_2. The proof is by induction on q. If $q = 2$, we use Lemma 4.17 b). So assume $q > 2$. Since G has no separating triangle, v_3 has degree 3. Now select two available colors in L_{v_3}, delete those colors from $L_{u_q} = \mathbb{Z}_5$, delete v_3 and all other neighbors of u_q on C of degree 3, and complete the proof by induction. □

4.5 $(\mathbb{Z}_5, 3)$-extendability

As in [40] we now characterize the near-triangulations that are not $(\mathbb{Z}_5, 3)$-extendable. Theorem 4.19 below is similar to Theorem 3 in [40] except that "generalized wheel" in [40] is replaced by "generalized multi-wheel". Figure 4.1 below shows an example of a graph which is a generalized multi-wheel but not a generalized wheel, whose precoloring does not extend to a coloring of the whole graph.

Theorem 4.19. *Let $\varphi : E(G) \to \mathbb{Z}_5$ be given where G is a plane near-triangulation with outer cycle $C : v_1 v_2 \cdots v_k v_1$. Assume that the vertices v_k, v_1 and v_2 are precolored, and for each $v \in C \setminus \{v_1, v_2, v_k\}$, F_v is a set containing at most two forbidden colors. For all other vertices v, F_v is empty. Then G has a $(\mathbb{Z}_5, \varphi)$-coloring $c : V(G) \to \mathbb{Z}_5$ which extends the precoloring of v_k, v_1, v_2 and which satisfies $c(v) \notin F_v$ for any $v \in C \setminus \{v_1, v_2, v_k\}$, unless G contains a subgraph G' which is a generalized multi-wheel whose principal path is $v_k v_1 v_2$, and all other vertices on the outer cycle of G' are on C and have precisely two forbidden colors.*

Proof. The proof is by induction on the number of vertices of G. For $k \leq 5$ the theorem follows from Theorem 4.11. So assume that $k > 5$. Suppose for contradiction that the theorem is false, and let G be a smallest counterexample.

Claim 1. C has no chord.

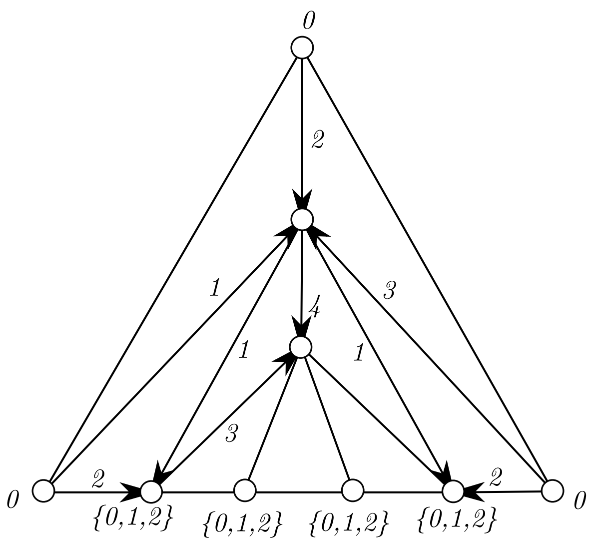

Figure 4.1: A generalized multi-wheel whose precoloring $c(v_k) = c(v_1) = c(v_2) = 0$ does not extend to a coloring of the whole graph. (Note, that unlabelled edges e have $\varphi(e) = 0$.).

Proof. Suppose for contradiction that v_iv_j is a chord of C, where $1 \leq j < i \leq k$. Then v_iv_j divides G into near-triangulations G_1, G_2, respectively. If G_2, say, does not contain v_1 then any $(\mathbb{Z}_5, \varphi)$-coloring of v_iv_j can be extended to G_2 by Theorem 4.10. Therefore G_1 has no $(\mathbb{Z}_5, \varphi)$-coloring. Now we apply the induction hypothesis to G_1 and obtain a contradiction. So assume that $j = 1$.

By Theorem 4.10, G_2 has a $(\mathbb{Z}_5, \varphi)$-coloring. That coloring cannot be extended to G_1. The induction hypothesis implies implies that G_1 satisfies the conclusion of Theorem 4.19, that is, G_1 contains a generalized multi-wheel. A similar argument shows that G_2 satisfies the conclusion of Theorem 4.19. Thus G contains a generalized multi-wheel. It only remains to be proved that L_{v_i} has only three available colors. But if $L_{v_i} \setminus \{\tau_{v_1}(v_i)\}$ has a subset consisting of three colors, then, by Theorem 4.10, each of G_1, G_2 can be $(\mathbb{Z}_5, \varphi)$-colored, and the color of v_i can be chosen in two distinct ways (among these three colors) for each of G_1, G_2, since we get one coloring c_1 from Theorem 4.10 and we get another coloring by replacing $c_1(v_i)$ by $\tau_{v_1}(v_i)$ in L_{v_i}. Hence G can be $(\mathbb{Z}_5, \varphi)$-colored, a contradiction which proves Claim 1. $\square$

Claim 2. G has no separating triangle and no separating 4-cycle.

Proof. Suppose for contradiction that G has a separating cycle C' of length 3 or 4. We consider first the case where C' has length 3. Delete $int(C')$ and denote the resulting graph by G'. If G' can be $(\mathbb{Z}_5, \varphi)$-colored, then so can G by Theorem 4.11. So we may assume that G' cannot be $(\mathbb{Z}_5, \varphi)$-colored. Then G' contains a generalized multi-wheel by the induction hypothesis, hence G contains such a generalized multi-wheel, a contradiction.

We consider next the case where C' has length 4. Choose C' such that $int(C')$ is maximal. Replace $int(C')$ by a single edge e and denote the resulting graph by G'. If G' can be $(\mathbb{Z}_5, \varphi)$-colored, then so can G by Theorem 4.11. So we may assume that G' cannot be $(\mathbb{Z}_5, \varphi)$-colored. Then G' contains a generalized multi-wheel satisfying the conclusion of Theorem 4.19 by the induction hypothesis. This generalized multi-wheel contains e because we previously assumed that G does not contain such a generalized multi-wheel. The maximality property of C' implies that e is not contained in a separating triangle of G'. Then the first part of Claim 2 implies that G' has no separating triangles at all. So, if we delete the edge e from G', then the resulting graph can be $(\mathbb{Z}_5, \varphi)$-colored by Lemma 4.16. By Theorem 4.11, G can be $(\mathbb{Z}_5, \varphi)$-colored, a contradiction which proves Claim 2. □

Claim 3. If u is a vertex in $int(C)$ which is joined to both v_i, v_j, where $2 \leq i \leq j-2 \leq k-2$, then u is joined to each of $v_i, v_{i+1}, \dots, v_j$.

Proof. Suppose for contradiction that there exist i', j' such that $i \leq i' \leq j'-2 \leq j-2$ and u is joined to $v_{i'}, v_{j'}$, but not joined to any of $v_{i'+1}, v_{i'+2}, \dots, v_{j'-1}$. Let C' be the cycle $uv_{i'}v_{i'+1}\cdots v_{j'}u$, and let C'' be the cycle $uv_{j'}v_{j'+1}\cdots v_k v_1 v_2 \cdots v_{i'}u$. We apply the induction hypothesis, first to $C'' \cup int(C'')$ and then to $C' \cup int(C')$. If $C' \cup int(C')$ is a generalized multi-wheel, then it is necessarily a multi-wheel, and then, by Lemma 4.15, there exists $\alpha \in \mathbb{Z}_5$ such that all colorings of $v_{i'}, u, v_{j'}$ which cannot be extended to G' satisfy $c(v_{j'}) - c(v_{i'}) = \alpha$. So before we apply the induction hypothesis to $C'' \cup int(C'')$ we add the edge $v_{i'}v_{j'}$ and we let $\varphi(v_{i'}v_{j'}) = \alpha$. Applying the induction hypothesis to this graph and then to $C' \cup int(C')$ either results in a $(\mathbb{Z}_5, \varphi)$-coloring of G, hence we get a contradiction which proves Claim 3, or else we conclude that $C'' \cup int(C'') \cup \{v_{i'}v_{j'}\}$ contains a generalized multi-wheel satisfying the conclusion of Theorem 4.19. This must contain the triangle $uv_{i'}v_{j'}u$ because of Claim 1 and the assumption that no vertex on the outer cycle of the generalized multi-wheel has more than three available colors, and as $C' \cup int(C')$ is a multi-wheel we conclude that G contains a generalized multi-wheel, a contradiction. □

Claim 4. G has no vertex in $int(C)$ which is joined to both v_2 and v_k.

Proof. Suppose for contradiction that some vertex u in $int(C)$ is joined to both v_2 and v_k. By Claim 3, u is joined to all vertices of C except possibly v_1. However, Claim 2 implies that u is joined to v_1, too. Hence G contains a spanning wheel. By Claim 2, G is a wheel. If some vertex of C has more than three available colors, then it is easy to $(\mathbb{Z}_5, \varphi)$-color G. This contradiction proves Claim 4. □

Claim 5. v_3 has degree at least 4.

Proof. Suppose for contradiction that v_3 has degree at most 3. By Claim 1, v_3 has degree precisely 3, and G has a vertex u in $int(C)$ joined to v_2, v_3, v_4. Let i be the largest number such that u is joined to v_i. The path v_2uv_i divides G into two near-triangulations G_1, G_2 where G_1 contains v_1. By Claims 2 and 3, G_2 is a broken wheel. By Claim 4, $i < k$.

Now we use the argument of the proof of Theorem 1 in [39]. We add to F_u two colors of $\tau_{v_3}(L_{v_3} \setminus \tau_{v_2}(v_3), u)$. We may assume that G_1 has no $(\mathbb{Z}_5, \varphi)$-coloring under this assignment of forbidden sets. For otherwise, that coloring could be extended to $G - v_3$ and hence also to G. Therefore the induction hypothesis implies that G_1 contains a generalized multi-wheel satisfying the conclusion of Theorem 4.19. By Claims 1 and 2, G_1 is a generalized multi-wheel.

Claim 4 implies that G_1 is not a multi-wheel. So G_1 has a chord. Claims 1 and 2 imply that this chord must be v_1u and, since the chord is unique, $G_1 - v_2$ is a multi-wheel. Claim 3 then implies that all inner vertices of $G_1 - v_2$ are joined to u. If $G_1 - v_2$ is a wheel, then G satisfies the assumption of Lemma 4.17 a) which implies that G has a $(\mathbb{Z}_5, \varphi)$-coloring, a contradiction. On the other hand, if $G_1 - v_2$ is not a wheel, then we color u such that v_3 still has two available colors. Corollary 4.18 (applied to G minus all those neighbors of u that have degree 3) now implies that G has a $(\mathbb{Z}_5, \varphi)$-coloring, a contradiction which proves Claim 5. □

By a similar argument we get

Claim 6. v_{k-1} has degree at least 4.

We now claim that

Claim 7. v_3 and v_{k-1} both have degree precisely 4, and v_3 and v_{k-1} have a common neighbor in $int(C)$.

Proof. Suppose for contradiction that Claim 7 is false. Let $v_2, u_1, \ldots, u_q, v_4$ be the neighbors of v_3 in clockwise order. Then $q \geq 2$, by Claim 5. Let $v_k, u'_1, \ldots, u'_q, v_{k-2}$ be the neighbors of v_{k-1} in anti-clockwise order. Then $q' \geq 2$, by Claim 6. Let u_iv_j be the unique edge such that i is minimum and j is maximum. By Claims 2 and 3, $i = q$, and $j \leq k - 2$. As in the proof of Claim 5, for each $1 \leq i \leq q$ we add to F_{u_i} two colors of $\tau_{v_3}(L_{v_3} \setminus \tau_{v_2}(v_3), u_i)$. And as in the proof of Claim 5, we conclude that $G - v_3$ contains a generalized multi-wheel G'. By Claims 1 and 2, we conclude that the outer cycle of this generalized multi-wheel G' must be $C' : v_1v_2u_1...u_qv_j...v_kv_1$. By Claim 4, G' cannot be a multi-wheel (because every multi-wheel has a vertex in the interior joined to all three vertices of the principal path). So G' has a chord. By Claims 1 and 2, there can be only one chord, namely v_1u_1. As $C'' : v_1u_1...u_qv_j...v_kv_1$ has no chord, it follows that C'' together with its interior is a multi-wheel which we call G_1. Then $int(C'')$ contains a vertex v joined to all vertices of the principal path $v_kv_1u_1$ of G_1. By an analogous argument (with v_{k-1} instead of v_3) we conclude that there exists a vertex w joined to v_2, v_1 and a neighbor of v_{k-1}. The only possibilities for v, w are: $w = u_1$ and $v = u'_1$. We now give u_1 a color such that $L_{v_3} \setminus \tau_{v_2}(v_3)$ still has at least two available colors. We delete v_2 and call the resulting graph G_2.

By repeating the arguments given for G_1 above, we see that G_2 is also a multi-wheel. If we apply Proposition 4.14 to the triangle containing the edge $u_1'u_2'$ (but not the vertex v_{k-1}) in G_2, then we conclude that u_1 is joined to u_2'. Similarly, u_1' is joined to u_2. This is possible only if $q = q' = 2$ and $u_2 = u_2'$. This contradiction proves Claim 7. □

We are now ready for the final contradiction. Using the proof of Claim 7 we obtain the structure of G': u_1, u_1' and u_2 $(= u_2')$ are the only vertices in $int(C)$, v_1 is joined by an edge to u_1 and u_1', and u_1 and u_1' are joined by an edge. By Claim 3, the cycle $u_2v_3v_4\cdots v_{k-1}u_2$ together with its interior is a broken wheel. Define G_2 as in the proof of Claim 7. By Corollary 1, G_2 is colorable, and by the construction of G_2, this coloring can be extended to G. We conclude that G is $(\mathbb{Z}_5, \varphi)$-colorable, a contradiction which completes the proof of Theorem 4.19. □

4.6 Further $\mathbb{Z}_5$-coloring properties of generalized multi-wheels

Lemma 4.20. *Let G be a generalized multi-wheel, and let $\varphi : E(G) \to \mathbb{Z}_5$. Assume that the vertex v_2 is precolored, and for each $v \in C \setminus \{v_1, v_2\}$, F_v is a set containing at most two forbidden colors. For all other vertices v, F_v is empty. Then it is possible to color v_k such that any coloring of v_1 (introducing no color conflict with v_2, v_k) can be extended to a $(\mathbb{Z}_5, \varphi)$-coloring $c : V(G) \to \mathbb{Z}_5$ of G which satisfies $c(v) \notin F_v$ for any $v \in C \setminus \{v_1, v_2, v_k\}$.*

Proof. We prove Lemma 4.20 by induction on the number of vertices of G.

If G is a multi-wheel, then Lemma 4.20 follows easily from Lemma 4.15. Assume that G is a generalized multi-wheel, but not a multi-wheel or a broken wheel. Then there exist $2 \le i \le j \le k$ such that G contains the edges v_1v_i, v_1v_j, and the cycle $C' := v_1v_iv_{i+1}\cdots v_jv_1$ and its interior form a multi-wheel. Let $G' := C' \cup int(C')$, and let G'' be the graph obtained from G by replacing G' by a triangle $v_1v_iv_jv_1$ (with v_iv_j directed towards v_j). By Lemma 4.15 there exists $\alpha \in \mathbb{Z}_5$ such that all colorings of v_i, v_1, v_j which cannot be extended to G' satisfy $c(v_j) - c(v_i) = \alpha$. Apply the induction hypothesis to G'' where we define $\varphi(v_iv_j) = \alpha$. Then the resulting coloring can be extended to G' by Lemma 4.15, hence Lemma 4.20 follows.

So we can assume that G is a broken wheel. In particular, v_1 is joined to v_3. By Proposition 4.5 we may assume that $\varphi(v_1v_i) = 0$ for each $2 \le i \le k$. Let α, β be two colors in $L_{v_3} \setminus \{\tau_{v_2}(v_3)\}$. Let γ, δ, ϵ be three colors in L_{v_k}. Suppose for contradiction that for each of these three colors it is possible to color v_1 such that the coloring cannot be extended to G. The color at v_1 must be one of α, β, since otherwise the coloring can be extended by Theorem 4.10 applied to $G - v_2$. So for two of the colors, γ, δ, ϵ, say γ, δ, it is the same color, say α, which is used at v_1. But now we get a contradiction to Theorem 4.10 applied to G, where v_1 has the color α, and v_k has the available colors α, γ, δ.

This completes the proof of Lemma 4.20. □

We define *generalized wheel strings*, *clean vertices*, and *broken wheel strings* as in [40]: If $G_1, G_2, \ldots, G_m$ are generalized wheels, then we define a *generalized wheel string* by identifying each principal neighbor of the major vertex in G_i with precisely one principal neighbor of the major vertex in G_{i-1}, G_{i+1}, respectively, for $i = 2, \ldots, k-1$. The two vertices which are principal neighbors of the major vertices in G_1 and G_m, respectively, and which have not been identified with any other vertex, are called *clean vertices*. If each of $G_1, G_2, \ldots, G_m$ is a broken wheel, then G is a *broken wheel string*.

We extend the first of these definitions as follows:

Definition 4.21. Let $G_1, G_2, \ldots, G_m$ be generalized multi-wheels. We define a *generalized multi-wheel string* by identifying each principal neighbor of the major vertex in G_i with precisely one principal neighbor of the major vertex in G_{i-1}, G_{i+1}, respectively, for $i = 2, \ldots, k-1$.

Given a generalized multi-wheel string, we now extend the definition of a *clean vertex* to be the principal neighbors of the major vertices in G_1 and G_m which have not been identified with any other vertex.

Lemma 4.22. *Let G be a generalized multi-wheel string, and let $\varphi : E(G) \to \mathbb{Z}_5$. Assume that the two clean vertices have forbidden sets containing at most three colors each, and that each non-clean vertex on the outer boundary has a forbidden set containing at most two colors. For all other vertices v, F_v is empty. Then it is possible to color the two clean vertices and all the cutvertices of G such that any coloring of the major vertices (introducing no color conflict) can be extended to a $(\mathbb{Z}_5, \varphi)$-coloring $c : V(G) \to \mathbb{Z}_5$ of G which satisfies $c(v) \notin F_v$ for all $v \in V(G)$.*

Proof. We prove Lemma 4.22 by induction on the number of vertices of G. Suppose for contradiction that G is a smallest counterexample.

Let G consist of the generalized multi-wheels $G_1, \ldots, G_m$ such that a principal neighbor of each of the major vertices in G_i and G_{i+1} are identified. Consider first the case where $m \geq 2$. Let x (repectively y) be the clean vertex in G_1 (respectively G_m). Let z be the common vertex of G_1 and G_2. Assume that $L_z = \{\alpha, \beta, \gamma\}$. We now apply the induction hypothesis to G_1. We may assume that x, z can be colored such that the conclusion of Lemma 4.22 holds. Assume that the color of z is α. Then we again apply the induction hypothesis to G_1 but now we only allow colors β, γ at z. So the coloring of x, z can be chosen in two ways in which z has two distinct colors. Applying the induction hypothesis to $G_2 \cup \ldots \cup G_m$ there are two distinct colorings of z, y (with z getting different colors) such that the conclusion of Lemma 4.22 holds. Now we let z receive a color that appears in both a coloring of x, z and a coloring of y, z. So we may assume that $m = 1$.

Let $x = v_2$, $y = v_k$ be the clean vertices in $G_1 = G$. Assume first that G is a generalized multi-wheel, but not a multi-wheel or a broken wheel. Then there exist $2 \leq i \leq j \leq k$ such that G contains the edges v_1v_i, v_1v_j, and the cycle $C' := v_1v_iv_{i+1}\cdots v_jv_1$ and its interior form a multi-wheel. Let $G' := C' \cup int(C')$, and let G'' be the graph obtained from G by replacing G' by a triangle $v_1v_iv_jv_1$ (with v_iv_j directed towards v_j). By Lemma 4.15 there exists $\alpha \in \mathbb{Z}_5$ such that the $(\mathbb{Z}_5, \varphi)$-colorings of v_j, v_1, v_i which cannot be extended to G' satisfy that $c(v_j) - c(v_i) = \alpha$. Apply the induction hypothesis to G'' where we define $\varphi(v_iv_j) = \alpha$. Then the resulting coloring can be extended to G' by Lemma 4.15, hence Lemma 4.20 follows in the case where G is a generalized multi-wheel, but not a multi-wheel and not a broken wheel.

So we can assume that G is either a multi-wheel or a broken wheel. We may assume that G is a broken wheel since otherwise Lemma 4.22 follows easily from Lemma 4.15. By Proposition 4.5 we may assume that $\varphi(v_1v_i) = 0$ for any $3 \leq i \leq k-1$, and also $\varphi(v_2v_3) = \varphi(v_{k-1}v_k) = 0$. Furthermore, we may assume that all vertices $v_3, \ldots, v_{k-1}$ have precisely three available colors, since otherwise it is easy to see that any coloring of v_2, v_k, v_1 can be extended. Let $L_{v_3} = \{\alpha, \beta, \gamma\}$, let $L_{v_2} = \{\alpha', \beta'\}$, and let $L_{v_k} = \{\alpha'', \beta''\}$. Now, there are four possible ways of coloring v_2, v_k. We may assume that none of them works, that is, for each of those four colorings, it is possible to color v_1 (introducing no color conflict with v_2, v_k) such that the resulting coloring cannot be extended to G. We say that these colors are the *bad colors* of v_1. Any bad color of v_1 must be in L_{v_i} for each $3 \leq i \leq k-1$, since otherwise we color $v_3, \ldots, v_{i-1}, v_{k-1}, \ldots, v_i$ in that order. In particular, the bad colors are among $\{\alpha, \beta, \gamma\}$. Thus, for at least two of the four possibilities, v_1 has the same bad color, say γ. The two possibilities must either be α', β'' and β', α'' or α', α'' and β', β'', since otherwise (if β', say, does not appear here) we apply Theorem 4.10 to G with v_1, v_2 colored γ, α', respectively, and $L_{v_k} = \{\alpha'', \beta'', \tau_{v_1}(v_k)\}$ to get a contradiction. Assume without loss of generality that the colorings α', β'', γ and β', α'', γ of v_2, v_k, v_1, respectively, cannot be extended to G. The same argument shows that γ cannot be the bad color of v_1 in three of the four possibilities. We shall now argue that $\{\alpha', \beta'\} = \{\alpha, \beta\}$: If we give v_1 color γ and v_k color α'' and then color $v_{k-1}, v_{k-2}, \ldots, v_4$ in that order, then the color at v_3 will be either α or β, say α. If α is not in $\{\alpha', \beta'\}$, then G is colorable with v_1 having color γ, a contradiction. If we next give v_1 color γ and v_k color β'', then the same argument implies that β is in $\{\alpha', \beta'\}$. (If we have any choices while coloring $v_{k-1}, v_{k-2}, \ldots, v_4$, then it is easy to see that G is colorable with v_1 having color γ, a contradiction. Thus β must be the available color at v_3 when v_4 has been colored.) We choose the notation such that $\alpha' = \alpha$ and $\beta' = \beta$. Now, if, say, β is not a bad color of v_1 (that is, α and γ are the only bad colors), then the coloring α, α'', α of v_2, v_k, v_1 does not extend to G, which gives a contradiction when we color $v_{k-1}, v_{k-2}, \ldots, v_3$ in that order (v_3 can be colored since v_1, v_2 have the same color and $\varphi(v_1v_3) = \varphi(v_2v_3) = 0$). Thus the colorings α, α'', β and (by a similar argument) β, β'', α of v_2, v_k, v_1 do not extend to G, and since α, β, γ are all bad colors of v_1 we conclude $L_{v_3} = L_{v_4} = \ldots = L_{v_{k-1}} = \{\alpha, \beta, \gamma\}$ by an observation made earlier. As above, we conclude that $\{\alpha'', \beta''\} = \{\alpha, \beta\}$. So v_2 and v_k have the same available colors, namely α, β.

Recall that none of the colorings α, β'', γ and α, α'', β of v_2, v_k, v_1 extend to G. We shall now obtain a contradiction by proving that at least one of them extends to

G. To prove this, let us now color v_2, v_k, v_1 by α, β'', γ. Then we color $v_3, \ldots, v_{k-1}$ in that order according to the following rule: if v_{i-1} has color $c(v_{i-1})$ and $\varphi(v_{i-1}v_i) \neq 0$ then we give v_i color $c(v_{i-1})$, and if $\varphi(v_{i-1}v_i) = 0$ then we give v_i the other available color (which is α or β) for $3 \leq i \leq k-1$. Since the coloring of v_2, v_k, v_1 does not extend, we must get a color conflict between v_{k-1} and v_k, that is, the color of v_{k-1} is that of v_k (since $\varphi(v_{k-1}v_k) = 0$), that is, β''. If this happens, we let the colors of v_2, v_k, v_1 be α, α'', β and now we give all vertices in $v_3, \ldots, v_{k-1}$ which have color β color γ instead. This coloring clearly works. □

4.7 Exponentially many $\mathbb{Z}_5$-colorings of planar graphs

In this section we prove the main result. The proof follows closely the analogous proof in [40].

Theorem 4.23. *Let G be an oriented plane near-triangulation with outer cycle C : $v_1v_2 \cdots v_kv_1$, and let $\varphi : E(G) \to \mathbb{Z}_5$. For each vertex v in G let F_v be a set of forbidden colors. Assume that the vertices v_k, v_1, v_2 or the vertices v_1, v_2 are precolored. If v is one of $v_3, v_4, \ldots, v_{k-1}$ (resp. $v_3, v_4, \ldots, v_k$), then F_v consists of at most two colors. For all other vertices v, F_v is empty. Let n denote the number of non-precolored vertices, and let r denote the number of vertices with precisely three available colors. Assume that G has a $(\mathbb{Z}_5, \varphi)$-coloring $c : V(G) \to \mathbb{Z}_5$ which satisfies $c(v) \notin F_v$ for any $v \in V(G)$. Then the number of such $(\mathbb{Z}_5, \varphi)$-colorings is at least $2^{n/9 - r/3}$, unless G has three precolored vertices and also contains a vertex u with precisely four available colors which is joined to the three precolored vertices and has only one available color distinct from $\tau_{v_k}(u), \tau_{v_1}(u), \tau_{v_2}(u)$.*

Proof. The proof is by induction on n. It is easy to verify the statement if $n = 1$ so we proceed to the induction step. Let f denote the number of vertices with precisely four available colors.

We assume that G is a counterexample such that n is minimum and, subject to this, r is maximal, and, subject to these conditions, f is minimum. We shall establish a number of properties of G which will lead to a contradiction. Clearly, $n > 3r$.

Claim 8. G has no separating triangle.

Proof. Suppose for contradiction that $xyzx$ is a separating triangle which divides G into near-triangulations G_1, G_2, respectively, where G_1 contains C. Then any $(\mathbb{Z}_5, \varphi)$-coloring of x, y, z can be extended to G_2 by Theorem 4.11. Let n_1 be the number of non-precolored vertices in G_1, and let n_2 be the number of vertices in $G_2 - x - y - z$. By the minimality of n, G_1 has at least $2^{n_1/9 - r/3}$ distinct $(\mathbb{Z}_5, \varphi)$-colorings. Each such coloring has at least $2^{n_2/9}$ extensions to G_2. As $n_1 + n_2 = n$, this proves Claim 8. □

Claim 9. G has no chord.

Proof. Suppose for contradiction that v_iv_j is a chord of C, where $1 \leq i < j \leq k$. Then v_iv_j divides G into near-triangulations G_1, G_2, respectively.

Consider first the case where G_2, say, does not contain a precolored vertex distinct from v_i, v_j. Then any $(\mathbb{Z}_5, \varphi)$-coloring of G_1 can be extended to G_2 by Theorem 4.10. We now obtain a contradiction by repeating the proof of Claim 8.

Assume next that $i = 1$ and that v_k is precolored. If each of G_1, G_2 is a generalized multi-wheel such that each non-precolored vertex on the outer cycle has precisely three available colors, then $r \geq n/3$, and there is nothing to prove. So assume that G_2, say, is not such a generalized multi-wheel. Moreover, it does not contain such a generalized multi-wheel because G has no separating triangles, by Claim 8, and every chord of G, if any, is incident with v_1, by the first part of the proof of Claim 9. Now, if $j < k - 1$ we repeat the proof of Claim 8. This proves Claim 9 unless $j = k - 1$, that is, G_2 is the triangle $v_1v_kv_{k-1}v_1$. So assume that this is the case.

Then we color v_{k-1}, and we apply the induction hypothesis to $G - v_k$. If v_{k-1} has precisely three available colors, then both n and r decreases, so Claim 9 follows. If v_{k-1} has at least four available colors, then only n decreases, but there are at least two choices for the color of v_{k-1} unless G is the exceptional case at the end of Theorem 4.23. So we need only consider the case where $G - v_k$ is the exceptional case at the end of Theorem 4.23, namely that G has a vertex with precisely four available colors joined to v_{k-1}, v_1, v_2. Then $k = 5$, and $n = 2$. As v_3 has at least four available colors, G has at least two $(\mathbb{Z}_5, \varphi)$-colorings. This proves Claim 9. □

Claim 10. Each non-precolored vertex on C has precisely three available colors.

Proof. Suppose for contradiction that Claim 10 is false. Select a set S of four available colors in L_{v_i} for some vertex v_i on C. Let S' be one of the four 3-element subsets of S. Now replace F_{v_i} by $\mathbb{Z}_5 \setminus S'$. By the maximality of r, the new G has at least $2^{n/9-(r+1)/3}$ distinct $(\mathbb{Z}_5, \varphi)$-colorings. As S' can be chosen in four ways, this results in $4 \cdot 2^{n/9-(r+1)/3}$ $(\mathbb{Z}_5, \varphi)$-colorings and each of these is counted three times. Thus we get at least $4 \cdot 2^{n/9-(r+1)/3}/3$ distinct $(\mathbb{Z}_5, \varphi)$-colorings, a contradiction which proves Claim 10. Note that G with its new lists of available colors cannot be a generalized multi-wheel because $n > 3r$, as noted earlier. □

Claim 11. v_k is precolored.

Proof. Suppose for contradiction that Claim 11 is false. The coloring of v_1, v_2 can be extended to G. We give v_k the color in that coloring. This decreases each of n, r by 1 and hence we obtain a contradiction to the minimality of n. Note that, by Claim 10, the new G cannot have a vertex with precisely four available colors joined to the three colored vertices. □

Claim 12. If u is a vertex in $int(C)$ joined to v_i, v_j, where $2 \leq i < j \leq k$, then v is also joined to each of $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$.

Proof. Suppose for contradiction that there exist i', j' such that $i \leq i' \leq j'-2 \leq j-2$ and u is joined to $v_{i'}, v_{j'}$, but not joined to any of $v_{i'+1}, v_{i'+2}, \ldots, v_{j'-1}$. Let C' be the cycle $uv_{i'}v_{i'+1} \cdots v_{j'}u$, and let C'' be the cycle $uv_{j'}v_{j'+1} \cdots v_k v_1 v_2 \cdots v_{i'}u$. We apply the induction hypothesis, first to the graph $G'' := C'' \cup int(C'')$ and then to the graph $G' := C' \cup int(C')$. This proves Claim 12 unless G' is a generalized multi-wheel. If G' is a generalized multi-wheel, then it is necessarily a multi-wheel, and then, by Lemma 4.15, there exists $\alpha \in \mathbb{Z}_5$ such that all colorings of $v_{i'}, u, v_{j'}$ which cannot be extended to G' satisfy $c(v_{j'}) - c(v_{i'}) = \alpha$. So before we apply the induction hypothesis to G'' we add the edge $v_{i'}v_{j'}$ and we let $\varphi(v_{i'}v_{j'}) = \alpha$. Apply the induction hypothesis to this graph and then to G'. If n' (respectively r') is the number of non-precolored vertices (respectively non-precolored vertices with precisely three available colors) of G'', then it is easy to see that $n'/9 - r'/3 \geq n/9 - r/3$. This contradiction proves Claim 12. □

Claim 12 implies that G does not contain an inserted wheel.

We may assume that

Claim 13. $k > 4$.

Proof. For, if $k = 3$, then we delete the edge v_2v_3. And if $k = 4$, then we color v_3 and delete it and use induction after having modified the available lists of the neighbors of v_3 accordingly. □

We now split the proof up into the following two cases.

Case 1. G does not contain a path $v_2u_1u_2 \cdots u_qv_k$ with the properties that

(i) each of $u_1, u_2, \ldots, u_q$ is a vertex in $int(C)$ joined to at least two vertices of $v_3, v_4, \ldots, v_{k-1}$, and

(ii) the cycle $v_1v_2u_1u_2 \cdots u_qv_kv_1$ and its interior form a generalized multi-wheel.

Case 2. G contains a path $v_2u_1u_2 \cdots u_qv_k$ with the above-mentioned properties (i) and (ii).

Note, that (ii) is equivalent to the following statement: the cycle $v_1v_2u_1u_2 \cdots u_q v_kv_1$ and its interior **contain** a generalized multi-wheel whose principal path is $v_kv_1v_2$ and all vertices on its outer cycle are on $v_1v_2u_1u_2 \cdots u_qv_kv_1$. This follows from Claim 8 and the fact that if such a subgraph exists, and there is an edge u_su_t for some $s < t$, then we may choose the path $v_2u_1 \cdots u_su_t \cdots u_qv_k$ in Case 2 instead of $v_2u_1u_2 \cdots u_qv_k$.

We first do Case 1. We shall prove that the number of $(\mathbb{Z}_5, \varphi)$-colorings is not just at least $2^{n/9-r/3}$ as required in Theorem 4.23, but at least $2^{(n+1)/9-r/3}$. This will be important in Case 2 which we shall reduce to Case 1 by deleting an appropriate vertex.

Let R be the set of vertices in $int(C)$ which are joined to at least two vertices of the path $C - v_k - v_1 - v_2$. By Claim 12, the union of the path $C - v_k - v_1 - v_2$ and R and the edges from R to C form a broken wheel string which we will call W.

Subcase 1.1. No two consecutive blocks in W are triangles.

We use Lemma 4.22 to color all the principal neighbors of the major vertices in W in such a way that, regardless of how the major vertices in W are colored, the coloring can be extended to W. This means that we can apply induction to $G' = G - v_3 - v_4 - \cdots - v_{k-1}$. Any $(\mathbb{Z}_5, \varphi)$-coloring of G' can be extended to G. By the induction hypothesis, the number of $(\mathbb{Z}_5, \varphi)$-colorings of G' is at least $2^{n'/9 - r'/3}$ where $n' = n - k + 3 = n - r$ and $r' = |R|$. The assumption of Subcase 1.1 implies that $r' \leq (2r - 1)/3$. Hence the number of $(\mathbb{Z}_5, \varphi)$-colorings of G' is at least $2^{(n+1)/9 - r/3}$.

Subcase 1.2. Two consecutive blocks in W are triangles.

Let w_1, w_2 be two vertices in R each joined to precisely two consecutive blocks of C. That is, there is a natural number i such that W contains the blocks $w_1 v_{i-1} v_i w_1$ and $w_2 v_i v_{i+1} w_2$. We now color successively v_3 and the cutvertices of W with increasing indices until we color v_{i-1}. Whenever we color a cutvertex, we do it such that the corresponding block of W can be colored regardless of how we color the major vertex. This is possible by Lemma 4.20. There are even two possibilities for coloring such a cutvertex of W whenever the preceding cutvertex is a neighbor of the cutvertex that is being colored. Then we color successively v_{k-1} and the cutvertices of W with decreasing indices until we color v_{i+1}. Again, there are even two possibilities for coloring such a cutvertex of W whenever the preceding cutvertex is a neighbor of the cutvertex that is being colored. (Also there are two possibilities for coloring each of v_3, v_{k-1}.) Finally we color v_i and apply the induction hypothesis to $G - v_3 - v_4 - \cdots - v_{k-1}$. Let r' be the number of vertices of R and let n' be the number of uncolored vertices of $G - v_3 - v_4 - \cdots - v_{k-1}$. Then $n' = n - k + 3 = n - r$.

The number of colorings of the vertices of W in the path $v_2 v_3 \cdots v_{k-1}$ is at least 2^t, where t is the number of blocks of W which are triangles.

For each of these there are at least $2^{n'/9 - r'/3}$ $(\mathbb{Z}_5, \varphi)$-colorings of $G - v_3 - v_4 - \cdots - v_{k-1}$, by the induction hypothesis. Let s be the number of blocks of W which are not triangles. Then $r' = s + t$ and $r \geq 2s + t + 1$. So the total number of colorings of G is at least $2^{n'/9 - r'/3 + t}$ which is greater than $2^{(n+1)/9 - r/3}$. This completes the proof in Case 1.

We now do Case 2. Let m be the smallest number such that u_q is joined to v_m. By Claim 12, u_q is joined to $v_m, v_{m+1}, \ldots, v_k$ (and possibly also to v_1). Again, we split up into two cases.

Subcase 2.1. v_1 is joined to u_q.

We select two colors α, β in $L_{v_{k-1}}$ distinct from $\tau_{v_k}(v_{k-1})$. We add the colors $\tau_{v_{k-1}}(\alpha, u_q)$, $\tau_{v_{k-1}}(\beta, u_q)$ to F_{u_q} and we delete the vertex v_{k-1} from G. Then we color u_q and delete also v_k. By the induction hypothesis, if the resulting graph G' has at least one $(\mathbb{Z}_5, \varphi)$-coloring, then it has at least $2^{(n-2)/9 - (r-1)/3}$ $(\mathbb{Z}_5, \varphi)$-colorings.

Each such coloring can be extended to v_{k-1} and the proof is complete. So assume that G' has no $(\mathbb{Z}_5, \varphi)$-coloring. By Theorem 4.19, G' contains a generalized multi-wheel. Clearly, $q \leq 2$. Furthermore, G' has no chords $v_1 v_i$ for $3 \leq i \leq k-2$. Hence either $q = 1$ in which case G is a wheel by Claims 8 and 12, or else $q = 2$ in which case $G' - v_{m+1} - v_{m+2} - \cdots - v_{k-2}$ is a wheel. But then $n - r \leq 2$ and there is nothing to prove.

Subcase 2.2. v_1 is not joined to u_q. Now G has a vertex w joined to v_k, u_q, u_l for some $l < q$ by the definition of a generalized multi-wheel. By Claim 12, w is not joined to v_2.

If $m < k-2$, then we select two colors α, β in $L_{v_{k-1}}$ distinct from $\tau_{v_k}(v_{k-1})$. We add the colors $\tau_{v_{k-1}}(\alpha, u_q), \tau_{v_{k-1}}(\beta, u_q)$ to F_{u_q} and we delete the vertices $v_{k-1}, v_{k-2}, \ldots, v_{m+1}$ from G. Then we use the induction hypothesis to obtain a contradiction because the resulting graph has a smaller r. So assume that $m = k-2$.

If $q = 1$, then w is joined to v_2, hence by Claim 12 both of w and u_1 are joined to all of $v_3, v_4, \ldots, v_{k-1}$ which is impossible. So assume that $q > 1$.

If u_{q-1} is joined to v_{k-2}, then we select two colors α, β in $L_{v_{k-1}}$ distinct from $\tau_{v_k}(v_{k-1})$. We add the colors $\tau_{v_{k-1}}(\alpha, u_q), \tau_{v_{k-1}}(\beta, u_q)$ to F_{u_q} and we delete the vertex v_{k-1} from G. The resulting graph G' satisfies the assumption in Case 1. We explain why: If G' contains a path $v_2 w_1 w_2 \cdots w_{q'} v_k$ such that each of $w_1, \ldots, w_{q'}$ is a vertex in the interior of $v_1 v_2 \cdots v_{k-2} u_q v_k v_1$, then $w_{q'}$ cannot be joined to any of $v_3, \ldots, v_{k-2}$, hence it is not joined to at least two vertices of $v_3, v_4, \ldots, v_{k-2}, u_q$. Thus the conclusion follows by repeating the proof in Case 1. The reason we can repeat the proof in Case 1 is that G' satisfies the analogue of Claim 12 when u_{q-1} is joined to v_{k-2}. Therefore we may assume that u_{q-1} is not joined to v_{k-2}.

Let i be the smallest number such that u_{q-1} is joined to v_i, and let j be the largest number u_{q-1} is joined to v_j. Then $j < k-2$. We select two colors α, β in $L_{v_{k-1}}$ distinct from $\tau_{v_k}(v_{k-1})$. We add the colors $\tau_{v_{k-1}}(\alpha, u_q), \tau_{v_{k-1}}(\beta, u_q)$ to F_{u_q} and we delete the vertex v_{k-1}. The path $v_j u_{q-1} u_q$ divides the resulting graph into two graphs G_1, G_2, where G_1 contains v_1. Assume first that G_2 is a generalized multi-wheel. If G_1 contains a generalized multi-wheel then $r \geq n/3$, so there is nothing to prove. If not, then we obtain a contradiction by applying the induction hypothesis to G_1. Let n' denote the number of non-precolored vertices in G_1, and let r' denote the number of vertices in G_1 with precisely three available colors. As G_2 is a generalized multi-wheel without chords, it is a multi-wheel and we have $n'/9 - r'/3 \geq n/9 - r/3$. We use the fact that, by Lemma 4.15, there exists $\alpha \in \mathbb{Z}_5$ such that all colorings of v_j, u_{q-1}, u_q which cannot be extended to G_2 satisfy $c(u_q) - c(v_j) = \alpha$. So before we apply the induction hypothesis to G_1 we add the edge $v_j u_q$ (directed towards u_q) and we let $\varphi(v_j u_q) = \alpha$. In this case the number of $(\mathbb{Z}_5, \varphi)$-colorings of G_1 is greater than or equal to $2^{n/9 - r/3}$, and any such coloring can be extended to G_2.

On the other hand, if G_2 is not a generalized multi-wheel, then we obtain a contradiction by applying the induction hypothesis first to G_1 and then to G_2. We lose a multiplicative factor $2^{1/9}$ because of the deleted vertex v_{k-1}. We make up for that before we apply induction to G_1 since we can delete one of the available colors of u_{q-1} in at least five different ways. In this way we gain a multiplicative factor $5/4$, and now the proof is complete, because $5/4 > 2^{1/9}$. □

Corollary 4.24. *Every planar simple graph with n vertices has at least $2^{n/9}$ $\mathbb{Z}_5$-colorings.*

CHAPTER 5

Group coloring with groups of order 4

This chapter contains results which will appear in the paper [29] by L. and Thomassen (under preparation).

Ordinary colorings and nowhere-zero flows both have the property that if it is possible to color or construct a nowhere-zero flow using one group, then it is also possible using another group of the same order. For colorings this is obvious, and for nowhere-zero flows it follows from a theorem by Tutte (see Theorem 1.34). As mentioned in Chapter 2 and [30], this does not hold for group colorings and group flows.

In this chapter we consider the following problem posed by Jaeger, Linial, Payan and Tarsi [17].

Problem 5.1. Do there exist simple planar graphs which are $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable? Do there exist simple planar graphs which are $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, but not $\mathbb{Z}_4$-colorable?

Using Hajos' construction we find simple planar graphs which are $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, and by counting the number of possible colorings we find simple planar graphs which are $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, but not $\mathbb{Z}_4$-colorable. By duality we thus also find planar 3-edge-connected graphs which are $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected, and we find planar 3-edge-connected graphs which are $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected, but not $\mathbb{Z}_4$-connected. Finally we find planar graphs which are $\mathbb{Z}_2^k$-colorable, but not Γ-colorable for any other group of order $|\Gamma| = 2^k$. By duality we thus also find planar graphs which are $\mathbb{Z}_2^k$-connected, but not Γ-connected for any other group of order $|\Gamma| = 2^k$.

5.1 Introduction

The problem of whether all $\mathbb{Z}_4$-connected graphs are also $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected, and vice versa, was first mentioned in the paper [17] by Jaeger, Linial, Payan and Tarsi.

Hušek, Mohelníková and Šámal [14] found that this is not the case. They describe three graphs. One is planar and $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected, and two non-planar graphs where one is $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected, and the other is $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected, but not $\mathbb{Z}_4$-connected. All three graphs contain several vertices of degree 2 and almost all proofs in the paper are computer-aided.

Han, Li, Li and Wang [12] extend their results to 3-edge-connected graphs. Using the graphs in [14], they construct non-planar, 3-edge-connected graphs which are $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected, as well as non-planar, 3-edge-connected graphs which are $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected, but not $\mathbb{Z}_4$-connected. They even construct non-planar, 3-edge-connected cubic graphs with these properties. The non-planarity is a result of the graphs being constructed using the non-planar graphs in [14].

In this chapter we use the planar graph in [14] to construct planar simple graphs (for group coloring) and planar 3-edge-connected graphs (for group connectivity), and we construct infinitely many such graphs.

The Hajos' construction is crucial when constructing these graphs, as well as when constructing many. The operation is defined as follows.

Definition 5.2. Let G_1 and G_2 be graphs, and let $e_1 = x_1y_1 \in E(G_1)$ and $e_1 = x_2y_2 \in E(G_2)$. The *Hajos' construction* on G_1 and G_2 with edges e_1 and e_2 is the graph G formed by deleting e_1 and e_2, contracting x_1 and x_2 to the vertex x, and adding the new edge e_0 between y_1 and y_2.

Additionally, we find a planar simple graph which is $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, but not $\mathbb{Z}_4$-colorable, answering an instance of Problem 5.1, and we generalize this result by finding planar graphs which are $\mathbb{Z}_2^k$-colorable, but not Γ-colorable for any other group of order $|\Gamma| = 2^k$.

5.1.1 Some lemmas

We first introduce two helpful lemmas, where the second is a repetition of Proposition 4.5. Let G be any graph, and let Γ be an Abelian group.

Lemma 5.3. *Let $\varphi : E(G) \to \Gamma$. Given $c : V(G) \to \Gamma$ and $\alpha \in \Gamma$ we define $c' : V(G) \to \Gamma$ by $c'(v) = c(v) + \alpha$ for all $v \in V(G)$. Then c is a (Γ, φ)-coloring if and only if c' is a (Γ, φ)-coloring.*

Proof. Clearly $c'(v) - c'(u) = c(v) - c(u) \neq \varphi(uw)$ for any edge $uv \in E(G)$. □

We say that c and c' are (Γ, φ)-*equivalent* on G.

Lemma 5.4. *Let $\varphi : E(G) \to \Gamma$. Given $v_0 \in V(G)$ and $\alpha \in \Gamma$ we define $\varphi' : E(G) \to \Gamma$ as follows:*

$$\varphi'(e) = \begin{cases} \varphi(e) + \alpha & \text{if } e \text{ is incident to } v_0 \text{ and directed towards } v_0, \\ \varphi(e) - \alpha & \text{if } e \text{ is incident to } v_0 \text{ and directed away from } v_0, \\ \varphi(e) & \text{otherwise.} \end{cases} \tag{5.1}$$

Then G is (Γ, φ)-colorable if and only if G is (Γ, φ')-colorable.

Proof. Given a (Γ, φ)-coloring c of G we define $c' : V(G) \to \Gamma$ by

$$c'(v) = \begin{cases} c(v) - \alpha & \text{if } v = v_0, \\ c(v) & \text{otherwise.} \end{cases} \tag{5.2}$$

for all $v \in V(G)$. Then c' is clearly a (Γ, φ')-coloring, so G is (Γ, φ')-colorable as wanted. The other implication follows from the same argument. □

We say that φ and φ' are Γ-*equivalent* on G and that φ' is obtained by *modifying* φ.

These two seemingly simple lemmas prove to be valuable when proving that a graph is (Γ, φ)-colorable (or non-Γ-colorable):

Observation 5.5. Given $\alpha_1, \ldots, \alpha_k \in \Gamma$ and any set of edges $E_0 = \{e_1, \ldots, e_k\} \subseteq E(G)$ such that $G[E_0]$ contains no cycles, we may assume that φ has $\varphi(e_i) = \alpha_i$ for $1 \leq i \leq k$ using the operation of Lemma 5.4 successively.

Furthermore, given $\beta \in \Gamma$ and any vertex $v_0 \in V(G)$, we may assume that any (Γ, φ)-coloring c has $c(v_0) = \beta$ using the operation of Lemma 5.3.

5.1.2 Hajos' construction

Theorem 5.6. *Let G_1 and G_2 be graphs, and let G be a Hajos' construction on G_1 and G_2 with edges $e_1 \in G_1$ and $e_2 \in G_2$. If G is Γ-colorable, then either G_1 or G_2 is also Γ-colorable. Equivalently, if G_1 and G_2 are both non-Γ-colorable, then G is non-Γ-colorable.*

Proof. Assume for contradiction that both G_1 and G_2 are non-Γ-colorable, but G is Γ-colorable. Let $\varphi_1 : E(G_1) \to \Gamma$ and $\varphi_2 : E(G_2) \to \Gamma$ be functions such that there is no (Γ, φ_1)-coloring of G_1 and there is no (Γ, φ_2)-coloring of G_2. By Observation 5.5, we can assume that $\varphi_1(e_1) = \varphi_2(e_2) = 0$. Define $\varphi : E(G) \to \Gamma$ such that $\varphi(e) = \varphi_1(e)$ for all $e \in E(G_1) \setminus \{e_1\}$, $\varphi(e) = \varphi_2(e)$ for all $e \in E(G_2) \setminus \{e_2\}$, and $\varphi(e_0) = 0$. Then G has a (Γ, φ)-coloring c, which must satisfy $c(y_1) \neq c(y_2)$. Thus either $c(y_1) \neq c(x)$ or $c(y_2) \neq c(x)$ which implies that either c restricted to G_1 is a (Γ, φ_1)-coloring of G_1 or c restricted to G_2 is a (Γ, φ_2)-coloring of G_2, a contradiction. □

Definition 5.7. Let Γ be an Abelian group. We say that a graph G without isolated vertices is Γ-*critical* if G is non-Γ-colorable, but for any edge $e \in E(G)$, $G - e$ is Γ-colorable.

Theorem 5.8. *Let G_1 and G_2 be Γ-critical graphs, and let G be a Hajos' construction on G_1 and G_2 with edges $e_1 \in G_1$ and $e_2 \in G_2$. Then G is Γ-critical.*

Proof. By Theorem 5.6 above G is not Γ-colorable. We now prove that for any $e \in G$ $G - e$ is Γ-colorable. We consider two cases. First we assume e is the new edge e_0. Let $\varphi : E(G - e_0) \to \Gamma$. Since $G_1 - e_1$ and $G_2 - e_2$ are both Γ-colorable we can find

(Γ,φ)-colorings c_1 and c_2 of $G_1 - e_1$ and $G_2 - e_2$, respectively, and we can assume $c_1(x) = c_2(x) = 0$ by Lemma 5.3. Combining c_1 and c_2 now gives a (Γ,φ)-coloring of $G - e_0$.

Otherwise, e is contained in, say, G_1. Assume e_0 is directed towards y_1 in G. Let $\varphi : E(G-e) \to \Gamma$, and assume $\varphi(e_0) = \gamma$. As $G_2 - e_2$ is Γ-colorable we can find a (Γ,φ)-coloring c_2 of $G_2 - e_2$, such that $c_2(x) = 0$ by Lemma 5.3. Define $\varphi' : E(G_1 - e) \to \Gamma$ such that φ' is equal to the restriction of φ on $G_1 - e$ and has $\varphi'(e_1) = c_2(y_2) + \gamma$. As $G_1 - e$ is Γ-colorable we can find a (Γ,φ')-coloring c_1 of $G_1 - e$, such that $c_1(x) = 0$ by Lemma 5.3. Combining c_1 and c_2 now gives a (Γ,φ)-coloring c of $G - e$, since $c(y_1) \neq c(y_2) + \gamma$. □

The argument in the second part of the proof above can also be used to prove the following:

Theorem 5.9. *Let G_1 be a Γ-colorable graph, let G_2 be a Γ-critical graph, and let G be a Hajos' construction on G_1 and G_2 with edges $e_1 \in G_1$ and $e_2 \in G_2$. Then G is Γ-colorable.*

Proof. Let $\varphi : E(G) \to \Gamma$. Assume $\varphi(e_0) = \gamma$ for some $\gamma \in \Gamma$ and e_0 is directed towards y_1. As $G_2 - e_2$ is Γ-colorable we can find a (Γ,φ)-coloring c_2 of $G_2 - e_2$, such that $c_2(x) = 0$ by Lemma 5.3. Define $\varphi' : E(G_1) \to \Gamma$ such that φ' is equal to the restriction of φ on G_1 and has $\varphi'(e_1) = c_2(y_2) + \gamma$. As G_1 is Γ-colorable we can find a (Γ,φ')-coloring c_1 of G_1, such that $c_1(x) = 0$ by Observation 5.5. Combining c_1 and c_2 now gives a (Γ,φ)-coloring c of G, since $c(y_1) \neq c(y_2) + \gamma$. □

5.2 A simple planar $\mathbb{Z}_4$-critical and $\mathbb{Z}_2 \times \mathbb{Z}_2$-critical graph

We now present a graph which will be used in the Hajos' construction of a planar simple graph which is $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable.

In the following section we let Γ be an Abelian group which can be either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. When we prove statements which are true for both $\mathbb{Z}_4$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorings, we write that we will prove them for Γ-colorings.

We define an operation as follows. Let G be a graph, and let $v_1, v_2, v_3 \in V(G)$ such that $v_1v_2v_3v_1$ forms a facial triangle. We obtain a new graph G' from G by adding the vertices u_1, u_2, u_3 and edges u_1u_2, u_2u_3, u_3u_1 as well as edges $v_1u_2, v_1u_3, v_2u_3, v_2u_1, v_3u_1, v_3u_2$ (see Figure 5.1). We say that we *insert a triangle* $u_1u_2u_3u_1$ into the triangle $v_1v_2v_3v_1$.

Given a forbidden function $\varphi : E(G) \to \Gamma$ and a coloring $c(v_1), c(v_2), c(v_3)$ of the vertices v_1, v_2, v_3 we define the sets of forbidden colors at u_1, u_2, u_3 to be $F_{u_1} := \{\tau_{v_2}(u_1), \tau_{v_3}(u_1)\}$, $F_{u_2} := \{\tau_{v_1}(u_2), \tau_{v_3}(u_2)\}$ and $F_{u_3} := \{\tau_{v_1}(u_3), \tau_{v_2}(u_3)\}$, respectively. Similarly, we define the sets of available colors at u_1, u_2, u_3 to be $L_{u_1} := \Gamma \setminus F_{u_1}$, $L_{u_2} := \Gamma \setminus F_{u_2}$ and $L_{u_3} := \Gamma \setminus F_{u_3}$, respectively.

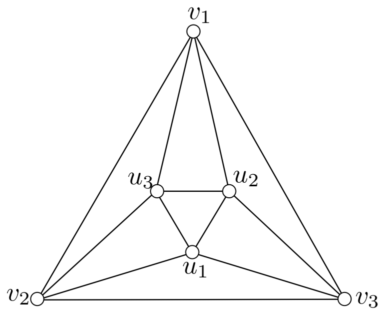

Figure 5.1: An inserted triangle.

We make the following observation: If $\varphi : E(G) \to \Gamma$ has $\varphi(u_1u_2) = \varphi(u_2u_3) = \varphi(u_3u_1) = 0$, and v_1, v_2, v_3 are (Γ, φ)-colored such that u_1, u_2, u_3 each have precisely two available colors where $L_{u_1} = L_{u_2} = L_{u_3}$, then clearly u_1, u_2, u_3 cannot be (Γ, φ)-colored without creating a color conflict. In fact, we can always assume that $L_{u_1} = L_{u_2} = L_{u_3}$ when u_1, u_2, u_3 cannot be colored.

Lemma 5.10. *Let $\varphi : E(G) \to \Gamma$. Let G be a graph which contains the inserted triangle $u_1u_2u_3u_1$ in the triangle $v_1v_2v_3v_1$. If the (Γ, φ)-coloring $c(v_1), c(v_2), c(v_3)$ of v_1, v_2, v_3 does not extend to u_1, u_2, u_3 then we may assume, after replacing φ by a Γ-equivalent φ' if necessary, that $\varphi(u_1u_2) = \varphi(u_2u_3) = \varphi(u_3u_1) = 0$ and $L_{u_1} = L_{u_2} = L_{u_3}$ (where the sets have precisely two elements).*

Equivalently, $F_{u_1} = F_{u_2} = F_{u_3}$ (where the sets have precisely two elements).

Proof. Assume first $\varphi(u_1u_2) + \varphi(u_2u_3) + \varphi(u_3u_1) = 0$. Modifying φ on u_1 and u_2 using the operation of Lemma 5.4 with $\alpha_1 = \varphi(u_1u_3)$ and $\alpha_2 = \varphi(u_2u_3)$ we get a Γ-equivalent function $\varphi' : E(G) \to \Gamma$ which has $\varphi'(u_1u_2) = \varphi'(u_2u_3) = \varphi'(u_3u_1) = 0$. Now assume without loss of generality that $L_{u_1} \neq L_{u_3}$, that is, we can assume there exists $\alpha \in L_{u_1} \setminus L_{u_3}$. Give u_1 color α and then color u_2 and u_3 in that order. The resulting coloring has no color conflicts, a contradiction.

Assume now $\varphi(u_1u_2) + \varphi(u_2u_3) + \varphi(u_3u_1) \neq 0$. Using the operation of Lemma 5.4 as above we get a Γ-equivalent function $\varphi' : E(G) \to \Gamma$ which has $\varphi'(u_1u_3) = \varphi'(u_2u_3) = 0$ and $\varphi'(u_1u_2) \neq 0$. Now if $L_{u_1} \cap L_{u_2} \neq \emptyset$ then choose $\alpha \in L_{u_1} \cap L_{u_2}$ and color u_1, u_2 with α. Then it is easy to also color u_3 without color conflicts. Otherwise $L_{u_1} \cap L_{u_2} = \emptyset$. Then either there exists $\alpha \in L_{u_1} \setminus L_{u_3}$, or there exists $\beta \in L_{u_2} \setminus L_{u_3}$. Assume without loss of generality that it is the first case. Color u_1 with α and then color u_2, u_3 in that order. The resulting coloring has no color conflicts, a contradiction. □

Furthermore, it is easy to see that if $c(v_1), c(v_2), c(v_3)$ is a coloring of v_1, v_2, v_3 which does not extend to u_1, u_2, u_3, then the coloring $c(v_1)+\alpha, c(v_2)+\alpha, c(v_3)+\alpha$ does not extend to the inserted triangle either for any $\alpha \in \Gamma$.

But, as the following lemma will show, these are the only colorings of v_1, v_2, v_3 which do not extend to the inserted triangle:

Lemma 5.11. *Let $\varphi : E(G) \to \Gamma$. Let G be a graph which contains the inserted triangle $u_1u_2u_3u_1$ in the triangle $v_1v_2v_3v_1$. If neither of the (Γ, φ)-colorings $c(v_1), c(v_2), c(v_3)$ and $c'(v_1), c'(v_2), c'(v_3)$ of v_1, v_2, v_3 extends to u_1, u_2, u_3 then there exists an $\alpha \in \Gamma$ such that $c(v_1) = c'(v_1)+\alpha$, $c(v_2) = c'(v_2)+\alpha$, $c(v_3) = c'(v_3)+\alpha$.*

Proof. Assume for contradiction that $c(v_1), c(v_2), c(v_3)$ and $c'(v_1), c'(v_2), c'(v_3)$ are (Γ, φ)-colorings of v_1, v_2, v_3, such that the conclusion of Lemma 5.11 is not satisfied. Then we may assume that $c'(v_1) = c(v_1)$, since otherwise we use the coloring $c'(v_1)+\beta, c'(v_2)+\beta, c'(v_3)+\beta$, where $\beta = c(v_1) - c'(v_1)$. Let L_{u_i}, F_{u_i} be the available and forbidden colors in u_i for $i = 1,2,3$, respectively, when v_1, v_2, v_3 are colored $c(v_1), c(v_2), c(v_3)$, and let L'_{u_i}, F'_{u_i} be the available and forbidden colors in u_i for $i = 1,2,3$, respectively, when v_1, v_2, v_3 are colored $c'(v_1), c'(v_2), c'(v_3)$. By Lemma 5.10 we can assume that $\varphi(u_1u_2) = \varphi(u_2u_3) = \varphi(u_3u_1) = 0$, $L_{u_1} = L_{u_2} = L_{u_3}$ and $L'_{u_1} = L'_{u_2} = L'_{u_3}$.

Consider first the case where $c'(v_2) \neq c(v_2)$, but $c'(v_3) = c(v_3)$ (the case where $c'(v_3) \neq c(v_3)$, but $c'(v_2) = c(v_2)$ is proven similarly). Then $L'_{u_2} = L_{u_2}$, but $L'_{u_1} \neq L_{u_1}$, a contradiction to Lemma 5.10.

Consider now the case where $c'(v_2) \neq c(v_2)$ and $c'(v_3) \neq c(v_3)$. Assume $c'(v_2) = c(v_2)+\gamma$ and $c'(v_3) = c(v_3)+\delta$ where $\gamma, \delta \neq 0$. Then if $F_{u_1} = \{a, b\}$ where $a, b \in \Gamma$ and $a \neq b$, we can assume $F'_{u_1} = \{a+\gamma, b+\delta\}$ and either $F'_{u_2} = \{a+\delta, b\}$ or $F'_{u_2} = \{a, b+\delta\}$. Thus either $\{a+\gamma, b+\delta\} = \{a+\delta, b\}$ or $\{a+\gamma, b+\delta\} = \{a, b+\delta\}$. It is easy to check that none of the combinations can be true, a contradiction which proves Lemma 5.11. □

Let K_5^{Δ} be the graph obtained from K_5 by deleting an edge and inserting triangles in four of the faces containing the same vertex. We shall refer to the five vertices of K_5 as v_1, v_2, v_3, v_4, v_5 (see Figure 5.2).

Theorem 5.12. *K_5^{Δ} is $\mathbb{Z}_4$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-critical.*

That $K_5^{\Delta} - e$ is $\mathbb{Z}_4$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable for every $e \in E(K_5^{\Delta})$ is too extensive to prove without using a computer, so we leave out the proof. Below follows the proof that K_5^{Δ} is neither $\mathbb{Z}_4$- nor $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable.

Proof. We first prove that K_5^{Δ} is not $\mathbb{Z}_4$-colorable. Define $\varphi : E(K_5^{\Delta}) \to \mathbb{Z}_4$ as in Figure 5.3 (for all unlabelled edges e, we let $\varphi(e) = 0$).

Now, consider all $(\mathbb{Z}_4, \varphi)$-colorings of the subgraph induced by v_1, v_2, v_3, v_4, v_5. By Lemma 5.3, we can assume that v_5 always has color 0. We can check that there are

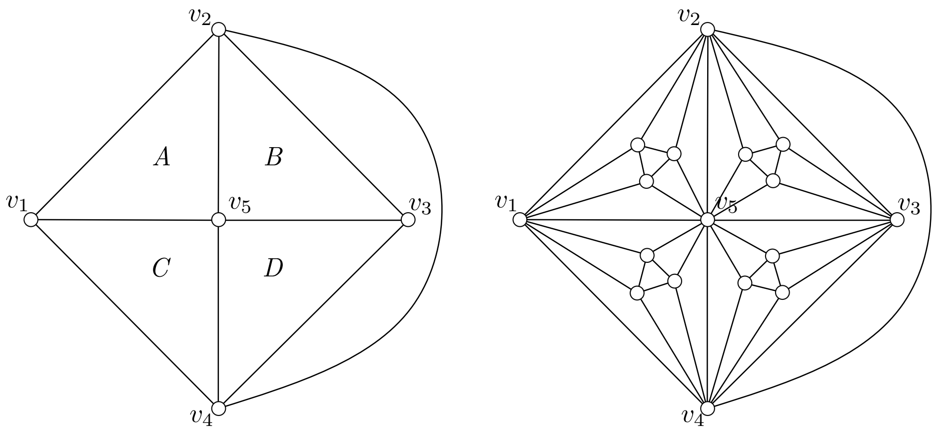

Figure 5.2: The graph $K_5 - e$ (on the left) and the graph K_5^{Δ} (on the right).

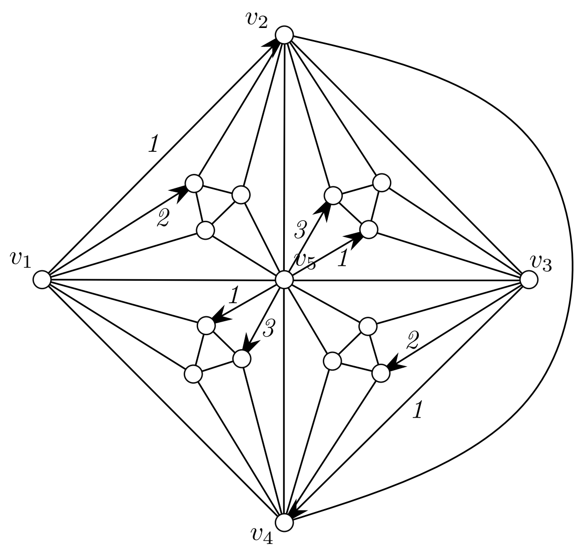

Figure 5.3: A φ-function for which K_5^{Δ} cannot be $\mathbb{Z}_4$-colored.

in total 16 $(\mathbb{Z}_4, \varphi)$-colorings, $c_1, c_2, \ldots, c_{16}$, and they are given by:

$$c_1(v_1) = 1, c_1(v_2) = 1, c_1(v_3) = 2, c_1(v_4) = 2, c_1(v_5) = 0 \tag{5.3}$$
$$c_2(v_1) = 1, c_2(v_2) = 1, c_2(v_3) = 3, c_2(v_4) = 2, c_2(v_5) = 0 \tag{5.4}$$
$$c_3(v_1) = 1, c_3(v_2) = 1, c_3(v_3) = 3, c_3(v_4) = 3, c_3(v_5) = 0 \tag{5.5}$$
$$c_4(v_1) = 1, c_4(v_2) = 3, c_4(v_3) = 2, c_4(v_4) = 2, c_4(v_5) = 0 \tag{5.6}$$
$$c_5(v_1) = 2, c_5(v_2) = 1, c_5(v_3) = 3, c_5(v_4) = 3, c_5(v_5) = 0 \tag{5.7}$$
$$c_6(v_1) = 2, c_6(v_2) = 2, c_6(v_3) = 1, c_6(v_4) = 1, c_6(v_5) = 0 \tag{5.8}$$

$$c_7(v_1) = 2, c_7(v_2) = 2, c_7(v_3) = 1, c_7(v_4) = 3, c_7(v_5) = 0 \tag{5.9}$$
$$c_8(v_1) = 2, c_8(v_2) = 2, c_8(v_3) = 3, c_8(v_4) = 1, c_8(v_5) = 0 \tag{5.10}$$
$$c_9(v_1) = 2, c_9(v_2) = 2, c_9(v_3) = 3, c_9(v_4) = 3, c_9(v_5) = 0 \tag{5.11}$$
$$c_{10}(v_1) = 3, c_{10}(v_2) = 1, c_{10}(v_3) = 2, c_{10}(v_4) = 2, c_{10}(v_5) = 0 \tag{5.12}$$
$$c_{11}(v_1) = 3, c_{11}(v_2) = 1, c_{11}(v_3) = 3, c_{11}(v_4) = 2, c_{11}(v_5) = 0 \tag{5.13}$$
$$c_{12}(v_1) = 3, c_{12}(v_2) = 2, c_{12}(v_3) = 1, c_{12}(v_4) = 1, c_{12}(v_5) = 0 \tag{5.14}$$
$$c_{13}(v_1) = 3, c_{13}(v_2) = 2, c_{13}(v_3) = 3, c_{13}(v_4) = 1, c_{13}(v_5) = 0 \tag{5.15}$$
$$c_{14}(v_1) = 3, c_{14}(v_2) = 3, c_{14}(v_3) = 1, c_{14}(v_4) = 1, c_{14}(v_5) = 0 \tag{5.16}$$
$$c_{15}(v_1) = 3, c_{15}(v_2) = 3, c_{15}(v_3) = 2, c_{15}(v_4) = 1, c_{15}(v_5) = 0 \tag{5.17}$$
$$c_{16}(v_1) = 3, c_{16}(v_2) = 3, c_{16}(v_3) = 2, c_{16}(v_4) = 2, c_{16}(v_5) = 0 \tag{5.18}$$

We will show that none of the colorings above can be extended to K_5^Δ by indicating a triangle whose inserted triangle cannot be colored:

- Colorings c_6, c_7, c_8, c_9 do not extend to the inserted triangle of A.
- Colorings c_2, c_3, c_5, c_{11} do not extend to the inserted triangle of B.
- Colorings c_1, c_4, c_{10}, c_{16} do not extend to the inserted triangle of C.
- Colorings $c_{12}, c_{13}, c_{14}, c_{15}$ do not extend to the inserted triangle of D.

In all cases the inserted triangle cannot be colored since only two colors (either 0 and 2 or 1 and 3) are available in the three vertices of the inserted triangle.

We now prove that K_5^Δ is not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable. Define $\varphi : E(K_5^\Delta) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ as in Figure 5.4 (for all unlabelled edges e, we let $\varphi(e) = 0$).

Now, consider all $(\mathbb{Z}_2 \times \mathbb{Z}_2, \varphi)$-colorings of the subgraph induced by v_1, v_2, v_3, v_4, v_5. By Lemma 5.3, we can assume that v_5 always has color 0. We can check that there are in total 20 $(\mathbb{Z}_2 \times \mathbb{Z}_2, \varphi)$-colorings, $c_1, c_2, \ldots, c_{20}$, and they are given by:

$$c_1(v_1) = 01, c_1(v_2) = 01, c_1(v_3) = 01, c_1(v_4) = 10, c_1(v_5) = 0 \tag{5.19}$$
$$c_2(v_1) = 01, c_2(v_2) = 01, c_2(v_3) = 10, c_2(v_4) = 10, c_2(v_5) = 0 \tag{5.20}$$
$$c_3(v_1) = 01, c_3(v_2) = 10, c_3(v_3) = 01, c_3(v_4) = 01, c_3(v_5) = 0 \tag{5.21}$$
$$c_4(v_1) = 01, c_4(v_2) = 10, c_4(v_3) = 10, c_4(v_4) = 01, c_4(v_5) = 0 \tag{5.22}$$
$$c_5(v_1) = 01, c_5(v_2) = 10, c_5(v_3) = 11, c_5(v_4) = 01, c_5(v_5) = 0 \tag{5.23}$$
$$c_6(v_1) = 01, c_6(v_2) = 11, c_6(v_3) = 10, c_6(v_4) = 01, c_6(v_5) = 0 \tag{5.24}$$
$$c_7(v_1) = 01, c_7(v_2) = 11, c_7(v_3) = 10, c_7(v_4) = 10, c_7(v_5) = 0 \tag{5.25}$$
$$c_8(v_1) = 01, c_8(v_2) = 11, c_8(v_3) = 11, c_8(v_4) = 01, c_8(v_5) = 0 \tag{5.26}$$
$$c_9(v_1) = 10, c_9(v_2) = 01, c_9(v_3) = 01, c_9(v_4) = 10, c_9(v_5) = 0 \tag{5.27}$$

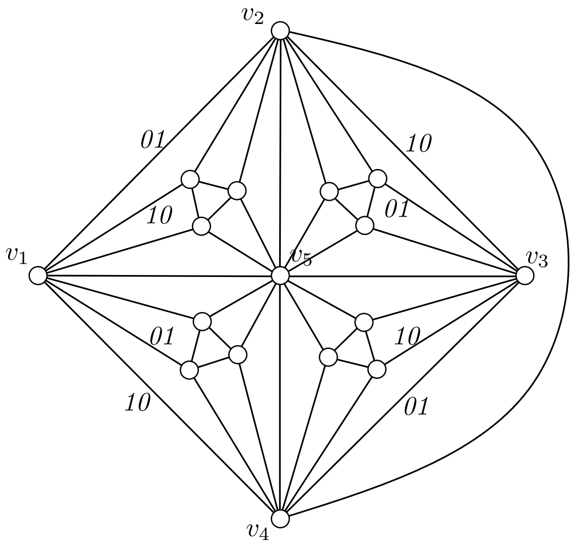

Figure 5.4: A φ-function for which K_5^Δ cannot be $\mathbb{Z}_2 \times \mathbb{Z}_2$-colored.

$$c_{10}(v_1) = 10, c_{10}(v_2) = 01, c_{10}(v_3) = 01, c_{10}(v_4) = 11, c_{10}(v_5) = 0 \tag{5.28}$$
$$c_{11}(v_1) = 10, c_{11}(v_2) = 01, c_{11}(v_3) = 10, c_{11}(v_4) = 10, c_{11}(v_5) = 0 \tag{5.29}$$
$$c_{12}(v_1) = 10, c_{12}(v_2) = 10, c_{12}(v_3) = 01, c_{12}(v_4) = 01, c_{12}(v_5) = 0 \tag{5.30}$$
$$c_{13}(v_1) = 10, c_{13}(v_2) = 10, c_{13}(v_3) = 01, c_{13}(v_4) = 11, c_{13}(v_5) = 0 \tag{5.31}$$
$$c_{14}(v_1) = 10, c_{14}(v_2) = 10, c_{14}(v_3) = 10, c_{14}(v_4) = 01, c_{14}(v_5) = 0 \tag{5.32}$$
$$c_{15}(v_1) = 10, c_{15}(v_2) = 10, c_{15}(v_3) = 11, c_{15}(v_4) = 01, c_{15}(v_5) = 0 \tag{5.33}$$
$$c_{16}(v_1) = 10, c_{16}(v_2) = 10, c_{16}(v_3) = 11, c_{16}(v_4) = 11, c_{16}(v_5) = 0 \tag{5.34}$$
$$c_{17}(v_1) = 11, c_{17}(v_2) = 01, c_{17}(v_3) = 01, c_{17}(v_4) = 10, c_{17}(v_5) = 0 \tag{5.35}$$
$$c_{18}(v_1) = 11, c_{18}(v_2) = 01, c_{18}(v_3) = 01, c_{18}(v_4) = 11, c_{18}(v_5) = 0 \tag{5.36}$$
$$c_{19}(v_1) = 11, c_{19}(v_2) = 01, c_{19}(v_3) = 10, c_{19}(v_4) = 10, c_{19}(v_5) = 0 \tag{5.37}$$
$$c_{20}(v_1) = 11, c_{20}(v_2) = 11, c_{20}(v_3) = 10, c_{20}(v_4) = 10, c_{20}(v_5) = 0 \tag{5.38}$$

We will show that none of the colorings above can be extended to K_5^Δ by indicating a triangle whose inserted triangle cannot be colored:

- Colorings $c_{12}, c_{13}, c_{14}, c_{15}, c_{16}$ do not extend to the inserted triangle of A.
- Colorings $c_1, c_9, c_{10}, c_{17}, c_{18}$ do not extend to the inserted triangle of B.
- Colorings $c_2, c_7, c_{11}, c_{19}, c_{20}$ do not extend to the inserted triangle of C.
- Colorings c_3, c_4, c_5, c_6, c_8 do not extend to the inserted triangle of D.

In all cases the inserted triangle cannot be colored since only two colors (either 0 and 01 or 0 and 10) are available in the three vertices of the inserted triangle. □

Observation 5.13. Note, that we may simplify the proof of the existence of either a simple $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable graph, or a simple $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, but not $\mathbb{Z}_4$-colorable graph when we already know one with multiple edges. Indeed, if we have a graph which can be proven to be neither $\mathbb{Z}_4$-colorable, nor $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, then we may remove edges until the subgraph is either $\mathbb{Z}_4$-colorable and non-$\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, or non-$\mathbb{Z}_4$-colorable and $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, or $\mathbb{Z}_4$-critical and $\mathbb{Z}_2 \times \mathbb{Z}_2$-critical. If we are in one of the first two situations, then we have already here achieved half of the goal of this section, and if we are in the third situation, we use Hajos' construction as described in the section below. (Note, that using this method we cannot decide which of the cases we end up in.)

5.3 A simple planar $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable graph

We will now use Hajos' construction to describe a simple planar graph which is $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable. Note that the proof is computer-aided since the graphs involved in the construction are checked by a computer.

The authors of [14] state without proof that the graph G depicted in Figure 5.5 is $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected. As it is planar we can construct the dual graph G^* which is $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, also depicted in Figure 5.5.

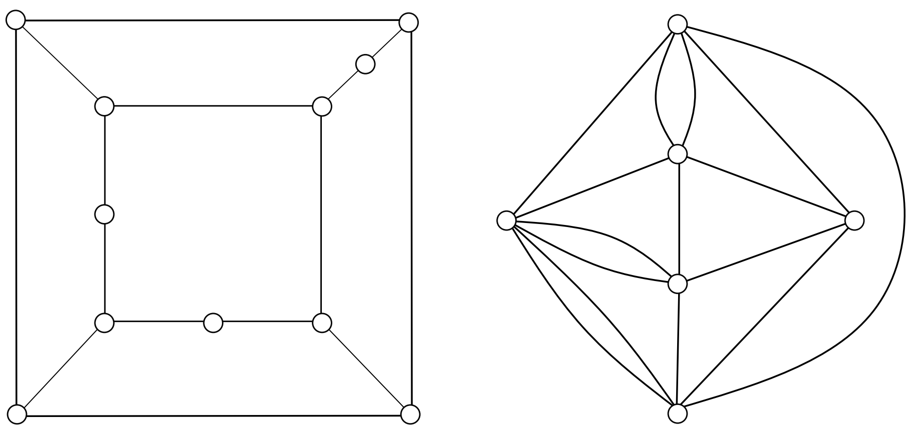

Figure 5.5: The graph G (on the left) and its dual G^* (on the right).

In the Figure 5.6 we see a forbidden function φ for which G^* cannot be $\mathbb{Z}_2 \times \mathbb{Z}_2$-colored. It is easy to check that it is indeed impossible to $\mathbb{Z}_2 \times \mathbb{Z}_2$-color the graph with this φ.

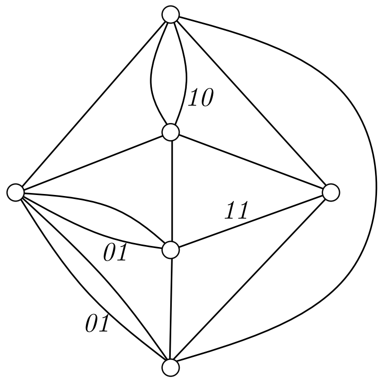

Figure 5.6: A φ-function for which G^* cannot be $\mathbb{Z}_2 \times \mathbb{Z}_2$-colored.

Consider now the graph H obtained from G^* by applying Hajos' construction with K_5^Δ three times where e_1 is one of the three multi-edges in G^* and e_2 is the edge v_1v_2.

Theorem 5.14. *H is a simple planar graph which is $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable.*

Proof. Clearly, all multi-edges disappear when the Hajos' construction is used on them, so H must be simple. By Theorem 5.9, H is $\mathbb{Z}_4$-colorable since G^* is $\mathbb{Z}_4$-colorable and K_5^Δ is $\mathbb{Z}_4$-critical. By Theorem 5.6, H is not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable since G^* is non-$\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable and K_5^Δ is $\mathbb{Z}_2 \times \mathbb{Z}_2$-critical and therefore non-$\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable. □

As we can keep applying Hajos' construction on G^* and K_5^Δ, we get the following corollary:

Corollary 5.15. *There exists an infinite family of simple planar graphs which are $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable.*

And when we take the dual graphs, we obtain the following result.

Corollary 5.16. *There exists an infinite family of 3-edge-connected planar graphs which are $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected.*

5.4 A simple planar $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, but not $\mathbb{Z}_4$-colorable graph

In this section we will describe a planar simple graph which is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -colorable, but not $\mathbb{Z}_4$-colorable. This is done without the aid of computers.

Let K_4^Δ be the graph obtained from K_4 by inserting triangles in each of the four faces. We will refer to the four vertices of K_4 as v_1, v_2, v_3, v_4 (see Figure 5.7). The facial triangles of K_4 will be referred to as follows: $A : v_1v_2v_3v_1$, $B : v_1v_2v_4v_1$, $C : v_1v_3v_4v_1$ and $D : v_2v_3v_4v_2$. Lastly, we will refer to the vertices of the inserted triangles as u_i^T where T is the facial triangle of K_4 in which u_i^T is inserted and i is the index of the vertex of T not adjacent to u_i^T (see Figure 5.7).

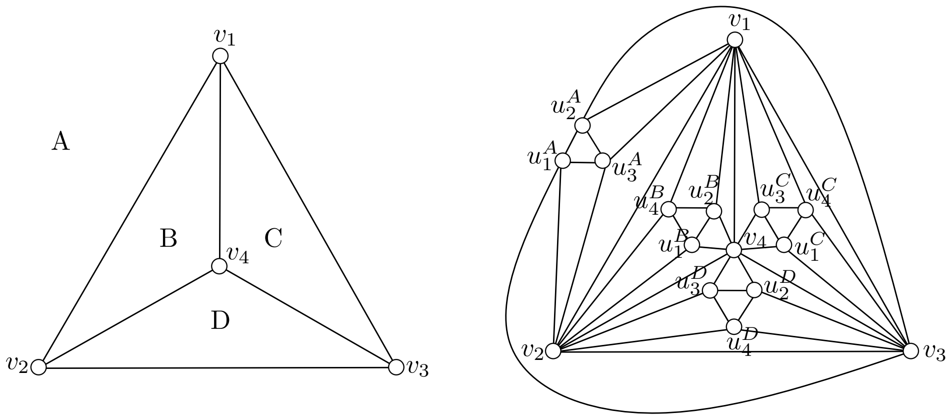

Figure 5.7: The graph K_4 (on the left) and the graph K_4^Δ (on the right).

Theorem 5.17. *K_4^Δ is not $\mathbb{Z}_4$-colorable, but it is $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable.*

Proof. We first proof that K_4^Δ is not $\mathbb{Z}_4$-colorable. To prove that it is not $\mathbb{Z}_4$-colorable we define $\varphi : E(K_4^\Delta) \to \mathbb{Z}_4$ as follows:

$$\varphi(v_2v_3) = \varphi(v_3v_4) = \varphi(v_4v_2) = 1, \tag{5.39}$$

$$\varphi(v_1u_3^A) = \varphi(v_1u_4^B) = \varphi(v_1u_3^C) = \varphi(v_2u_3^D) = \varphi(v_3u_4^D) = \varphi(v_4u_2^D) = 1, \tag{5.40}$$

$$\varphi(v_1u_2^A) = \varphi(v_1u_2^B) = \varphi(v_1u_4^C) = \varphi(v_2u_4^D) = \varphi(v_3u_2^D) = \varphi(v_4u_3^D) = 3, \tag{5.41}$$

and for all other edges $e \in E(K_4^\Delta)$, $\varphi(e) = 0$ (see Figure 5.8).

Now, consider all $(\mathbb{Z}_4, \varphi)$-colorings of the subgraph induced by v_1, v_2, v_3, v_4. By Lemma 5.3, we can assume that v_1 always has color 0. We can check that there are

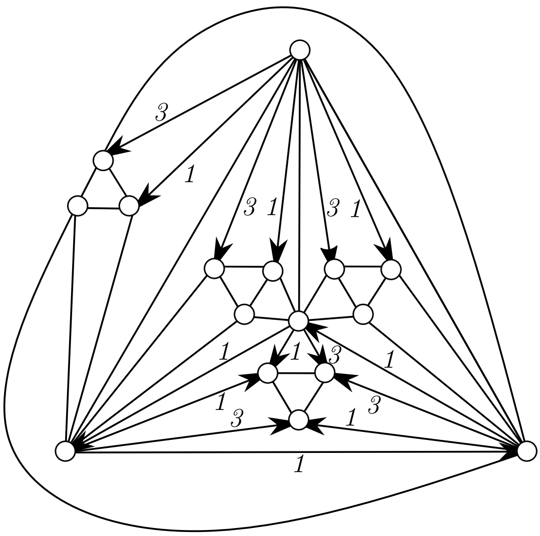

Figure 5.8: A φ-function for which K_4^Δ cannot be colored.

in total 12 $(\mathbb{Z}_4, \varphi)$-colorings, $c_1, c_2, \ldots, c_{12}$, and they are given by:

$$c_1(v_1) = 0, c_1(v_2) = 1, c_1(v_3) = 1, c_1(v_4) = 1 \tag{5.42}$$
$$c_2(v_1) = 0, c_2(v_2) = 1, c_2(v_3) = 1, c_2(v_4) = 3 \tag{5.43}$$
$$c_3(v_1) = 0, c_3(v_2) = 1, c_3(v_3) = 3, c_3(v_4) = 1 \tag{5.44}$$
$$c_4(v_1) = 0, c_4(v_2) = 1, c_4(v_3) = 3, c_4(v_4) = 2 \tag{5.45}$$
$$c_5(v_1) = 0, c_5(v_2) = 1, c_5(v_3) = 3, c_5(v_4) = 3 \tag{5.46}$$
$$c_6(v_1) = 0, c_6(v_2) = 2, c_6(v_3) = 1, c_6(v_4) = 3 \tag{5.47}$$
$$c_7(v_1) = 0, c_7(v_2) = 2, c_7(v_3) = 2, c_7(v_4) = 2 \tag{5.48}$$
$$c_8(v_1) = 0, c_8(v_2) = 3, c_8(v_3) = 1, c_8(v_4) = 1 \tag{5.49}$$
$$c_9(v_1) = 0, c_9(v_2) = 3, c_9(v_3) = 1, c_9(v_4) = 3 \tag{5.50}$$
$$c_{10}(v_1) = 0, c_{10}(v_2) = 3, c_{10}(v_3) = 2, c_{10}(v_4) = 1 \tag{5.51}$$
$$c_{11}(v_1) = 0, c_{11}(v_2) = 3, c_{11}(v_3) = 3, c_{11}(v_4) = 1 \tag{5.52}$$
$$c_{12}(v_1) = 0, c_{12}(v_2) = 3, c_{12}(v_3) = 3, c_{12}(v_4) = 3 \tag{5.53}$$

We will show that none of the colorings above can be extended to K_4^Δ by indicating a triangle whose inserted triangle cannot be colored:

- Colorings c_1, c_7, c_{12} does not extend to the inserted triangle of D.
- Colorings c_2, c_6, c_9 does not extend to the inserted triangle of C.

- Colorings c_3, c_4, c_5 does not extend to the inserted triangle of B.
- Colorings c_8, c_{10}, c_{11} does not extend to the inserted triangle of A.

In all cases the inserted triangle cannot be colored since only two colors (in most cases only 0 and 2) are available in the three vertices of the inserted triangle.

To prove that is K_4^Δ is $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable we follow a similar strategy. We divide into seven cases in which we will count the number of colorings of the underlying graph K_4 and also the number of colorings each inserted triangle may forbid. The proof will be complete when we can deduce that not all possible colorings can be forbidden at once.

As above, we use Lemma 5.3 to assume that v_1 always has color 0. Furthermore, we can assume that $\varphi(v_1v_2) = \varphi(v_1v_3) = \varphi(v_1v_4) = 0$ by Lemma 5.4. Thus we force $c(v_2), c(v_3), c(v_4)$ to be in $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{0\}$. As mentioned the inserted triangles may forbid more than one coloring. But luckily the colorings forbidden by an inserted triangle are closely related. In fact it follows from Lemma 5.11 that:

- The inserted triangle in face A can forbid at most 3 colorings where $c(v_1), c(v_2)$, $c(v_3)$ are constant and $c(v_4)$ is one of the three non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, if the inserted triangle in face A forbids precisely 3 colorings then v_2 and v_3 are colored such that $\tau_{v_2}(v_4) = \tau_{v_3}(v_4) = 0$.
- The inserted triangle in face B can forbid at most 3 colorings where $c(v_1), c(v_2)$, $c(v_4)$ are constant and $c(v_3)$ is one of the three non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, if the inserted triangle in face B forbids precisely 3 colorings then v_2 and v_4 are colored such that $\tau_{v_2}(v_3) = \tau_{v_4}(v_3) = 0$.
- The inserted triangle in face C can forbid at most 3 colorings where $c(v_1), c(v_3)$, $c(v_4)$ are constant and $c(v_2)$ is one of the three non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, if the inserted triangle in face C forbids precisely 3 colorings then v_3 and v_4 are colored such that $\tau_{v_3}(v_2) = \tau_{v_4}(v_2) = 0$.
- The inserted triangle in face D can forbid at most 3 colorings of the form $c(v_1), c(v_2) + \alpha, c(v_3) + \alpha, c(v_4) + \alpha$ where $\alpha \in \mathbb{Z}_2 \times \mathbb{Z}_2$ (note that not all four are valid colorings since $c(v_2), c(v_3), c(v_4) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{0\}$). In fact, if the inserted triangle in face D forbids precisely 3 colorings then $c(v_2) = c(v_3) = c(v_4)$.

We now go through the cases:

Case 1. $\varphi(v_2v_3) = \varphi(v_2v_4) = \varphi(v_3v_4) = 0$. In this case there are 6 colorings of K_4^Δ in which v_1 has color 0. The inserted triangles can clearly only forbid one coloring each, hence the graph is colorable.

Case 2. Two of $\varphi(v_2v_3), \varphi(v_2v_4), \varphi(v_3v_4)$ are 0, while the last is non-zero. In this case there are 10 colorings of K_4^Δ in which v_1 has color 0. We will argue that the inserted triangles can only forbid two colorings each, hence at most 8 of the 10 colorings are forbidden so the graph is colorable. Assume without loss of contradiction that $\varphi(v_2v_4) = \varphi(v_3v_4) = 0$. Then as $c(v_2), c(v_3), c(v_4)$ are non-zero, we must have $\tau_{v_2}(v_4), \tau_{v_4}(v_3), \tau_{v_4}(v_2) \neq 0$ so the inserted triangles of faces A, B, C cannot forbid three colorings each. Neither can the inserted triangle in face D forbid three colorings since v_2, v_3, v_4 cannot receive the same color.

Case 3. One of $\varphi(v_2v_3), \varphi(v_2v_4), \varphi(v_3v_4)$ is 0, while the last two are distinct. In this case there are 11 colorings of K_4^Δ in which v_1 has color 0. We will argue that three of the inserted triangles can only forbid two colorings each, hence at most 9 of the 11 colorings are forbidden so the graph is colorable. Assume without loss of generality that $\varphi(v_2v_3) = 0$. Then as $c(v_2)$ and $c(v_3)$ are non-zero, we must have $\tau_{v_2}(v_3), \tau_{v_3}(v_2) \neq 0$ so the inserted triangles of face B and face C cannot forbid three colorings each. Neither can the inserted triangle in face D forbid three colorings since v_2, v_3, v_4 cannot receive the same color.

Case 4. One of $\varphi(v_2v_3), \varphi(v_2v_4), \varphi(v_3v_4)$ is 0, while the last two are similar. In this case there are 10 colorings of K_4^Δ in which v_1 has color 0. We will argue that the inserted triangles can only forbid two colorings each, hence at most 8 of the 10 colorings are forbidden so the graph is colorable. Assume $\varphi(v_2v_3) = 0, \varphi(v_2v_4) = a, \varphi(v_3v_4) = a$ where $a \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{0\}$. Then as $c(v_2)$ and $c(v_3)$ are non-zero, we must have $\tau_{v_2}(v_3), \tau_{v_3}(v_2) \neq 0$ so the inserted triangles of face B and face C cannot forbid three colorings each. Now assume that the inserted triangle of A forbids three colorings. Then there is a coloring $c(v_2), c(v_3)$ such that $\tau_{v_2}(v_4) = \tau_{v_3}(v_4) = 0$, hence $c(v_2) = a, c(v_3) = a$. But this is impossible as $\varphi(v_2v_3) = a$. Thus we conclude that the inserted triangles of faces A, B, C cannot forbid three colorings each. Neither can the inserted triangle in face D forbid three colorings since v_2, v_3, v_4 cannot receive the same color.

Case 5. $\varphi(v_2v_3), \varphi(v_2v_4), \varphi(v_3v_4) \neq 0$, and all three are distinct. In this case there are 11 colorings of K_4^Δ in which v_1 has color 0. We will argue that three of the inserted triangles can only forbid two colorings each, hence at most 9 of the 11 colorings are forbidden so the graph is colorable. Assume $\varphi(v_2v_3) = a, \varphi(v_2v_4) = b, \varphi(v_3v_4) = c$ where $a, b, c \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{0\}$ are distinct. Suppose without loss of contradiction that the inserted triangle of A forbids three colorings. Then there is a coloring $c(v_2), c(v_3)$ such that $\tau_{v_2}(v_4) = \tau_{v_3}(v_4) = 0$, hence $c(v_2) = b, c(v_3) = c$. But this is impossible as $\varphi(v_2v_3) = a$. Thus we conclude that the inserted triangles of faces A, B, C cannot forbid three colorings each.

Case 6. $\varphi(v_2v_3), \varphi(v_2v_4), \varphi(v_3v_4) \neq 0$, and two of them are similar. In this case there are 13 colorings of K_4^Δ in which v_1 has color 0. As the inserted triangles can forbid at most 12 of them, the graph is colorable.

Case 7. $\varphi(v_2v_3), \varphi(v_2v_4), \varphi(v_3v_4) \neq 0$, and all three are similar. In this case there are 15 colorings of K_4^Δ in which v_1 has color 0. As the inserted triangles can forbid at most 12 of them, the graph is colorable. □

5.5 A planar $\mathbb{Z}_2^k$-colorable graph which is not colorable for any non-isomorphic group of the same order

In this section we expand the results proven in Section 5.4. For any k we will describe a graph which is $\mathbb{Z}_2^k$-colorable, but not Γ-colorable for any other Abelian group Γ of order 2^k.

We define a family of graphs G_k as follows: Let G_2 denote the graph K_4^Δ as defined in the section above. For $k > 2$, let G_k be obtained from G_2 by replacing each edge in an appropriate spanning tree by multiple edges as follows. Each edge v_4v_i, $i = 1, 2, 3$, in K_4 is replaced by $2^k - 3$ edges. If u is a vertex in an inserted triangle, then we select a neighbor v_i of u in T_4, and we replace the edge uv_i by $2^k - 3$ edges. We select the neighbors such that no two vertices of an inserted triangle have multi-edges to the same vertex of K_4. All other edges have multiplicity 1.

We first prove a lemma which extends Lemma 5.11 to inserted triangles in which some of the edges are replaced by multi-edges.

Lemma 5.18. *Let $k \geq 2$ be a natural number, and let $\varphi : E(G) \to \mathbb{Z}_2^k$. Let G be a graph which contains the inserted triangle $u_1u_2u_3u_1$ in the triangle $v_1v_2v_3v_1$, where there are $2^k - 3$ edges between each pair v_1 and u_2, v_2 and u_3, and v_3 and u_1. If neither of the (Γ, φ)-colorings $c(v_1), c(v_2), c(v_3)$ and $c'(v_1), c'(v_2), c'(v_3)$ of v_1, v_2, v_3 extends to u_1, u_2, u_3, then there exists an $\alpha \in \Gamma$ such that $c(v_1) = c'(v_1) + \alpha$, $c(v_2) = c'(v_2) + \alpha$, $c(v_3) = c'(v_3) + \alpha$.*

Proof. We denote the set of $\varphi(e)$ where e joins v_1 and u_2 by $\Phi(v_1u_2)$. $\Phi(v_2u_3)$ and $\Phi(v_3u_1)$ are defined similarly. By modifying φ if necessary, we may assume that $c(v_1) = c(v_2) = c(v_3) = 0$. By Lemma 5.10, we may also assume that $\varphi(u_1u_2) = \varphi(u_2u_3) = \varphi(u_3u_1) = 0$ and that $\Phi(v_3u_1) \cup \{\varphi(v_2u_1)\} = \Phi(v_1u_2) \cup \{\varphi(v_3u_2)\} = \Phi(v_2u_3) \cup \{\varphi(v_1u_3)\}$. Put $\varphi(v_2u_1) = \gamma, \varphi(v_3u_2) = \delta, \varphi(v_1u_3) = \epsilon$.

Assume for contradiction that Lemma 5.18 is false. Then there exists a coloring c' of the triangle $v_1v_2v_3v_1$ which cannot be extended to $u_1u_2u_3u_1$ such that c' is not constant on v_1, v_2, v_3. If two of $c'(v_1), c'(v_2), c'(v_3)$ are equal, say $c'(v_1)$ and $c'(v_2)$, then we can assume $c'(v_1) = c'(v_2) = 0$, and then $L'_{u_2} = L_{u_2}$, but $L'_{u_1} \neq L_{u_1}$, a contradiction to Lemma 5.10.

So we may assume that $c'(v_1), c'(v_2), c'(v_3)$ are distinct, say $c'(v_1) = 0, c'(v_2) = a, c'(v_3) = b$, where $a, b \in \mathbb{Z}_2^k \setminus \{0\}$ and $a \neq b$. After the c-coloring, the color δ is forbidden at u_2 and hence at each of u_1, u_2, u_3. After the c'-coloring the forbidden colors at u_2 are the same as after the c-coloring, except that δ is replaced by $\delta + b$. Since ϵ is forbidden at u_3 (and hence at each of u_1, u_2, u_3) after the c'-coloring, and δ is not, we conclude that δ, ϵ are distinct. By similar arguments, all of γ, δ, ϵ are distinct. In particular, $k > 2$, and each multiple edge incident with one of u_1, u_2, u_3 contains precisely two of γ, δ, ϵ. It follows that there exists a subset Γ_0 of $\mathbb{Z}_2^k$ such that

$$\Phi(v_1u_2) = \{\gamma, \epsilon\} \cup \Gamma_0,$$
$$\Phi(v_2u_3) = \{\gamma, \delta\} \cup \Gamma_0,$$
$$\Phi(v_3u_1) = \{\delta, \epsilon\} \cup \Gamma_0.$$

After the c'-coloring of v_1, v_2, v_3, the set of forbidden colors at u_2 is $\Phi(v_1u_2) \cup \{\varphi(v_3u_2)+b\} = \{\delta+b, \gamma, \epsilon\} \cup \Gamma_0$. After the c'-coloring of v_1, v_2, v_3, the set of forbidden colors at u_1 is $(\Phi(v_3u_1) + b) \cup \{\varphi(v_2u_1) + a\} = \{\gamma + a, \delta + b, \epsilon + b\} \cup (b + \Gamma_0)$. This implies that

$$\{\gamma, \epsilon\} \cup \Gamma_0 = \{\gamma + a, \epsilon + b\} \cup (b + \Gamma_0).$$

Summing over the elements on each side of the equation and adding $\gamma + \epsilon + \sum \Gamma_0$ we get that $0 = a + b + (2^k - 5)b$ as Γ_0 has $2^k - 5$ elements. Hence $a = 0$. But we assumed that $a \neq 0$, a contradiction which proves Lemma 5.10. □

Now follows the main result of this section. We shall use the τ-notation as well as the definition of the matching graph, both of which are described in Chapter 4.

Theorem 5.19. *For every natural number k, G_k is $\mathbb{Z}_2^k$-colorable, but not Γ-colorable for any other Abelian group Γ of order 2^k.*

Proof. Consider first the case where $k = 2$ and $\Gamma = \mathbb{Z}_4$. Let $\varphi(v_4v_1) = \varphi(v_4v_2) = \varphi(v_4v_3) = 0$ and $\varphi(v_1v_2) = \varphi(v_2v_3) = \varphi(v_3v_1) = 1$. Now define the remaining values of φ such that

(i) the triangle inside $v_1v_2v_3v_1$ cannot be colored when the coloring of v_1, v_2, v_3 is equivalent to the type $0, 0, 0$ (that is, v_1, v_2, v_3 have the same color),

(ii) the triangle inside $v_4v_iv_jv_4$ cannot be colored when the coloring of v_4, v_i, v_j is equivalent to the type $0, 1, 3$, for each pair $(i, j) = (1, 2), (2, 3), (3, 1)$.

With this φ it follows from the proof of Theorem 5.17 that G_2 is not $\mathbb{Z}_4$-colorable.

Consider next the case where $k > 2$, and Γ is not isomorphic to $\mathbb{Z}_2^k$. Then Γ contains a subgroup isomorphic to $\mathbb{Z}_4$. (This follows since any Abelian group can be written as the sum of cyclic groups by the Fundamental Theorem of Finite Abelian Groups. Hence if Γ contains an element of order larger than 2, then it also contains an element of order 4. Thus Γ contains a subgroup of order 4 generated by this element.) Also, G_k contains a subgraph isomorphic to G_2. We define φ on the edges of this subgraph G_2 as in the previous case using the subgroup isomorphic to $\mathbb{Z}_4$. Consider now a multiple edge where precisely one of the $2^k - 3$ edges has received a φ-value. The φ-values of the remaining $2^k - 4$ edges will be precisely the elements in $\Gamma \setminus \mathbb{Z}_4$.

If G_k has a Γ-coloring c allowed by φ, then we may assume that $c(v_0) = 0$, which then implies that all other c-values are in $\mathbb{Z}_4$ because the multiple edges correspond to a spanning tree in G_2. But, this is impossible by the case $k = 2$ and $\Gamma = \mathbb{Z}_4$.

Consider now the case where $k \geq 2$, and Γ is isomorphic to $\mathbb{Z}_2^k$. We let $\varphi : E(G) \to \Gamma$, and we shall describe a Γ-coloring c which is allowed by φ. By modifying φ, if necessary, we may assume that, for some three edges $e_1 \in v_4v_1, e_2 \in v_4v_2, e_3 \in v_4v_3$, we have $\varphi(e_1) = \varphi(e_2) = \varphi(e_3) = 0$. We first color v_1, v_2, v_3, v_4, and then we extend that coloring to each of the inserted triangles using Lemma 5.18.

We give v_4 the color 0. After the coloring of v_4 each vertex v_i, $i = 1, 2, 3$, has a set L_{v_i} of 3 available colors. We form the matching graph M_{G_k} by taking the disjoint union of $L_{v_1}, L_{v_2}, L_{v_3}$ as vertices and adding an edge between α in L_{v_i} and β in L_{v_j} $(i \neq j)$ if $\tau_{v_i}(\alpha, v_j) = \beta$ (that is, α on v_i prevents β on v_j). By Lemma 5.18 we conclude that there are at least two possible ways of giving v_1 a color in L_{v_1} such that, regardless of how we give v_2 a color in L_{v_2} (not forbidden by the color of v_1), we can extend the coloring to the inserted triangle of $v_4v_1v_2v_4$. A similar argument holds with v_3 instead of v_2. Combining these two observations we conclude that we can give v_1 a color α_1 in L_{v_1} such that, regardless of how we color v_2, v_3 (in a way allowed by φ), we can extend the coloring to the inserted triangle of $v_4v_1v_2v_4$ and also the inserted triangle of $v_4v_1v_3v_4$. We now try to extend the coloring of v_1 with color α_1 to v_2, v_3 such that we can also extend to the inserted triangle of $v_4v_2v_3v_4$ and $v_1v_2v_3v_1$, respectively. This is possible unless the two sets $L_{v_2} \setminus \{\tau_{v_1}(\alpha_1, v_2)\}$ and $L_{v_3} \setminus \{\tau_{v_1}(\alpha_1, v_3)\}$ each has precisely two colors, and the graph M_{G_k} has two edges joining these sets. So, we may assume that L_{v_2} contains $\tau_{v_1}(\alpha_1, v_2)$, and $L(v_3)$ contains $\tau_{v_1}(\alpha_1, v_3)$. Also, M_{G_k} has at least two edges between L_{v_2} and L_{v_3} which are not incident with any of $\tau_{v_1}(\alpha_1, v_2) \in L(v_2), \tau_{v_1}(\alpha_1, v_3) \in L(v_3)$. If there is a third edge between L_{v_2} and L_{v_3}, that edge must join between $\tau_{v_1}(\alpha_1, v_2)$ and $\tau_{v_1}(\alpha_1, v_3)$. That edge is present in M_{G_k} if and only if $\varphi(v_1v_2) + \varphi(v_2v_3) + \varphi(v_3v_1) = 0$ by Proposition 4.8. We can repeat this argument for each of v_2, v_3 instead of v_1. It follows that, if $\varphi(v_1v_2) + \varphi(v_2v_3) + \varphi(v_3v_1) = 0$, then M_{G_k} consists of three pairwise disjoint triangles.

If $\varphi(v_1v_2) + \varphi(v_2v_3) + \varphi(v_3v_1) \neq 0$, then we add an edge between $\tau_{v_1}(\alpha_1, v_2)$ and $\tau_{v_1}(\alpha_1, v_3)$ and call this a *red edge*. In this case the two edges in M_{G_k} joining L_{v_2} and L_{v_3} together with the red edge form a matching. We add two other similar red edges and let M'_{G_k} denote M_{G_k} with the three red edges added. If M_{G_k} consists of three triangles, we put $M'_{G_k} = M_{G_k}$. In either case M'_{G_k} consists of three pairwise disjoint triangles.

If $M'_{G_k} \neq M_{G_k}$, we define $\alpha_2 \in L_{v_2}$, respectively $\alpha_3 \in L_{v_3}$ in the same way as we defined $\alpha_1 \in L_{v_1}$. If we give v_i the color α_i, $i = 1, 2, 3$, then that coloring can be extended to every inserted triangle of a triangle containing v_4. We may therefore assume that it cannot be extended to the inserted triangle of $v_1v_2v_3v_1$ (in the case where $M'_{G_k} \neq M_{G_k}$.)

Consider the colorings of v_1, v_2, v_3 obtained by choosing one color from each triangle of M'_{G_k}. There are 6 such colorings. Suppose now for contradiction that none of them extend to all four inserted triangles. For each triangle of G_2 containing v_4, at most one of those 6 colorings cannot be extended to its inserted triangle. Hence at least 3 of those 6 colorings cannot be extended to the inserted triangle of $v_1v_2v_3v_1$. Since any two of those 3 colorings differ by a constant, by Lemma 5.18, it follows that precisely 3 colorings of v_1, v_2, v_3 obtained by choosing one color from each triangle of M'_{G_k} cannot be extended to the inserted triangle of $v_1v_2v_3v_1$. We have earlier observed that, if $M'_{G_k} \neq M_{G_k}$, then the coloring of v_i by α_i, $i = 1, 2, 3$, is one of them. This statement also holds if $M'_{G_k} = M_{G_k}$, and we define α_2, α_3 appropriately. We now consider the two other colorings that cannot be extended to the inserted triangle of $v_1v_2v_3v_1$. One must be the coloring of v_1, v_2, v_3 with colors $\tau_{v_2}(\alpha_2, v_1), \tau_{v_3}(\alpha_3, v_2), \tau_{v_1}(\alpha_1, v_3)$, respectively. The other must be the coloring of v_1, v_2, v_3 with colors, $\tau_{v_3}(\alpha_3, v_1), \tau_{v_1}(\alpha_1, v_2), \tau_{v_2}(\alpha_2, v_3)$, respectively. Now any two of the three color triples $\mathbf{t_1} = (\alpha_1, \alpha_2, \alpha_3), \mathbf{t_2} = (\tau_{v_2}(\alpha_2, v_1), \tau_{v_3}(\alpha_3, v_2), \tau_{v_1}(\alpha_1, v_3)), \mathbf{t_3} = (\tau_{v_3}(\alpha_3, v_1), \tau_{v_1}(\alpha_1, v_2), \tau_{v_2}(\alpha_2, v_3))$ differ by a constant. Since $\mathbf{t_1}$ and $\mathbf{t_2}$ differ by a constant, the sum $\mathbf{t_1} + \mathbf{t_2}$ is a constant vector. The constant must equal $\alpha_i + \tau_{v_j}(\alpha_j, v_i)$ whenever $1 \leq i, j \leq 3$, $i \neq j$. Focussing on the first coordinates we conclude that $\mathbf{t_2}$ is obtained from $\mathbf{t_1}$ by adding $\alpha_1 + \tau_{v_2}(\alpha_2, v_1)$, and $\mathbf{t_3}$ is obtained from $\mathbf{t_1}$ by adding $\alpha_1 + \tau_{v_3}(\alpha_3, v_1)$. As $\tau_{v_2}(\alpha_2, v_1) = \tau_{v_3}(\alpha_3, v_1) = \alpha_1$ we conclude that $\mathbf{t_2} = \mathbf{t_3}$.

This contradiction completes the proof. □

Note that the matching graph M_{G_2} and the auxiliary graph M'_{G_2} may also be defined when we try to $\mathbb{Z}_4$-color G_2. All arguments from the proof of Theorem 5.19 carry over (with a modified proof of Lemma 5.18), except the final contradiction. This may serve as an explanation of why the two groups of order 4 behave differently. Also, the structure of M_{G_2}, M'_{G_2} can be used to find the φ that does not allow a $\mathbb{Z}_4$-coloring of G_2.

By dualizing Theorem 5.19 we obtain the following:

Corollary 5.20. *For every natural number k there exists a planar graph which is $\mathbb{Z}_2^k$-connected, but not Γ-connected for any other group Γ of order 2^k.*

CHAPTER 6

Conclusion

We conclude this thesis by providing a brief summary of the results and by pointing out some unresolved questions.

In Chapter 2 (based on [30]) we consider the weak group connectivity number and the connection between this and the group connectivity number. We prove the existence of graphs for which the group connectivity number is almost twice as big as the weak group connectivity number. We also bound the group connectivity number by a function on the weak group connectivity number. It is natural to ask whether the difference between the two numbers always is at most a factor 2.

In Chapter 2 we also consider group connectivity and group coloring for small graphs. We prove that $\mathbb{Z}_3$-connectivity implies Γ-connectivity for all groups of order $|\Gamma| \geq 4$, and we ask whether the same is true for group colorings. We have proved that $\mathbb{Z}_3$-colorability implies Γ-colorability for each Γ of order at least 5. But order 4 is open here. (As proven in Chapter 5, it is not true that $\mathbb{Z}_4$-colorability and $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorability are equivalent, not even for simple planar graphs - but it is not known whether there are $\mathbb{Z}_3$- and $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable graphs, or $\mathbb{Z}_3$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, but not $\mathbb{Z}_4$-colorable graphs. None of the examples in Chapter 5 are $\mathbb{Z}_3$-colorable.)

The following questions are of similar interest: Does $\mathbb{Z}_4$-connectivity and/or $\mathbb{Z}_2 \times \mathbb{Z}_2$-connectivity imply $\mathbb{Z}_5$-connectivity? And does $\mathbb{Z}_4$-colorability and/or $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorability imply $\mathbb{Z}_5$-colorability?

In Chapter 3 (based on [6]) we prove that there are exponentially many group flows in 3-edge-connected graphs for groups of order at least 8, and that there are many $\mathbb{Z}_6$-flows. It is natural to ask whether there are also exponentially many $\mathbb{Z}_6$- and $\mathbb{Z}_7$-flows in 3-edge-connected graphs. The result in Chapter 4 on $\mathbb{Z}_5$-colorings and $\mathbb{Z}_5$-flows in planar graphs indicates that there should indeed also be many $\mathbb{Z}_6$- and $\mathbb{Z}_7$-flows.

In Chapter 4 (based on [28]) it is proven that all simple planar graphs have exponentially many $\mathbb{Z}_5$-colorings. This work is based on [40] in which it is proven that planar graphs have exponentially many 5-list-colorings. As pointed out in [30] and [35] the proof that all planar graphs of girth at least 5 are 3-list-colorable in [37] is easily adapted to prove that all planar graphs of girth at least 5 are $\mathbb{Z}_3$-colorable. As also pointed out in [30], the work in [41] (in which it is proven that planar graphs of girth at least 5 have exponentially many 3-list-colorings) extends, almost word for word, to the more general fact that planar graphs of girth at least 5 have exponentially many $\mathbb{Z}_3$-colorings. It even extends to DP-colorings (introduced in [9] and defined in Chapter 3) with 3 colors. But we do not know whether all simple planar graphs have exponentially many DP-colorings with 5 colors, since the proofs in Chapter 4 do not work for DP-colorings in general.

As mentioned in Chapter 2 it is conjectured in [20] that the list chromatic number is always smaller than or equal to the group chromatic number. The proofs in [28] indicate that group coloring is indeed harder that list coloring since the family of graphs which are not 3-extendable is a subfamily of the family of graphs which are not $(\mathbb{Z}_5, 3)$-extendable.

In Chapter 5 we prove that there are simple planar graphs which are $\mathbb{Z}_4$-colorable, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable and that there are simple planar graphs which are $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorable, but not $\mathbb{Z}_4$-colorable. We even extend this result to prove that there are graphs which are Γ-colorable for $|\Gamma| = 2^k$ if and only if $\Gamma = \mathbb{Z}_2^k$.

The more general question is when a graph is Γ-colorable (or Γ-connected), but not Γ'-colorable (or Γ'-connected) for Abelian groups Γ, Γ' where $|\Gamma| \leq |\Gamma'|$. We would like to see more examples of graphs where this is the case, particular examples in which computers are not used for verification. This might illuminate what mechanisms are at work.

The $\mathbb{Z}_5$-Connectivity Conjecture 1.42 and the $\mathbb{Z}_3$-Connectivity Conjecture 1.43 are the main open questions in the area of group connectivity. These are particularly interesting because, if true, they imply Tutte's 5-Flow Conjecture and 3-Flow Conjecture, respectively. The results that 3-edge-connected graphs even have many $\mathbb{Z}_6$-flows and that planar graphs have exponentially many $\mathbb{Z}_5$-flows seem to indicate that the $\mathbb{Z}_5$-Connectivity Conjecture is true.

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