



## Epistemic Logic: Completeness of Modal Logics

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# Epistemic Logic: Completeness of Modal Logics

Asta Halkjær From

November 30, 2021

## Abstract

This work is a formalization of epistemic logic with countably many agents. It includes proofs of soundness and completeness for the axiom system K. The completeness proof is based on the textbook "Reasoning About Knowledge" by Fagin, Halpern, Moses and Vardi (MIT Press 1995) [2]. The extensions of system K (T, KB, K4, S4, S5) and their completeness proofs are based on the textbook "Modal Logic" by Blackburn, de Rijke and Venema (Cambridge University Press 2001) [1].

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**theory** *Epistemic-Logic* **imports** *HOL-Library.Countable* **begin**

## 1 Syntax

**type-synonym**  $id = string$

**datatype**  $\langle 'i\ fm \rangle$   
=  $FF (\perp)$   
|  $Pro\ id$   
|  $Dis\ \langle 'i\ fm \rangle\ \langle 'i\ fm \rangle$  (**infixr**  $\vee\ 30$ )  
|  $Con\ \langle 'i\ fm \rangle\ \langle 'i\ fm \rangle$  (**infixr**  $\wedge\ 35$ )  
|  $Imp\ \langle 'i\ fm \rangle\ \langle 'i\ fm \rangle$  (**infixr**  $\longrightarrow\ 25$ )  
|  $K\ 'i\ \langle 'i\ fm \rangle$

**abbreviation**  $TT (\top)$  **where**

$\langle TT \equiv \perp \longrightarrow \perp \rangle$

**abbreviation**  $Neg (\neg - [40]\ 40)$  **where**

$\langle Neg\ p \equiv p \longrightarrow \perp \rangle$

**abbreviation**  $\langle L\ i\ p \equiv \neg K\ i\ (\neg p) \rangle$

## 2 Semantics

**datatype**  $\langle 'i, 'w \rangle\ kripke =$   
 $Kripke (\mathcal{W}: \langle 'w\ set \rangle) (\pi: \langle 'w \Rightarrow id \Rightarrow bool \rangle) (\mathcal{K}: \langle 'i \Rightarrow 'w \Rightarrow 'w\ set \rangle)$

**primrec**  $semantics :: \langle \langle 'i, 'w \rangle\ kripke \Rightarrow 'w \Rightarrow 'i\ fm \Rightarrow bool \rangle$

$(-, - \models - [50, 50]\ 50)$  **where**  
 $\langle (M, w \models \perp) = False \rangle$   
|  $\langle (M, w \models Pro\ x) = \pi\ M\ w\ x \rangle$   
|  $\langle (M, w \models (p \vee q)) = ((M, w \models p) \vee (M, w \models q)) \rangle$   
|  $\langle (M, w \models (p \wedge q)) = ((M, w \models p) \wedge (M, w \models q)) \rangle$   
|  $\langle (M, w \models (p \longrightarrow q)) = ((M, w \models p) \longrightarrow (M, w \models q)) \rangle$   
|  $\langle (M, w \models K\ i\ p) = (\forall v \in \mathcal{W}\ M \cap \mathcal{K}\ M\ i\ w. M, v \models p) \rangle$

## 3 S5 Axioms

**definition**  $reflexive :: \langle \langle 'i, 'w \rangle\ kripke \Rightarrow bool \rangle$  **where**

$\langle reflexive\ M \equiv \forall i. \forall w \in \mathcal{W}\ M. w \in \mathcal{K}\ M\ i\ w \rangle$

**definition**  $symmetric :: \langle \langle 'i, 'w \rangle\ kripke \Rightarrow bool \rangle$  **where**

$\langle symmetric\ M \equiv \forall i. \forall v \in \mathcal{W}\ M. \forall w \in \mathcal{W}\ M. v \in \mathcal{K}\ M\ i\ w \longleftrightarrow w \in \mathcal{K}\ M\ i\ v \rangle$

**definition**  $transitive :: \langle \langle 'i, 'w \rangle\ kripke \Rightarrow bool \rangle$  **where**

$\langle transitive\ M \equiv \forall i. \forall u \in \mathcal{W}\ M. \forall v \in \mathcal{W}\ M. \forall w \in \mathcal{W}\ M. \dots \rangle$

$$w \in \mathcal{K} M i v \wedge u \in \mathcal{K} M i w \longrightarrow u \in \mathcal{K} M i v$$

**abbreviation** *equivalence* ::  $\langle ('i, 'w) \text{ kripke} \Rightarrow \text{bool} \rangle$  **where**  
 $\langle \text{equivalence } M \equiv \text{reflexive } M \wedge \text{symmetric } M \wedge \text{transitive } M \rangle$

**lemma** *Imp-intro* [*intro*]:  $\langle (M, w \models p \Longrightarrow M, w \models q) \Longrightarrow M, w \models (p \longrightarrow q) \rangle$   
 $\langle \text{proof} \rangle$

**theorem** *distribution*:  $\langle M, w \models (K i p \wedge K i (p \longrightarrow q)) \longrightarrow K i q \rangle$   
 $\langle \text{proof} \rangle$

**theorem** *generalization*:  
**assumes** *valid*:  $\langle \forall (M :: ('i, 'w) \text{ kripke}) w. M, w \models p \rangle$   
**shows**  $\langle (M :: ('i, 'w) \text{ kripke}), w \models K i p \rangle$   
 $\langle \text{proof} \rangle$

**theorem** *truth*:  
**assumes**  $\langle \text{reflexive } M \rangle \langle w \in \mathcal{W} M \rangle$   
**shows**  $\langle M, w \models (K i p \longrightarrow p) \rangle$   
 $\langle \text{proof} \rangle$

**theorem** *pos-introspection*:  
**assumes**  $\langle \text{transitive } M \rangle \langle w \in \mathcal{W} M \rangle$   
**shows**  $\langle M, w \models (K i p \longrightarrow K i (K i p)) \rangle$   
 $\langle \text{proof} \rangle$

**theorem** *neg-introspection*:  
**assumes**  $\langle \text{symmetric } M \rangle \langle \text{transitive } M \rangle \langle w \in \mathcal{W} M \rangle$   
**shows**  $\langle M, w \models (\neg K i p \longrightarrow K i (\neg K i p)) \rangle$   
 $\langle \text{proof} \rangle$

## 4 Normal Modal Logic

**primrec** *eval* ::  $\langle (\text{id} \Rightarrow \text{bool}) \Rightarrow ('i \text{ fm} \Rightarrow \text{bool}) \Rightarrow 'i \text{ fm} \Rightarrow \text{bool} \rangle$  **where**  
 $\langle \text{eval } - \text{ } \perp = \text{False} \rangle$   
 $\mid \langle \text{eval } g \text{ } (\text{Pro } x) = g \ x \rangle$   
 $\mid \langle \text{eval } g \ h \ (p \vee q) = (\text{eval } g \ h \ p \vee \text{eval } g \ h \ q) \rangle$   
 $\mid \langle \text{eval } g \ h \ (p \wedge q) = (\text{eval } g \ h \ p \wedge \text{eval } g \ h \ q) \rangle$   
 $\mid \langle \text{eval } g \ h \ (p \longrightarrow q) = (\text{eval } g \ h \ p \longrightarrow \text{eval } g \ h \ q) \rangle$   
 $\mid \langle \text{eval } - \ h \ (K i p) = h \ (K i p) \rangle$

**abbreviation**  $\langle \text{tautology } p \equiv \forall g \ h. \text{eval } g \ h \ p \rangle$

**inductive** *AK* ::  $\langle ('i \text{ fm} \Rightarrow \text{bool}) \Rightarrow 'i \text{ fm} \Rightarrow \text{bool} \rangle$   $(- \vdash - [50, 50] 50)$   
**for**  $A :: \langle 'i \text{ fm} \Rightarrow \text{bool} \rangle$  **where**  
 $A1: \langle \text{tautology } p \Longrightarrow A \vdash p \rangle$   
 $\mid A2: \langle A \vdash (K i p \wedge K i (p \longrightarrow q)) \longrightarrow K i q \rangle$   
 $\mid Ax: \langle A \ p \Longrightarrow A \vdash p \rangle$   
 $\mid R1: \langle A \vdash p \Longrightarrow A \vdash (p \longrightarrow q) \Longrightarrow A \vdash q \rangle$

| R2:  $\langle A \vdash p \implies A \vdash K i p \rangle$

## 5 Soundness

**lemma** *eval-semantics*:  $\langle eval (pi w) (\lambda q. Kripke W pi r, w \models q) p = (Kripke W pi r, w \models p) \rangle$   
 $\langle proof \rangle$

**lemma** *tautology*:  
**assumes**  $\langle tautology p \rangle$   
**shows**  $\langle M, w \models p \rangle$   
 $\langle proof \rangle$

**theorem** *soundness*:  
**fixes**  $M :: \langle 'i, 'w \rangle kripke$   
**assumes**  $\langle \bigwedge (M :: \langle 'i, 'w \rangle kripke) w p. A p \implies P M \implies w \in \mathcal{W} M \implies M, w \models p \rangle$   
**shows**  $\langle A \vdash p \implies P M \implies w \in \mathcal{W} M \implies M, w \models p \rangle$   
 $\langle proof \rangle$

## 6 Derived rules

**lemma** *K-A2'*:  $\langle A \vdash (K i (p \longrightarrow q) \longrightarrow K i p \longrightarrow K i q) \rangle$   
 $\langle proof \rangle$

**lemma** *K-map*:  
**assumes**  $\langle A \vdash (p \longrightarrow q) \rangle$   
**shows**  $\langle A \vdash (K i p \longrightarrow K i q) \rangle$   
 $\langle proof \rangle$

**lemma** *K-LK*:  $\langle A \vdash (L i (\neg p) \longrightarrow \neg K i p) \rangle$   
 $\langle proof \rangle$

**primrec** *imply* ::  $\langle 'i fm list \Rightarrow 'i fm \Rightarrow 'i fm \rangle$  **where**  
 $\langle imply [] q = q \rangle$   
 $\mid \langle imply (p \# ps) q = (p \longrightarrow imply ps q) \rangle$

**lemma** *K-imply-head*:  $\langle A \vdash imply (p \# ps) p \rangle$   
 $\langle proof \rangle$

**lemma** *K-imply-Cons*:  
**assumes**  $\langle A \vdash imply ps q \rangle$   
**shows**  $\langle A \vdash imply (p \# ps) q \rangle$   
 $\langle proof \rangle$

**lemma** *K-right-mp*:  
**assumes**  $\langle A \vdash imply ps p \rangle \langle A \vdash imply ps (p \longrightarrow q) \rangle$   
**shows**  $\langle A \vdash imply ps q \rangle$

$\langle proof \rangle$

**lemma** *tautology-imp-ysuperset*:

**assumes**  $\langle set\ ps \subseteq set\ qs \rangle$

**shows**  $\langle tautology\ (imply\ ps\ r \longrightarrow imply\ qs\ r) \rangle$

$\langle proof \rangle$

**lemma** *K-imp-weaken*:

**assumes**  $\langle A \vdash imply\ ps\ q \rangle$   $\langle set\ ps \subseteq set\ ps' \rangle$

**shows**  $\langle A \vdash imply\ ps'\ q \rangle$

$\langle proof \rangle$

**lemma** *imp-append*:  $\langle imply\ (ps\ @\ ps')\ q = imply\ ps\ (imply\ ps'\ q) \rangle$

$\langle proof \rangle$

**lemma** *K-ImpI*:

**assumes**  $\langle A \vdash imply\ (p\ \#\ G)\ q \rangle$

**shows**  $\langle A \vdash imply\ G\ (p \longrightarrow q) \rangle$

$\langle proof \rangle$

**lemma** *K-Boole*:

**assumes**  $\langle A \vdash imply\ ((\neg\ p)\ \#\ G)\ \perp \rangle$

**shows**  $\langle A \vdash imply\ G\ p \rangle$

$\langle proof \rangle$

**lemma** *K-DisE*:

**assumes**  $\langle A \vdash imply\ (p\ \#\ G)\ r \rangle$   $\langle A \vdash imply\ (q\ \#\ G)\ r \rangle$   $\langle A \vdash imply\ G\ (p \vee q) \rangle$

**shows**  $\langle A \vdash imply\ G\ r \rangle$

$\langle proof \rangle$

**lemma** *K-mp*:  $\langle A \vdash imply\ (p\ \#\ (p \longrightarrow q)\ \#\ G)\ q \rangle$

$\langle proof \rangle$

**lemma** *K-swap*:

**assumes**  $\langle A \vdash imply\ (p\ \#\ q\ \#\ G)\ r \rangle$

**shows**  $\langle A \vdash imply\ (q\ \#\ p\ \#\ G)\ r \rangle$

$\langle proof \rangle$

**lemma** *K-DisL*:

**assumes**  $\langle A \vdash imply\ (p\ \#\ ps)\ q \rangle$   $\langle A \vdash imply\ (p'\ \#\ ps)\ q \rangle$

**shows**  $\langle A \vdash imply\ ((p \vee p')\ \#\ ps)\ q \rangle$

$\langle proof \rangle$

**lemma** *K-distrib-K-imp*:

**assumes**  $\langle A \vdash K\ i\ (imply\ G\ q) \rangle$

**shows**  $\langle A \vdash imply\ (map\ (K\ i)\ G)\ (K\ i\ q) \rangle$

$\langle proof \rangle$

## 7 Completeness

### 7.1 Consistent sets

**definition** *consistent* ::  $\langle ('i\ fm \Rightarrow bool) \Rightarrow 'i\ fm\ set \Rightarrow bool \rangle$  **where**  
 $\langle consistent\ A\ S \equiv \nexists S'.\ set\ S' \subseteq S \wedge A \vdash imply\ S' \perp \rangle$

**lemma** *inconsistent-subset*:

**assumes**  $\langle consistent\ A\ V \rangle \langle \neg\ consistent\ A\ (\{p\} \cup V) \rangle$   
**obtains**  $V'$  **where**  $\langle set\ V' \subseteq V \rangle \langle A \vdash imply\ (p \# V') \perp \rangle$   
 $\langle proof \rangle$

**lemma** *consistent-deriv*:

**assumes**  $\langle consistent\ A\ V \rangle \langle A \vdash p \rangle$   
**shows**  $\langle consistent\ A\ (\{p\} \cup V) \rangle$   
 $\langle proof \rangle$

**lemma** *consistent-consequent*:

**assumes**  $\langle consistent\ A\ V \rangle \langle p \in V \rangle \langle A \vdash (p \longrightarrow q) \rangle$   
**shows**  $\langle consistent\ A\ (\{q\} \cup V) \rangle$   
 $\langle proof \rangle$

**lemma** *consistent-consequent'*:

**assumes**  $\langle consistent\ A\ V \rangle \langle p \in V \rangle \langle tautology\ (p \longrightarrow q) \rangle$   
**shows**  $\langle consistent\ A\ (\{q\} \cup V) \rangle$   
 $\langle proof \rangle$

**lemma** *consistent-disjuncts*:

**assumes**  $\langle consistent\ A\ V \rangle \langle (p \vee q) \in V \rangle$   
**shows**  $\langle consistent\ A\ (\{p\} \cup V) \vee consistent\ A\ (\{q\} \cup V) \rangle$   
 $\langle proof \rangle$

**lemma** *exists-finite-inconsistent*:

**assumes**  $\langle \neg\ consistent\ A\ (\{\neg\ p\} \cup V) \rangle$   
**obtains**  $W$  **where**  $\langle \{\neg\ p\} \cup W \subseteq \{\neg\ p\} \cup V \rangle \langle (\neg\ p) \notin W \rangle \langle finite\ W \rangle \langle \neg\ consistent\ A\ (\{\neg\ p\} \cup W) \rangle$   
 $\langle proof \rangle$

**lemma** *inconsistent-imply*:

**assumes**  $\langle \neg\ consistent\ A\ (\{\neg\ p\} \cup set\ G) \rangle$   
**shows**  $\langle A \vdash imply\ G\ p \rangle$   
 $\langle proof \rangle$

### 7.2 Maximal consistent sets

**definition** *maximal* ::  $\langle ('i\ fm \Rightarrow bool) \Rightarrow 'i\ fm\ set \Rightarrow bool \rangle$  **where**  
 $\langle maximal\ A\ S \equiv \forall p.\ p \notin S \longrightarrow \neg\ consistent\ A\ (\{p\} \cup S) \rangle$

**theorem** *deriv-in-maximal*:

**assumes**  $\langle consistent\ A\ V \rangle \langle maximal\ A\ V \rangle \langle A \vdash p \rangle$



**shows**  $\langle p \in V \rangle$   
 $\langle \text{proof} \rangle$

**theorem** *exactly-one-in-maximal*:  
**assumes**  $\langle \text{consistent } A \ V \rangle \langle \text{maximal } A \ V \rangle$   
**shows**  $\langle p \in V \longleftrightarrow (\neg p) \notin V \rangle$   
 $\langle \text{proof} \rangle$

**theorem** *consequent-in-maximal*:  
**assumes**  $\langle \text{consistent } A \ V \rangle \langle \text{maximal } A \ V \rangle \langle p \in V \rangle \langle (p \longrightarrow q) \in V \rangle$   
**shows**  $\langle q \in V \rangle$   
 $\langle \text{proof} \rangle$

**theorem** *ax-in-maximal*:  
**assumes**  $\langle \text{consistent } A \ V \rangle \langle \text{maximal } A \ V \rangle \langle A \ p \rangle$   
**shows**  $\langle p \in V \rangle$   
 $\langle \text{proof} \rangle$

**theorem** *mcs-properties*:  
**assumes**  $\langle \text{consistent } A \ V \rangle$  **and**  $\langle \text{maximal } A \ V \rangle$   
**shows**  $\langle A \vdash p \implies p \in V \rangle$   
**and**  $\langle p \in V \longleftrightarrow (\neg p) \notin V \rangle$   
**and**  $\langle p \in V \implies (p \longrightarrow q) \in V \implies q \in V \rangle$   
 $\langle \text{proof} \rangle$

### 7.3 Lindenbaum extension

**instantiation** *fm* :: (countable) countable **begin**  
**instance**  $\langle \text{proof} \rangle$   
**end**

**primrec** *extend* ::  $\langle ('i \text{ fm} \implies \text{bool}) \implies 'i \text{ fm set} \implies (\text{nat} \implies 'i \text{ fm}) \implies \text{nat} \implies 'i \text{ fm set} \rangle$   
**where**  
 $\langle \text{extend } A \ S \ f \ 0 = S \rangle \mid$   
 $\langle \text{extend } A \ S \ f \ (\text{Suc } n) =$   
 $\quad \langle \text{if consistent } A \ (\{f \ n\} \cup \text{extend } A \ S \ f \ n)$   
 $\quad \text{then } \{f \ n\} \cup \text{extend } A \ S \ f \ n$   
 $\quad \text{else } \text{extend } A \ S \ f \ n \rangle$

**definition** *Extend* ::  $\langle ('i \text{ fm} \implies \text{bool}) \implies 'i \text{ fm set} \implies (\text{nat} \implies 'i \text{ fm}) \implies 'i \text{ fm set} \rangle$   
**where**  
 $\langle \text{Extend } A \ S \ f \equiv \bigcup n. \text{extend } A \ S \ f \ n \rangle$

**lemma** *Extend-subset*:  $\langle S \subseteq \text{Extend } A \ S \ f \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *extend-bound*:  $\langle (\bigcup n \leq m. \text{extend } A \ S \ f \ n) = \text{extend } A \ S \ f \ m \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *consistent-extend*:  $\langle \text{consistent } A \ S \implies \text{consistent } A \ (\text{extend } A \ S \ f \ n) \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *UN-finite-bound*:  
**assumes**  $\langle \text{finite } A \rangle \langle A \subseteq (\bigcup n. f \ n) \rangle$   
**shows**  $\langle \exists m :: \text{nat}. A \subseteq (\bigcup n \leq m. f \ n) \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *consistent-Extend*:  
**assumes**  $\langle \text{consistent } A \ S \rangle$   
**shows**  $\langle \text{consistent } A \ (\text{Extend } A \ S \ f) \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *maximal-Extend*:  
**assumes**  $\langle \text{surj } f \rangle$   
**shows**  $\langle \text{maximal } A \ (\text{Extend } A \ S \ f) \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *maximal-extension*:  
**fixes**  $V :: \langle 'i :: \text{countable} \rangle \text{ fm set} \rangle$   
**assumes**  $\langle \text{consistent } A \ V \rangle$   
**obtains**  $W$  **where**  $\langle V \subseteq W \rangle \langle \text{consistent } A \ W \rangle \langle \text{maximal } A \ W \rangle$   
 $\langle \text{proof} \rangle$

## 7.4 Canonical model

**abbreviation**  $pi :: \langle 'i \text{ fm set} \Rightarrow id \Rightarrow \text{bool} \rangle$  **where**  
 $\langle pi \ V \ x \equiv \text{Pro } x \in V \rangle$

**abbreviation**  $known :: \langle 'i \text{ fm set} \Rightarrow 'i \Rightarrow 'i \text{ fm set} \rangle$  **where**  
 $\langle known \ V \ i \equiv \{p. K \ i \ p \in V\} \rangle$

**abbreviation**  $reach :: \langle ('i \text{ fm} \Rightarrow \text{bool}) \Rightarrow 'i \Rightarrow 'i \text{ fm set} \Rightarrow 'i \text{ fm set set} \rangle$  **where**  
 $\langle reach \ A \ i \ V \equiv \{W. known \ V \ i \subseteq W\} \rangle$

**abbreviation**  $mcss :: \langle ('i \text{ fm} \Rightarrow \text{bool}) \Rightarrow 'i \text{ fm set set} \rangle$  **where**  
 $\langle mcss \ A \equiv \{W. consistent \ A \ W \wedge maximal \ A \ W\} \rangle$

**lemma** *truth-lemma*:  
**fixes**  $A$  **and**  $p :: \langle ('i :: \text{countable}) \text{ fm} \rangle$   
**defines**  $\langle M \equiv \text{Kripke } (mcss \ A) \ pi \ (reach \ A) \rangle$   
**assumes**  $\langle \text{consistent } A \ V \rangle$  **and**  $\langle \text{maximal } A \ V \rangle$   
**shows**  $\langle (p \in V \longleftrightarrow M, V \models p) \wedge ((\neg p) \in V \longleftrightarrow M, V \models \neg p) \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *canonical-model*:  
**assumes**  $\langle \text{consistent } A \ S \rangle$  **and**  $\langle p \in S \rangle$   
**defines**  $\langle V \equiv \text{Extend } A \ S \ \text{from-nat} \rangle$  **and**  $\langle M \equiv \text{Kripke } (mcss \ A) \ pi \ (reach \ A) \rangle$   
**shows**  $\langle M, V \models p \rangle$  **and**  $\langle \text{consistent } A \ V \rangle$  **and**  $\langle \text{maximal } A \ V \rangle$

⟨proof⟩

## 7.5 Completeness

**lemma** *imply-completeness*:

**assumes** *valid*: ⟨ $\forall (M :: ('i :: countable, 'i \text{ fm set}) \text{ kripke}). \forall w \in \mathcal{W} M.$

⟨ $\forall q \in G. M, w \models q \longrightarrow M, w \models p$ ⟩

**shows** ⟨ $\exists \text{qs. set qs} \subseteq G \wedge (A \vdash \text{imply qs } p)$ ⟩

⟨proof⟩

**theorem** *completeness*:

**assumes** ⟨ $\forall (M :: ('i :: countable, 'i \text{ fm set}) \text{ kripke}). \forall w \in \mathcal{W} M. M, w \models p$ ⟩

**shows** ⟨ $A \vdash p$ ⟩

⟨proof⟩

## 8 System K

**abbreviation** *SystemK* :: ⟨*i fm*  $\Rightarrow$  bool⟩ ( $\vdash_K$  - [50] 50) **where**

⟨ $\vdash_K p \equiv (\lambda-. \text{False}) \vdash p$ ⟩

**lemma** *soundness<sub>K</sub>*: ⟨ $\vdash_K p \Longrightarrow w \in \mathcal{W} M \Longrightarrow M, w \models p$ ⟩

⟨proof⟩

**abbreviation** *valid<sub>K</sub>*  $p \equiv \forall (M :: (\text{nat}, \text{nat fm set}) \text{ kripke}). \forall w \in \mathcal{W} M. M, w \models p$

**theorem** *main<sub>K</sub>*: ⟨*valid<sub>K</sub>*  $p \longleftrightarrow \vdash_K p$ ⟩

⟨proof⟩

**corollary**

**assumes** ⟨*valid<sub>K</sub>*  $p$ ⟩ **and** ⟨ $w \in \mathcal{W} M$ ⟩

**shows** ⟨ $M, w \models p$ ⟩

⟨proof⟩

## 9 System T

Also known as System M

**inductive** *AxT* :: ⟨*i fm*  $\Rightarrow$  bool⟩ **where**

⟨*AxT* ( $K \ i \ p \longrightarrow p$ )⟩

**abbreviation** *SystemT* :: ⟨*i fm*  $\Rightarrow$  bool⟩ ( $\vdash_T$  - [50] 50) **where**

⟨ $\vdash_T p \equiv \text{AxT} \vdash p$ ⟩

**lemma** *soundness-AxT*: ⟨*AxT*  $p \Longrightarrow \text{reflexive } M \Longrightarrow w \in \mathcal{W} M \Longrightarrow M, w \models p$ ⟩

⟨proof⟩

**lemma** *soundness<sub>T</sub>*: ⟨ $\vdash_T p \Longrightarrow \text{reflexive } M \Longrightarrow w \in \mathcal{W} M \Longrightarrow M, w \models p$ ⟩

⟨proof⟩

**lemma** *AxT-reflexive*:

**assumes**  $\langle \forall p. AxT\ p \longrightarrow A\ p \rangle$  **and**  $\langle consistent\ A\ V \rangle$  **and**  $\langle maximal\ A\ V \rangle$   
**shows**  $\langle V \in reach\ A\ i\ V \rangle$

$\langle proof \rangle$

**lemma** *mcs<sub>T</sub>-reflexive*:

**assumes**  $\langle \forall p. AxT\ p \longrightarrow A\ p \rangle$   
**shows**  $\langle reflexive\ (Kripke\ (mcss\ A)\ pi\ (reach\ A)) \rangle$

$\langle proof \rangle$

**lemma** *imply-completeness-T*:

**assumes** *valid*:  $\langle \forall (M :: ('i :: countable, 'i\ fm\ set)\ kripke). \forall w \in \mathcal{W}\ M. reflexive\ M \longrightarrow (\forall q \in G. M, w \models q) \longrightarrow M, w \models p \rangle$

**shows**  $\langle \exists qs. set\ qs \subseteq G \wedge (AxT \vdash imply\ qs\ p) \rangle$

$\langle proof \rangle$

**lemma** *completeness<sub>T</sub>*:

**assumes**  $\langle \forall (M :: ('i :: countable, 'i\ fm\ set)\ kripke). \forall w \in \mathcal{W}\ M. reflexive\ M \longrightarrow M, w \models p \rangle$

**shows**  $\langle \vdash_T\ p \rangle$

$\langle proof \rangle$

**abbreviation** *valid<sub>T</sub>*  $p \equiv \forall (M :: (nat, nat\ fm\ set)\ kripke). \forall w \in \mathcal{W}\ M. reflexive\ M \longrightarrow M, w \models p$

**theorem** *main<sub>T</sub>*:  $\langle valid_T\ p \longleftrightarrow \vdash_T\ p \rangle$

$\langle proof \rangle$

**corollary**

**assumes**  $\langle reflexive\ M \rangle$   $\langle w \in \mathcal{W}\ M \rangle$

**shows**  $\langle valid_T\ p \longrightarrow M, w \models p \rangle$

$\langle proof \rangle$

## 10 System KB

**inductive** *AxB* ::  $\langle 'i\ fm \Rightarrow bool \rangle$  **where**

$\langle AxB\ (p \longrightarrow K\ i\ (L\ i\ p)) \rangle$

**abbreviation** *SystemKB* ::  $\langle 'i\ fm \Rightarrow bool \rangle$  ( $\vdash_{KB}$  - [50] 50) **where**

$\langle \vdash_{KB}\ p \equiv AxB \vdash p \rangle$

**lemma** *soundness-AxB*:  $\langle AxB\ p \Longrightarrow symmetric\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p \rangle$

$\langle proof \rangle$

**lemma** *soundness<sub>KB</sub>*:  $\langle \vdash_{KB}\ p \Longrightarrow symmetric\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p \rangle$

$\langle proof \rangle$

**lemma** *AxB-symmetric'*:

**assumes**  $\langle \forall p. AxB\ p \longrightarrow A\ p \rangle$   $\langle$ consistent  $A\ V \rangle$   $\langle$ maximal  $A\ V \rangle$   $\langle$ consistent  $A\ W \rangle$   $\langle$ maximal  $A\ W \rangle$   
**and**  $\langle W \in reach\ A\ i\ V \rangle$   
**shows**  $\langle V \in reach\ A\ i\ W \rangle$   
 $\langle$ proof $\rangle$

**lemma** *AxB-symmetric*:

**assumes**  $\langle \forall p. AxB\ p \longrightarrow A\ p \rangle$   $\langle$ consistent  $A\ V \rangle$   $\langle$ maximal  $A\ V \rangle$   $\langle$ consistent  $A\ W \rangle$   $\langle$ maximal  $A\ W \rangle$   
**shows**  $\langle W \in reach\ A\ i\ V \longleftrightarrow V \in reach\ A\ i\ W \rangle$   
 $\langle$ proof $\rangle$

**lemma** *mcs<sub>KB</sub>-symmetric*:

**assumes**  $\langle \forall p. AxB\ p \longrightarrow A\ p \rangle$   
**shows**  $\langle$ symmetric (Kripke (mcs  $A$ )  $pi$  (reach  $A$ )) $\rangle$   
 $\langle$ proof $\rangle$

**lemma** *imply-completeness-KB*:

**assumes** *valid*:  $\langle \forall (M :: ('i :: countable, 'i\ fm\ set)\ kripke). \forall w \in \mathcal{W}\ M.$   
*symmetric*  $M \longrightarrow (\forall q \in G. M, w \models q) \longrightarrow M, w \models p \rangle$   
**shows**  $\langle \exists qs. set\ qs \subseteq G \wedge (AxB \vdash imply\ qs\ p) \rangle$   
 $\langle$ proof $\rangle$

**lemma** *completeness<sub>KB</sub>*:

**assumes**  $\langle \forall (M :: ('i :: countable, 'i\ fm\ set)\ kripke). \forall w \in \mathcal{W}\ M. symmetric\ M$   
 $\longrightarrow M, w \models p \rangle$   
**shows**  $\langle \vdash_{KB}\ p \rangle$   
 $\langle$ proof $\rangle$

**abbreviation**  $\langle valid_{KB}\ p \equiv \forall (M :: (nat, nat\ fm\ set)\ kripke). \forall w \in \mathcal{W}\ M. symmetric\ M$   
 $\longrightarrow M, w \models p \rangle$

**theorem** *main<sub>KB</sub>*:  $\langle valid_{KB}\ p \longleftrightarrow \vdash_{KB}\ p \rangle$   
 $\langle$ proof $\rangle$

**corollary**

**assumes**  $\langle symmetric\ M \rangle$   $\langle w \in \mathcal{W}\ M \rangle$   
**shows**  $\langle valid_{KB}\ p \longrightarrow M, w \models p \rangle$   
 $\langle$ proof $\rangle$

## 11 System K4

**inductive** *Ax4* ::  $\langle 'i\ fm \Rightarrow bool \rangle$  **where**  
 $\langle Ax4\ (K\ i\ p \longrightarrow K\ i\ (K\ i\ p)) \rangle$

**abbreviation** *SystemK4* ::  $\langle 'i\ fm \Rightarrow bool \rangle$  ( $\vdash_{K4}$  - [50] 50) **where**  
 $\langle \vdash_{K4}\ p \equiv Ax4 \vdash p \rangle$

**lemma** *soundness-Ax4*:  $\langle Ax4\ p \Longrightarrow transitive\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p \rangle$

⟨proof⟩

**lemma** *soundness<sub>K4</sub>*: ⟨ $\vdash_{K4} p \implies \text{transitive } M \implies w \in \mathcal{W} M \implies M, w \models p$ ⟩  
⟨proof⟩

**lemma** *Ax4-transitive*:

**assumes** ⟨ $\forall p. Ax4\ p \longrightarrow A\ p$ ⟩ ⟨*consistent*  $A\ V$ ⟩ ⟨*maximal*  $A\ V$ ⟩  
**and** ⟨ $W \in \text{reach } A\ i\ V$ ⟩ ⟨ $U \in \text{reach } A\ i\ W$ ⟩  
**shows** ⟨ $U \in \text{reach } A\ i\ V$ ⟩

⟨proof⟩

**lemma** *mcs<sub>K4</sub>-transitive*:

**assumes** ⟨ $\forall p. Ax4\ p \longrightarrow A\ p$ ⟩  
**shows** ⟨*transitive* (*Kripke* (*mcs*  $A$ )  $pi$  (*reach*  $A$ ))⟩  
⟨proof⟩

**lemma** *imply-completeness-K4*:

**assumes** *valid*: ⟨ $\forall (M :: ('i :: \text{countable}, 'i\ \text{fm}\ \text{set})\ \text{kripke}). \forall w \in \mathcal{W} M.$   
 $\text{transitive } M \longrightarrow (\forall q \in G. M, w \models q) \longrightarrow M, w \models p$ ⟩  
**shows** ⟨ $\exists qs. \text{set } qs \subseteq G \wedge (Ax4 \vdash \text{imply } qs\ p)$ ⟩

⟨proof⟩

**lemma** *completeness<sub>K4</sub>*:

**assumes** ⟨ $\forall (M :: ('i :: \text{countable}, 'i\ \text{fm}\ \text{set})\ \text{kripke}). \forall w \in \mathcal{W} M. \text{transitive } M$   
 $\longrightarrow M, w \models p$ ⟩  
**shows** ⟨ $\vdash_{K4} p$ ⟩  
⟨proof⟩

**abbreviation** ⟨*valid<sub>K4</sub>*  $p \equiv \forall (M :: (\text{nat}, \text{nat}\ \text{fm}\ \text{set})\ \text{kripke}). \forall w \in \mathcal{W} M. \text{transitive } M \longrightarrow M, w \models p$ ⟩

**theorem** *main<sub>K4</sub>*: ⟨*valid<sub>K4</sub>*  $p \longleftrightarrow \vdash_{K4} p$ ⟩  
⟨proof⟩

**corollary**

**assumes** ⟨*transitive*  $M$ ⟩ ⟨ $w \in \mathcal{W} M$ ⟩  
**shows** ⟨*valid<sub>K4</sub>*  $p \longrightarrow M, w \models p$ ⟩  
⟨proof⟩

## 12 System S4

**abbreviation** *Or* :: ⟨ $('a \Rightarrow \text{bool}) \Rightarrow ('a \Rightarrow \text{bool}) \Rightarrow 'a \Rightarrow \text{bool}$ ⟩ (**infixl**  $\oplus$ ) 65)

**where**

⟨ $A \oplus A' \equiv \lambda x. A\ x \vee A'\ x$ ⟩

**abbreviation** *SystemS4* :: ⟨ $'i\ \text{fm} \Rightarrow \text{bool}$ ⟩ ( $\vdash_{S4} - [50] 50$ ) **where**

⟨ $\vdash_{S4} p \equiv AxT \oplus Ax4 \vdash p$ ⟩

**lemma** *soundness-AxT4*: ⟨ $(AxT \oplus Ax4) p \implies \text{reflexive } M \wedge \text{transitive } M \implies w$ ⟩

$\in \mathcal{W} M \implies M, w \models p$   
 ⟨proof⟩

**lemma** *soundness<sub>S4</sub>*:  $\langle \vdash_{S4} p \implies \text{reflexive } M \wedge \text{transitive } M \implies w \in \mathcal{W} M \implies M, w \models p \rangle$   
 ⟨proof⟩

**lemma** *imply-completeness-S4*:

**assumes** *valid*:  $\langle \forall (M :: ('i :: \text{countable}, 'i \text{ fm set}) \text{kripke}). \forall w \in \mathcal{W} M. \text{reflexive } M \longrightarrow \text{transitive } M \longrightarrow (\forall q \in G. M, w \models q) \longrightarrow M, w \models p \rangle$   
**shows**  $\langle \exists qs. \text{set } qs \subseteq G \wedge (AxT \oplus Ax4 \vdash \text{imply } qs \ p) \rangle$   
 ⟨proof⟩

**lemma** *completeness<sub>S4</sub>*:

**assumes**  $\langle \forall (M :: ('i :: \text{countable}, 'i \text{ fm set}) \text{kripke}). \forall w \in \mathcal{W} M. \text{reflexive } M \longrightarrow \text{transitive } M \longrightarrow M, w \models p \rangle$   
**shows**  $\langle \vdash_{S4} p \rangle$   
 ⟨proof⟩

**abbreviation**  $\langle \text{valid}_{S4} p \equiv \forall (M :: (\text{nat}, \text{nat fm set}) \text{kripke}). \forall w \in \mathcal{W} M. \text{reflexive } M \longrightarrow \text{transitive } M \longrightarrow M, w \models p \rangle$

**theorem** *main<sub>S4</sub>*:  $\langle \text{valid}_{S4} p \longleftrightarrow \vdash_{S4} p \rangle$   
 ⟨proof⟩

**corollary**

**assumes**  $\langle \text{reflexive } M \rangle \langle \text{transitive } M \rangle \langle w \in \mathcal{W} M \rangle$   
**shows**  $\langle \text{valid}_{S4} p \longrightarrow M, w \models p \rangle$   
 ⟨proof⟩

## 13 System S5

**abbreviation** *SystemS5* ::  $\langle 'i \text{ fm} \Rightarrow \text{bool} \rangle (\vdash_{S5} - [50] 50)$  **where**  
 $\langle \vdash_{S5} p \equiv AxT \oplus AxB \oplus Ax4 \vdash p \rangle$

**abbreviation** *AxTB4* ::  $\langle 'i \text{ fm} \Rightarrow \text{bool} \rangle$  **where**  
 $\langle AxTB4 \equiv AxT \oplus AxB \oplus Ax4 \rangle$

**lemma** *soundness-AxTB4*:  $\langle AxTB4 \ p \implies \text{equivalence } M \implies w \in \mathcal{W} M \implies M, w \models p \rangle$   
 ⟨proof⟩

**lemma** *soundness<sub>S5</sub>*:  $\langle \vdash_{S5} p \implies \text{equivalence } M \implies w \in \mathcal{W} M \implies M, w \models p \rangle$   
 ⟨proof⟩

**lemma** *imply-completeness-S5*:

**assumes** *valid*:  $\langle \forall (M :: ('i :: \text{countable}, 'i \text{ fm set}) \text{kripke}). \forall w \in \mathcal{W} M. \text{equivalence } M \longrightarrow (\forall q \in G. M, w \models q) \longrightarrow M, w \models p \rangle$   
**shows**  $\langle \exists qs. \text{set } qs \subseteq G \wedge (AxTB4 \vdash \text{imply } qs \ p) \rangle$

$\langle \text{proof} \rangle$

**lemma** *completeness<sub>S5</sub>*:

**assumes**  $\langle \forall (M :: ('i :: \text{countable}, 'i \text{ fm set}) \text{kripke}). \forall w \in \mathcal{W} M. \text{equivalence } M \longrightarrow M, w \models p \rangle$

**shows**  $\langle \vdash_{S5} p \rangle$

$\langle \text{proof} \rangle$

**abbreviation**  $\langle \text{valid}_{S5} p \equiv \forall (M :: (\text{nat}, \text{nat fm set}) \text{kripke}). \forall w \in \mathcal{W} M. \text{equivalence } M \longrightarrow M, w \models p \rangle$

**theorem** *main<sub>S5</sub>*:  $\langle \text{valid}_{S5} p \longleftrightarrow \vdash_{S5} p \rangle$

$\langle \text{proof} \rangle$

**corollary**

**assumes**  $\langle \text{equivalence } M \rangle \langle w \in \mathcal{W} M \rangle$

**shows**  $\langle \text{valid}_{S5} p \longrightarrow M, w \models p \rangle$

$\langle \text{proof} \rangle$

### 13.1 Traditional formulation

**inductive** *SystemS5'* ::  $\langle 'i \text{ fm} \Rightarrow \text{bool} \rangle (\vdash_{S5'} - [50] 50)$  **where**

*A1'*:  $\langle \text{tautology } p \Longrightarrow \vdash_{S5'} p \rangle$

| *A2'*:  $\langle \vdash_{S5'} (K i (p \longrightarrow q)) \longrightarrow K i p \longrightarrow K i q \rangle$

| *AT'*:  $\langle \vdash_{S5'} (K i p \longrightarrow p) \rangle$

| *A5'*:  $\langle \vdash_{S5'} (\neg K i p \longrightarrow K i (\neg K i p)) \rangle$

| *R1'*:  $\langle \vdash_{S5'} p \Longrightarrow \vdash_{S5'} (p \longrightarrow q) \Longrightarrow \vdash_{S5'} q \rangle$

| *R2'*:  $\langle \vdash_{S5'} p \Longrightarrow \vdash_{S5'} K i p \rangle$

**lemma** *S5'-trans*:  $\langle \vdash_{S5'} ((p \longrightarrow q) \longrightarrow (q \longrightarrow r)) \longrightarrow p \longrightarrow r \rangle$

$\langle \text{proof} \rangle$

**lemma** *S5'-L*:  $\langle \vdash_{S5'} (p \longrightarrow L i p) \rangle$

$\langle \text{proof} \rangle$

**lemma** *S5'-B*:  $\langle \vdash_{S5'} (p \longrightarrow K i (L i p)) \rangle$

$\langle \text{proof} \rangle$

**lemma** *S5'-KL*:  $\langle \vdash_{S5'} (K i p \longrightarrow L i p) \rangle$

$\langle \text{proof} \rangle$

**lemma** *S5'-map-K*:

**assumes**  $\langle \vdash_{S5'} (p \longrightarrow q) \rangle$

**shows**  $\langle \vdash_{S5'} (K i p \longrightarrow K i q) \rangle$

$\langle \text{proof} \rangle$

**lemma** *S5'-map-L*:

**assumes**  $\langle \vdash_{S5'} (p \longrightarrow q) \rangle$

**shows**  $\langle \vdash_{S5'} (L i p \longrightarrow L i q) \rangle$



⟨proof⟩

**lemma** *S5'-L-dual*:  $\langle \vdash_{S5'} (\neg L i (\neg p) \longrightarrow K i p) \rangle$   
⟨proof⟩

**lemma** *S5'-4*:  $\langle \vdash_{S5'} (K i p \longrightarrow K i (K i p)) \rangle$   
⟨proof⟩

**lemma** *S5-S5'*:  $\langle \vdash_{S5} p \implies \vdash_{S5'} p \rangle$   
⟨proof⟩

**lemma** *S5'-S5*:  
  **fixes**  $p :: \langle ('i :: countable) fm \rangle$   
  **shows**  $\langle \vdash_{S5'} p \implies \vdash_{S5} p \rangle$   
⟨proof⟩

**theorem** *main<sub>S5'</sub>*:  $\langle valid_{S5} p \longleftrightarrow \vdash_{S5'} p \rangle$   
⟨proof⟩

## 14 Acknowledgements

The formalization is inspired by Berghofer's formalization of Henkin-style completeness.

- Stefan Berghofer: First-Order Logic According to Fitting. <https://www.isa-afp.org/entries/FOL-Fitting.shtml>

end

## References

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