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Rigid-plastic optimization of solid reinforced concrete structures

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Abstract

Design of solid reinforced concrete structures, e.g., piers and pile caps, is primarily performed using simple hand calculation methods such as the strut-and-tie method. This may lead to designs which are conservative regarding the load-bearing capacity and material consumption, and in turn may result in more expensive designs with an unnecessarily high environmental impact. Finite Element Limit Analysis (FELA), that is, finite element analysis, which employs a rigid-plastic material model, has previously been used for capacity calculations by load optimization of trusses, beams, membranes, plates, shells, and solids. The method utilizes the material of the structure efficiently and is able to show capacities greater than the results of simple plastic hand calculations. FELA has also previously been used in material layout optimization of reinforced concrete membranes, plates, and shells. This paper presents FELA for material optimization of solid reinforced concrete structures. The formulation uses the Modified Mohr-Coulomb material model for concrete, and the reinforcement is modeled using the so-called smeared approach. The optimization procedure seeks to carry the load by varying the amount of smeared reinforcement in each of the three predefined orthogonal directions. The smeared reinforcement variables can either be optimized freely or in distinct zones, for instance, in a zone for bending reinforcement. The material layout is also indirectly optimized by allowing the concrete compressive strength to be a variable. The optimization procedure is applied to the case of a typical four-pile cap.

Keywords: Finite Element Limit Analysis, FELA, material layout optimization, solid reinforced concrete, pile cap.

1. Introduction

The design of efficient structures demands professional skill. Efficient designs can be made using previous experience, engineering sense, and rule of thumb. However, with complicated geometry and loading conditions, the experience is not always enough to find optimal designs. For these situations, the help of numerical tools is often sought.

One way to search for optimal designs of structures is using topology optimization. Topology optimization seeks to find the optimal distribution of an isotropic linearly elastic material under a given load by minimizing compliance (Bendsøe & Kikuchi, 1988). This method produces interesting results and valuable information about load paths under service load conditions. However, the method is not suitable for reinforced concrete structures since the inclusion of reinforcement and the formation of cracks makes the material anisotropic. Furthermore, in the ultimate limit state, design of concrete structures must be performed using plasticity methods to enable an optimal utilization of the materials. This is especially true for solid reinforced concrete structures such as piers and pile caps.

Finite Element Limit Analysis (FELA) has previously been used for material layout optimization of reinforced concrete structures modeled as membranes (Poulsen & Damkilde, 2000), plates, and shells.
2. Finite Element Limit Analysis for material layout optimization

The Finite Element Limit Analysis (FELA) is a combination of domain discretization of the Finite Element Method (FEM), with limit analysis based on the extremum principles (Drucker, Prager, & Greenberg, 1952; Gvozdev, 1960). The method was first proposed by Anderheggen & Knöpfel (1972) and since it has seen a great deal of development. The FELA method applied in this paper is based on the lower bound theorem. The goal of the method is to minimize material usage, idealized as a cost, under the constraints of stress equilibrium of the domain and the materials' yield conditions. It can be set up as an optimization problem in the following way:

\[
\begin{align*}
\text{min. } & \sum_{i=1}^{\text{nel}} V_i (c_i (\rho_{s,x,i} + \rho_{s,y,i} + \rho_{s,z,i}) + c_i \rho_{c,i}) & \text{Cost/objective function} \\
\text{s. t. } & H \beta_j = R_j, \ j = 1, ..., \text{nlc} & \text{Stress equilibrium} \\
& (\sigma_{c,i,j}, \sigma_{s,i,j}) \in f_{rc}, \ i = 1, ..., \text{nel}, \ j = 1, ..., \text{nlc} & \text{Yield conditions of the materials} \\
& \rho_{s,x,i} \in [0, u_{s,x,i}], \ \rho_{s,y,i} \in [0, u_{s,y,i}], \ \rho_{s,z,i} \in [0, u_{s,z,i}], \ \rho_{c,i} \in [0,1], \ i = 1, ..., \text{nel} & \text{Limits on the densities/ratios}
\end{align*}
\]

where \( \text{nlc} \) is the number of load cases, and \( \text{nel} \) is the number of elements. Since the stress equilibrium and yield condition of the materials are ensured for each load case, separately, the problem's size is roughly proportional to the number of load cases. Consequently, a limited number of load cases can be considered simultaneously.

The optimization problem is said to be convex when certain criteria are fulfilled. Convex optimization problems can be solved efficiently by software exploiting the problem's structure. The above problem is indeed convex, since the objective function is affine, the equality constraints are affine, and the inequality constraints are convex.

Each part of the above formulation will be described in detail in the following sections.

2.1. Stress equilibrium

Stress equilibrium is ensured by (2) for each separate load case. The equilibrium between the external loading \( \mathbf{R} \) and the stresses \( \mathbf{\sigma} \), for load case \( j \), is ensured via the equilibrium matrix \( \mathbf{H} \). The equilibrium matrix describes the inter-element equilibrium of the domain and consists of contributions from each element via the element equilibrium matrix \( \mathbf{h} \). The element utilized in this paper is the so-called Normal Traction element, which can be seen in Figure 1 (Andersen, Poulsen, & Olesen, 2021). The Normal Traction element is a constant stress tetrahedron. The element provides normal traction equilibrium in the traction nodes and shear traction equilibrium in the corner nodes. The element contains one stress point, which is given by:

\[
\mathbf{\sigma}^T = [\sigma_{xx} \ \sigma_{yy} \ \sigma_{zz} \ \sigma_{xy} \ \sigma_{xz} \ \sigma_{yz}]
\]

Equivalent to a full triaxial stress state tensor. The equilibrium matrix only depends on the geometric layout of the domain discretization, and as such, it is identical for all load cases. However, since the load
vector, \( \mathbf{R}_j \), differs for each load case, the domain stresses differ, too. The element stresses for a load case are collected in the vector \( \mathbf{\beta}_j \):

\[
\mathbf{\beta}_j = \begin{bmatrix} \sigma_{j,1} \\
\vdots \\
\sigma_{j,n_{\text{el}}} \end{bmatrix}
\]  

(6)

Where \( \sigma_{j,i} \) is the stress state for the \( i \)'th element of the \( j \)'th load case. Each row of the load vector \( \mathbf{R}_j \) corresponds to the equilibrium of the equations for each of the system force nodes and traction nodes.

2.2. Reinforced concrete stress point with material optimization

In the following, the reinforced concrete material model for material optimization is presented, which corresponds to the yield conditions of the materials (3) of the optimization problem. The model uses a separation of stresses into concrete stresses and smeared reinforcement stresses. For the concrete stresses, the Modified Mohr-Coulomb yield criterion is utilized, and for the smeared reinforcement, a simple linear uniaxial yield criterion is adopted.

2.2.1. Separation of concrete and reinforcement stresses

The stresses are represented in Cauchy stress tensor form and the relation between the total stress tensor \( \sigma_\text{\(\square\)} \), concrete stress tensor \( \sigma_{\text{\(\square\)},c} \), and smeared reinforcement stress tensor \( \sigma_{\text{\(\square\)},s} \) is given by:

\[ \sigma_\text{\(\square\)} = \sigma_{\text{\(\square\)},c} + \sigma_{\text{\(\square\)},s} \]  

(7)

where

\[
\sigma_\text{\(\square\)} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ 
\sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ 
\sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}, \quad \sigma_{\text{\(\square\)},c} = \begin{bmatrix} \sigma_{c,xx} & \sigma_{c,xy} & \sigma_{c,xz} \\ 
\sigma_{c,xy} & \sigma_{c,yy} & \sigma_{c,yz} \\ 
\sigma_{c,xz} & \sigma_{c,yz} & \sigma_{c,zz} \end{bmatrix}, \quad \sigma_{\text{\(\square\)},s} = \begin{bmatrix} \sigma_{s,xx} & 0 & 0 \\ 
0 & \sigma_{s,yy} & 0 \\ 
0 & 0 & \sigma_{s,zz} \end{bmatrix}
\]

(8)

As can be seen, the smeared reinforcement is assumed to carry normal forces, only. This way of separating the reinforced concrete stresses into concrete and smeared reinforcement stresses is similar to the analytical work of Andreasen & Nielsen (1985) and in a FELA framework by Larsen (2010).

2.2.2. Concrete as a modified Mohr-Coulomb material

The modified Mohr-Coulomb yield criterion has a sliding and a separation yield criterion given by the following formula:

\[ \sigma_1 \leq \rho_c f_{c,\text{eff}} \]  

(9)

\[ k\sigma_1 - \sigma_3 \leq \rho_c f_{c,\text{eff}} \]  

(10)

where \( \sigma_1 \) and \( \sigma_3 \) are the largest and smallest principal stress, respectively. Here \( f_{c,\text{eff}} \) is the effective concrete tensile strength, \( f_{c,\text{eff}} \) is the effective concrete compressive strength, \( k \) is the frictional parameter, usually taken as 4, and \( \rho_c \) is the concrete density factor for the material point. The concrete density factor is one of the optimization parameters of the objective function and has the additional requirement that it should be between 0 and 1. The parameter scales the strength, that is, in areas with a low value of the concrete density factor, a lower grade concrete can be used, or maybe the concrete can be completely removed.
The Modified Mohr-Coulomb yield criteria (9)-(10) are defined in principal stresses. In a convex optimization framework, it can be modeled as two linear matrix inequalities (LMI)’s operating on the concrete stress tensor and two additional linear inequalities (Larsen, 2010):

\[
\begin{align*}
\sigma_{c,c} + (k\alpha_1)\mathbf{I}_3 & \geq 0 \quad (11) \\
\sigma_{c,c} + \alpha_2\mathbf{I}_3 & \geq 0 \quad (12) \\
\alpha_2 - & \rho_c f_{c,\text{eff}} \leq 0 \quad (13) \\
k(\alpha_1 + \alpha_2) - & \rho_c f_{c,\text{eff}} \leq 0 \quad (14)
\end{align*}
\]

where \( \mathbf{I}_3 \) is the identity matrix of order 3, \( \alpha_1 \) and \( \alpha_2 \) are auxiliary variables, and \( \mathbf{0}_3 \) is the zero tensor of order 3. Equations (11) and (12) are LMI’s, and the symbol \( \geq \) denotes a generalized inequality. In this case, the symbol means that the left-hand side of the equations should be positive semidefinite.

2.2.3. Smeared reinforcement as linear constraints

The smeared reinforcement stresses are modeled with the following uni-axial relations:

\[
\begin{align*}
-f_s c_\rho s_x \rho_s x & \leq \sigma s_{xx} \leq f_s c_\rho s_x \\
-f_s c_\rho s_y \rho_s y & \leq \sigma s_{yy} \leq f_s c_\rho s_y \\
-f_s c_\rho s_z \rho_s z & \leq \sigma s_{zz} \leq f_s c_\rho s_z
\end{align*}
\]

where \( f_s c_\rho \) is the reinforcement compressive strength, \( f_s \) is the reinforcement tensile strength, and \( \rho_s x, \rho_s y, \text{ and } \rho_s z \) are the reinforcement ratios in the three cardinal directions, which are also optimization variables. The reinforcement ratios should range from 0 to set maximum densities in the three directions, \( u_s x, u_s y, \text{ and } u_s z \). These limits are accounted for in the limits on the densities/ratios (4) of the optimization problem.

2.3. Objective function

The objective function (1) is a measure of the total cost of a given material layout. For each element, a summation of the reinforcement ratios multiplied by the unit cost of reinforcement, \( c_r \), and the concrete density multiplied by the concrete unit cost, \( c_c \), all weighted with the element volume, \( V \), is performed. The objective function is linear which it must be for the formulation to remain a convex optimization problem. The linearity of the problem enables the optimization procedure to find solutions with any density in the range of the limits. This solution is perfectly reasonable for the reinforcement ratios since we can include the necessary amount of reinforcement rods to obtain the given ratio. However, intermediate values of the concrete densities do not necessarily make much sense. One could argue that intermediate values correspond to a lower grade of concrete, but the casting of such structures could prove troublesome in practice. However, the problem is quite limited since the resulting designs turn out to be mostly black or white, that is, with a lot of concrete densities being either 0/white or 1/black.

The two variables \( c_r \) and \( c_c \) can signify different kinds of costs. A logical choice would be the monetary cost, but this can be hard to define, especially for the reinforcement, where a large part of the cost is in the installation. A price per volume is therefore hard to define. An interesting prospect could be to use each of the materials’ CO₂-footprint since this value would be quite easily connected to the volume. Another issue to consider is the ratio between the reinforcement cost and the concrete cost, \( c_r/c_c \). If this value is too low, the optimization algorithm will push for a design where compression is carried by confinement, i.e., by adding reinforcement orthogonal to the loading direction, since this would be cheaper. A value of \( c_r/c_c \) greater than \( 4f/r_f \) is recommended. For this value, even when compression in two perpendicular directions are present, it will be cheaper to carry compression by manipulating \( \rho_c \), rather than by confinement.

3. Example four-pile cap

An example of a four-pile cap is used to test the developed material layout optimization framework. The framework is programmed in Matlab and uses the commercial convex solver Mosek 9.2. The geometric model and meshing are created via GMSH, and the results are visualized using Paraview.
3.1. Model and mesh

A sketch of the pile cap model can be seen together with the geometric parameters in Figure 2. The model is separated into three parts: the piles, the pile cap, and the pier. The part which is sought to be material optimized is the pile cap itself. The pier and piles are only included to properly model the boundary conditions. The material properties and optimization parameters of the parts can be seen in Table 1. The piles are left out of the material optimization step by setting the cost of the material in these parts to 0. A small but negligible tensile strength of the concrete is included to improve the numerical performance. The cost of the materials has the unit \([1/m^3]\) resulting in costs of the designs which are unitless.

The meshing of the model is performed using GMSH (Geuzaine & Remacle, 2009). The mesh resolution is controlled by means of characteristic lengths, \(l_c\). The characteristic length is a measure of the side length of the tetrahedral elements, which the program aims at producing. The characteristic length can be varied throughout the model, however, in the present case this is not utilized. Figure 3 shows the geometric model in GMSH with the geometric parameters from Figure 2. An example of a mesh with \(l_c=400\) mm can be seen in Figure 4, the mesh contains 37827 elements and 7851 nodes.

Table 1 Material properties and optimization parameters of the model. *The reinforcement ratios for the piles are only included in the outside layer.

<table>
<thead>
<tr>
<th></th>
<th>(f_{c,eff})</th>
<th>(f_{t,eff})</th>
<th>(c_c)</th>
<th>(f_c)</th>
<th>(f_t)</th>
<th>(u_{s,x})</th>
<th>(u_{s,y})</th>
<th>(u_{s,z})</th>
<th>(c_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pile cap and pier</td>
<td>25</td>
<td>0.001</td>
<td>1</td>
<td>0</td>
<td>500</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>100</td>
</tr>
<tr>
<td>Piles*</td>
<td>25</td>
<td>0.001</td>
<td>0</td>
<td>0</td>
<td>500</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 2 Geometry of the four-pile cap model.

Figure 3 Geometric model in GMSH.

Figure 4 Mesh with \(l_c=400\) mm.
3.1.1. Load application and supports

The pile cap model is supported vertically at the bottom of the piles and in three additional nodes to hinder rigid body motion. The vertical support in the piles makes it possible to transfer moment through the piles. The load is applied on top of the short pier. The top of the pier is separated into four different zones. By applying a varying intensity of normal force in the four sections, it is possible to apply any combination of normal force $N$, and moment about each of the axis, $M_x$ and $M_y$. This is visualized in Figure 5.

Five different load cases are considered: pure normal compression and normal compression with either positive or negative moment about the two axis. The load cases can be seen in Table 2. The load cases are considered individually and combined. Due to the double symmetric model, load case 2-3 and 4-5, respectively, will produce equal but mirrored results. However, they all need to be included in the combined optimization to get a doubly symmetric result.

![Figure 5 Load application at the top of the short pier.](image)

<table>
<thead>
<tr>
<th>Load Case</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$ [MN]</td>
<td>-200</td>
<td>-150</td>
<td>-150</td>
<td>-150</td>
<td>-150</td>
</tr>
<tr>
<td>$M_x$ [MNm]</td>
<td>0</td>
<td>-150</td>
<td>150</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$M_y$ [MNm]</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-150</td>
<td>150</td>
</tr>
</tbody>
</table>

3.1.2. Convergence analysis

To verify that a sufficiently fine mesh resolution is used, a convergence analysis is performed on a model with load case 1. Characteristic lengths of $l_c = [800\text{mm}, 600\text{mm}, 500\text{mm}, 400\text{mm}, 300\text{mm}, 250\text{mm}, 200\text{mm}]$, respectively, is used in the calculations. A plot of the cost as a function of the number of elements can be seen in Figure 6. The difference between the coarsest mesh and the finest mesh is only 5.6%, and for the mesh with $l_c=400\text{mm}$ the difference to the finest mesh is less than 1.0%. For the rest of this paper, a mesh of $l_c=400\text{mm}$ is used. This resolution is also adequate to provide detailed plots of the material layout.

![Figure 6 Convergence plot for load case 1.](image)
3.2. Results

In the following, the results of the optimization problems are presented. The optimization procedure is performed with the five different load cases separately and with all load cases combined into one problem. First, the costs of the problems are compared and discussed, after which the material layouts are studied. Finally, the distributions of the densities are analyzed.

3.2.1. Comparison of costs of designs

The cost, that is, the result of the objective function (1), for each of the load situations can be seen in Table 3. The most expensive load case is number 1. This load case has a large compressive normal force but no moment. The second most expensive load cases are 2 and 3, which have normal compressive force and moment about the weak axis of the pier. The least expensive load cases are 4 and 5, which have the same normal force as load cases 2 and 3, but this time with moment about the strong axis of the pier. As expected, the load cases 2-3 and 4-5, respectively, have the same cost due to the load cases being symmetric, and also as expected, the load cases with moment about the weak axis are more expensive. The combined load situation with all five load cases is only around 5% more expensive than load case 1 by itself. In the next section, the material layouts are studied in more detail.

<table>
<thead>
<tr>
<th>Load case(s)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>1,2,3,4,5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost [-]</td>
<td>224.11</td>
<td>181.79</td>
<td>181.89</td>
<td>173.68</td>
<td>173.06</td>
<td>236.29</td>
</tr>
</tbody>
</table>

3.2.2. Comparison of material layout

The material layout represented by the concrete density and the reinforcement ratios can be visualized as a volume plot. This visualization is made by applying a colormap and an opacity map to the chosen density or ratio variable. The opacity map will be transparent for densities and ratios of 0 and opaque for ratios at the other extreme.

Figure 7 and Figure 8 show the material layout for two of the single load case situations, namely load case 1 and 2, while Figure 9 shows the material layout for the combined load situation. In Figure 7 top left, the concrete layout can be seen. The compressive force from the pier gives rise to four legs or struts originating at the pier and ending at each of the four piles. The three other plots show the reinforcement ratios in the three cardinal directions. For reinforcement in the x- and y-directions, two pairwise bands of reinforcement are placed over the piles, or rather they are placed at the very edge of the piles. Figure
7 bottom right shows the vertical reinforcement. Besides the reinforcement in the piles that was left out of the optimization, almost no vertical reinforcement is present. The rather small amount of reinforcement that is present is placed where the struts meet the piles. By placing vertical reinforcement there, and with reinforcement already present in the x- and y-directions, the concrete is confined in these zones. By confining the concrete, the load from the pier can be transferred to a smaller part of the pile, and this limits the inclination of the compressive strut, saving reinforcement in the x- and y-directions.

Figure 8 shows the material layout of load case 4. This load case combines a compressive normal force with a negative moment about the strong axis of the pier. This results in a material layout that still has four struts. However, now the struts are not all similar since the resultant of the compressive force on the pier is no longer concentric. Likewise, the reinforcement layout is also different from load case 1 since more reinforcement is required between two of the piles due to the excentric load.

![Figure 8 Material layout for load case 4.](image1)

![Figure 9 Material layout for the combined load situation.](image2)
Figure 9 shows the resulting material layout for the combined load situations with all five load cases. The layout is like the material layout of load case 1. However, this was also the single load case with the highest cost. It differs mainly from load case 1 by having a zone of intermediate concrete densities under the pier. A low degree of vertical reinforcement also seems to be present in this zone. This zone is necessary to transfer the compression from the load cases, which include moments.

3.2.3. Comparison of density distributions

When performing material layout optimization, a so-called black or white layout is preferable. That is, either no-material/white, which refers to a density of 0, or material/black, which refers to a density of 1. Histograms of the distribution of the concrete density parameter $\rho_c$ for load cases 1, 4, and the combined load situation can be seen in Figure 10. For all three situations, the largest bin corresponds to a density close to zero, and the next largest corresponds to a density close to 1, which is desirable when searching for a black or white material layout. However, some intermediate densities do exist in the results. The intermediate densities are slightly more pronounced in the combined load situations, which was also the conclusion from the discussion of Figure 7-Figure 9.

![Histograms of the distribution of the concrete density parameter $\rho_c$.](image)

Figure 10 Distribution of the concrete density parameter $\rho_c$.

4. Conclusion

A Finite Element Limit Analysis framework for material layout optimization of solid reinforced concrete structures has been presented. The framework uses the modified Mohr-Coulomb yield criterion with an included density variable to model the concrete and linear constraints with reinforcement ratios as variables to model the reinforcement.

The framework was used on a four-pile cap model, subjected to different load situations. The load situations consisted of a combination of normal force and moment about the two axes of the plane. Five load cases were considered individually and combined into one optimization problem. By considering all the load cases simultaneously, the cost of the design only increased by about 5% compared to the most expensive single load case. The obtained concrete distribution was mostly black or white, supplemented with small parts of intermediate densities.

Acknowledgements

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