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Holopainen, Ilkka; Markvorsen, Steen; Palmer, Vicente

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p-CAPACITY AND p-HYPERBOLICITY OF SUBMANIFOLDS

ILKKA HOLOPAINEN[†], STEEN MARKVORSEN[#], AND VICENTE PALMER[*]

ABSTRACT. We use drifted Brownian motion in warped product model spaces as comparison constructions to show p-hyperbolicity of a large class of submanifolds for $p \geq 2$. The condition for p-hyperbolicity is expressed in terms of upper support functions for the radial sectional curvatures of the ambient space and for the radial convexity of the submanifold. In the process of showing p-hyperbolicity we also obtain explicit lower bounds on the p-capacity of finite annular domains of the submanifolds in terms of the drifted 2-capacity of the corresponding annuli in the respective comparison spaces.

1. INTRODUCTION

In [12] the first named author solved the asymptotic Dirichlet problem at infinity for the p-Laplacian in Cartan–Hadamard manifolds of pinched negative sectional curvature. As a consequence, such a manifold admits a wealth of non-constant bounded p-harmonic functions. On the other hand, there are no non-constant positive p-harmonic functions on a complete Riemannian manifold with non-negative Ricci curvature; see e.g. [1]. The purpose of the present paper is to initiate the study of the p-Laplacian and the existence of p-harmonic functions of various types on submanifolds. In this paper we concentrate on p-hyperbolicity of submanifolds.

To describe the problem we are dealing with, suppose that S is a Riemannian submanifold of an ambient Riemannian manifold N. We look for the most general intrinsic geometric condition on N and the most general extrinsic geometric condition on S which together will assure that S is p-hyperbolic. Recall that a Riemannian manifold M is called *p-hyperbolic*, with $1 < p < \infty$, if there exists a compact set $K \subset M$ of positive p-capacity $\mathrm{Cap}_p(K, M)$ relative to M. Here the p-capacity of K is defined by

$$\mathrm{Cap}_p(K, M) = \inf_u \int_M \|\nabla u\|^p \, d\mu,$$

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where the infimum is taken over all real-valued functions $u \in C_0^\infty(M)$, with $u \geq 1$ in K. In case $p = 2$, the p-hyperbolicity of M is equivalent both to the existence of a positive Green's kernel for the Laplace-Beltrami operator and to the transience of M, (see the works [16] and [6]). Using the particular 2-capacity condition alluded to above, the two last named authors have obtained geometric criteria for 2-hyperbolicity of minimal - or close to minimal - submanifolds in manifolds with sectional curvatures bounded from above, (see [19] and [20]).

In the general case of $1 < p < \infty$, the p-hyperbolicity of M is known to be equivalent to the existence of a (positive) Green's function $g = g(\cdot, y)$ for the p-Laplace equation, i.e. a certain positive solution (in the sense of distributions) of

$$-\operatorname{div}\left(\|\nabla g\|^{p-2}\nabla g\right) = \delta_y, \quad y \in M.$$

A third equivalent criterion for the p-hyperbolicity of M is the existence of a non-constant positive p-supersolution of the p-Laplace equation; see [8] and [9]. We refer to [1], [10], and [11] for further studies on p-hyperbolicity and various Liouville-type results and to [20] for a study of the geometric conditions which have been previously applied to extend the intrinsic analysis of hyperbolicity to the extrinsic analysis which is the main concern of the present paper.

1.1. **Outline of the paper.** In Section 2 we describe some of the basic properties of the p-Laplacian and present the corresponding maximum principle, which will be fundamental for the comparison technique applied in this paper. Section 3 is devoted to set up a so-called comparison constellation, which is essentially molded from curvature restrictions and a model space construction. In Section 4 we formulate our main result together with three of its corollaries. They are all proved in Sections 6, 7, and 8. A technical tool, the drifted 2-capacity of model spaces is defined and analyzed in Section 5. Finally, in Section 9 we present an alternative proof of the main theorem based directly on finite capacity comparison results.

2. The p-Laplacian

Let M be a non-compact Riemannian manifold, with the Riemannian metric $\langle \cdot, \cdot \rangle$ and the Riemannian volume form $d\mu$. We say that a vector field $\nabla u \in L^1_{\text{loc}}(M)$ is a *distributional gradient* of a function $u \in L^1_{\text{loc}}(M)$ if

$$\int_M \langle \nabla u, V \rangle \, d\mu = -\int_M u \operatorname{div} V \, d\mu$$

for all compactly supported vector fields $V \in C_0^1(M)$. Let $W^{1,p}(M)$, $1 \leq p < \infty$, be the Sobolev space of all functions $u \in L^p(M)$ whose distributional gradient ∇u belongs to $L^p(M)$. We equip $W^{1,p}(M)$ with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. The corresponding local space $W^{1,p}_{\text{loc}}(M)$ is defined in an obvious manner. The space $W^{1,p}_0(M)$ is the closure of $C_0^\infty(M)$ in $W^{1,p}(M)$.

Let $1 < p < \infty$. A function $u \in W^{1,p}_{\text{loc}}(M)$ is a (weak) solution to the p-Laplace equation

$$-\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) = 0 \tag{2.1}$$

in M if

$$\int_M \langle \|\nabla u\|^{p-2}\nabla u, \nabla\phi\rangle \, d\mu = 0 \tag{2.2}$$

for all $\phi \in C_0^\infty(M)$. If, moreover, $\|\nabla u\| \in L^p(M)$, it is equivalent to require (2.2) for all $\phi \in W_0^{1,p}(M)$. Continuous solutions of (2.1) are called *p-harmonic*. Here the continuity assumption makes no restriction since every solution of (2.1) has a continuous representative by the fundamental work of Serrin [24]. In fact, p-harmonic functions have locally Hölder-continuous first order derivatives by regularity results due to Ural'tseva [28] and Lewis [14]; see also DiBenedetto [2], Evans [3], Tolksdorf [25], and Uhlenbeck [27]. Furthermore, if $D \subset M$ is a precompact open set with $C^{1,\alpha}$ boundary ($\alpha \leq 1$), $h \in C^{1,\alpha}(\partial D)$, and u is p-harmonic in D with boundary values h, then $u \in C^{1,\beta}(\bar{D})$, with $\beta = \beta(\alpha, p, \dim M)$, by Lieberman [15].

A function $u \in W_{\mathrm{loc}}^{1,p}(M)$ is called a *p-supersolution* in M if

$$\int_M \langle \|\nabla u\|^{p-2}\nabla u, \nabla\phi\rangle \, d\mu \geq 0$$

for all non-negative $\phi \in C_0^\infty(M)$. Similarly, a function $v \in W_{\mathrm{loc}}^{1,p}(M)$ is called a *p-subsolution* in M if

$$\int_M \langle \|\nabla v\|^{p-2}\nabla v, \nabla\phi\rangle \, d\mu \leq 0$$

for all non-negative $\phi \in C_0^\infty(M)$. A fundamental feature of solutions of (2.1) is the following *maximum (or comparison) principle* which will be instrumental for the comparison technique presented below in Sections 4 and 5: If $u \in W^{1,p}(M)$ is a p-supersolution, $v \in W^{1,p}(M)$ is a p-subsolution, and $\max(v-u, 0) \in W_0^{1,p}(M)$, then $u \geq v$ a.e. in M. In particular, if $D \subset M$ is a precompact open set, $u \in C(\bar{D})$ is a p-supersolution, $v \in C(\bar{D})$ is a p-subsolution, and $u \geq v$ in ∂D, then $u \geq v$ in D. For the reader's convenience we recall the short proof of the comparison principle from [7, 3.18]. The proof is based on the following elementary inequality: Let $a \neq b$ denote two vectors in a given tangent space T_xM and suppose that $1 < p < \infty$. Then

$$\langle \|a\|^{p-2}a - \|b\|^{p-2}b, a - b\rangle > 0.$$

Suppose then that $u \in W^{1,p}(M)$ is a p-supersolution and $v \in W^{1,p}(M)$ is a p-subsolution such that $\phi = \max(v-u, 0) \in W_0^{1,p}(M)$. Since

$$\begin{aligned} 0 &\geq \int_M \langle \|\nabla v\|^{p-2}\nabla v, \nabla\phi\rangle \, d\mu - \int_M \langle \|\nabla u\|^{p-2}\nabla u, \nabla\phi\rangle \, d\mu \\ &= \int_{\{u<v\}} \langle \|\nabla v\|^{p-2}\nabla v - \|\nabla u\|^{p-2}\nabla u, \nabla v - \nabla u\rangle \, d\mu \geq 0, \end{aligned}$$

we have $\nabla\phi = 0$ a.e. in M as required.

3. Comparison Constellations

We assume throughout the paper that S^m is a non-compact connected complete Riemannian submanifold of a complete Riemannian manifold N^n. Furthermore, we assume that N^n possesses at least one pole. Recall that a pole is a point o such that the exponential map $\exp_o\colon T_oN^n \to N^n$ is a diffeomorphism. For example, an Hadamard–Cartan manifold has everywhere non-positive sectional curvatures and since it is also by definition simply connected, *every* point is a pole. The rôle of the pole o is precisely to serve as the origin of a smooth distance function r from o: For every $x \in N^n \setminus \{o\}$ we define $r(x) = \operatorname{dist}_N(o,x)$, and this distance is realized by the length of a unique geodesic from o to x, which is the *radial geodesic from o*. We also denote by r the restriction $r|_S : S \to \mathbb{R}_+ \cup \{0\}$. This restriction is called the *extrinsic distance function* from o in S^m. The gradients of r in N and S are denoted by $\nabla^N r$ and $\nabla^S r$, respectively. Let us remark that $\nabla^S r(x)$ is just the tangential component in S of $\nabla^N r(x)$, for all $x \in S$. Then we have the following basic relation:

$$\nabla^N r = \nabla^S r + (\nabla^N r)^\perp,$$

where $(\nabla^N r)^\perp(x)$ is perpendicular to T_xS for all $x \in S$.

3.1. **Curvature restrictions.** The sectional curvatures of N *along* the radial geodesics from o are called the o-radial sectional curvatures of N.

Definition 3.1. Let o be a point in a Riemannian manifold M and let $x \in M \setminus \{o\}$. The sectional curvature $K_M(\sigma_x)$ of the two-plane $\sigma_x \in T_xM$ is then called an *o-radial sectional curvature* of M at x if σ_x contains the tangent vector to a minimal geodesic from o to x. We denote these curvatures by $K_{o,M}(\sigma_x)$.

The o-radial sectional curvatures of N control the second order behavior of $r(x)$ in N via the classical Jacobi field index theory. Indeed, a bound on the o-radial sectional curvatures gives a bound on the Hessian of radial functions, $\operatorname{Hess}^N(f(r))$, as proved by Greene and Wu [5, Theorem A]; see Theorem 3.14 below. The submanifold S and the restricted radial functions $f(r)|_S$ inherit this second order bound to the S-intrinsic Hessian, $\operatorname{Hess}^S f(r)$, and therefore also to the Laplacian $\Delta^S f(r)$ of such modified distance functions.

The mean curvatures H_S of S also appear in the Laplacian $\Delta^S f(r)$ via its radially weighted component, which we define as follows:

Definition 3.2. The *o-radial mean convexity* $\mathcal{C}(x)$ of S in N, is defined in terms of the inner product of H_S with the N-gradient of the distance function $r(x)$ as follows:

$$\mathcal{C}(x) \;=\; -\langle \nabla^N r(x), H_S(x)\rangle, \quad x \in S,$$

where $H_S(x)$ denotes the mean curvature vector of S in N, i.e. the mean trace of the second fundamental form α_x. With respect to an orthonormal basis $\{X_1, ..., X_m\}$ of T_xS at $x \in S$ we have

$$H_S(x) = \frac{1}{m}\sum_{i=1}^{m} \alpha_x\left(X_i, X_i\right).$$

We will assume, that $\mathcal{C}(x)$ is bounded from above by a function $h(r(x))$ which only depends on the distance r from o:

$$\mathcal{C}(x) \le h(r(x)), \quad x \in S.$$

Moreover, for $p > 2$ we shall also need a particular inequality for the second fundamental form of S in N in the direction of the gradient $\nabla^N r(x)$. This gives rise to the following definition:

Definition 3.3. The *o-radial component* $\mathcal{B}(x)$ *of the second fundamental form* of S in N, is defined in terms of the following inner product:

$$\mathcal{B}(x) = -\langle \nabla^N r(x), \alpha_x(U_r, U_r)\rangle,$$

where

$$U_r = \nabla^S(r(x))/\|\nabla^S r(x)\| \in T_x S \subset T_x N$$

is the unit tangent vector to S in the direction of $\nabla^S r(x)$ (resp. tacitly assumed to be 0 in case $\nabla^S r(x) = 0$).

We assume that $\mathcal{B}(x)$ is bounded from above by a function $\lambda(r(x))$ which only depends on the distance r from o:

$$\mathcal{B}(x) \leq \lambda(r(x)).$$

Finally, we also impose an upper control on the 'radiality' of the submanifold, i.e. a local measure of how much the submanifold is extending away from the pole o:

Definition 3.4. The *o-radial tangency* $\mathcal{T}(x)$ of S in N is defined as follows:

$$\mathcal{T}(x) = \|\nabla^S r(x)\|$$

for all $x \in S$.

We assume that this S-gradient of the restricted distance function $r|_S$ has an upper radial support function $g(r) \leq 1$:

$$\mathcal{T}(x) \leq g(r(x)).$$

Definition 3.5. Given a connected and complete m-dimensional submanifold S^m in a complete Riemannian manifold N^n with a pole o, we denote the *extrinsic metric balls* of (sufficiently large) radius R and center o by $D_R(o)$. They are defined as any connected component of the intersection

$$B_R(o) \cap S = \{x \in S \colon r(x) < R\},$$

where $B_R(o)$ denotes the open geodesic ball of radius R centered at the pole o in N^n. Using these extrinsic balls we define the o-centered extrinsic annuli

$$A_{\rho,R}(o) = D_R(o) \setminus \bar{D}_\rho(o)$$

in S^m for $\rho < R$, where $D_R(o)$ is the component of $B_R(o) \cap S$ containing $D_\rho(o)$.

The upper bounding functions $h(r)$, $g(r)$, and $\lambda(r)$ together with a suitable control on the o-radial sectional curvatures of the ambient space will eventually control the p-Laplacian of restricted radial functions on S. In particular, we consider potential functions stemming from capacity calculations of radially symmetric comparison spaces and transplant them to S via the distance function r in N. Such transplantations are then compared with the 'correct' potentials on extrinsic metric balls of S. The maximum principle for the p-Laplacian Δ_p^S then finally gives the comparison result for capacities in S. Concerning the general strategy and types of results (in the case of $p = 2$) we refer to [17], [22], and [18].

We now collect the previous ingredients and formulate the general framework for our p-hyperbolicity comparison result:

Definition 3.6. Let N^n denote a Riemannian manifold with a pole o and distance function $r = r(x) = \operatorname{dist}_N(o,x)$. Let S^m denote a connected complete submanifold in N^n and assume that there is an extrinsic ball $D_\rho(o)$ which is precompact with smooth boundary $\partial D_\rho(o)$ in S^m. Let M_w^m denote a w-model with warping function $w : \pi(M_w^m) \to \mathbb{R}_+$ and center o_w; see Definition 3.9. Then the triple $\{N^n, S^m, M_w^m\}$ is called a *comparison constellation* on the interval $[0,R]$ if the o-radial sectional curvatures of N are bounded from above by the o_w-radial sectional curvatures of M_w^m:

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)} \tag{3.1}$$

for all x with $r = r(x) \in [0,R]$ and, moreover, the radial tangency $\mathcal{T}$ and the radial convexity functions $\mathcal{B}$ and $\mathcal{C}$ of the submanifold S^m are all bounded from above by smooth radial functions $g(r)$, $\lambda(r)$, and $h(r)$, respectively:

$$\begin{aligned}
&\mathcal{T}(x) \leq g(r(x)), \\
&\mathcal{B}(x) \leq \lambda(r(x)), \text{ and} \\
&\mathcal{C}(x) \leq h(r(x)) \text{ for all } x \in S^m \text{ with } r(x) \in [0,R].
\end{aligned} \tag{3.2}$$

Remark 3.7. This definition of comparison constellation extends a previous definition considered in [20]. In that paper, the triple $\{N^n, S^m, M_w^m\}$ is called a *comparison constellation* if inequality (3.1) holds and if in addition only the following condition holds in replacement of inequalities (3.2) for some bounding radial function $h(r)$:

$$\mathcal{C}(x) \leq h(r(x)) \leq \frac{w'(r(x))}{w(r(x))} \text{ for all } x \in S^m.$$

It is proved in [20] that under these conditions S^m is 2-hyperbolic if

$$\int_\rho^\infty \frac{\mathcal{G}^m(r)}{w^{m-1}(r)} dr < \infty,$$

where

$$\mathcal{G}(r) = \exp\left(\int_\rho^r h(t)dt\right).$$

3.2. **Warped products and model spaces.** Warped products are generalized manifolds of revolution, see e.g. [21]. Let (B^k, g_B) and (F^l, g_F) denote two Riemannian manifolds and let $w\colon B \to \mathbb{R}_+$ be a positive real function on B. We assume throughout that w is at least C^1 with piecewise continuous second order derivatives. We consider the product manifold $M^{k+l} = B \times F$ and denote the projections onto the factors by $\pi\colon M \to B$ and $\sigma\colon M \to F$, respectively. The metric g on M is then defined by the following w-modified (warped) product metric

$$g = \pi^*(g_B) + (w \circ \pi)^2 \sigma^*(g_F).$$

Definition 3.8. The Riemannian manifold $(M,g) = (B^k \times F^l, g)$ is called a *warped product* with *warping function* w, base manifold B and fiber F. We write as follows: $M_w^m = B^k \times_w F^l$.

Definition 3.9 (See [6], [5])**.** A $w-$model M_w^m is a smooth warped product with base $B^1 = [0, \Lambda[\subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{m-1} = \mathbb{S}_1^{m-1}$ (i.e. the unit $(m-1)$-sphere with standard metric), and warping function $w\colon [0, \Lambda[\to \mathbb{R}_+ \cup \{0\}$, with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where π denotes the projection onto B^1, is called the *center point* of the model space. If $\Lambda = \infty$, then o_w is a pole of M_w^m.

Proposition 3.10. *The simply connected space forms $\mathbb{K}^m(b)$ of constant curvature b are $w-$models with warping functions*

$$w(r) = Q_b(r) = \begin{cases} \frac{1}{\sqrt{b}} \sin(\sqrt{b}\, r) & \text{if } b > 0 \\ \quad r & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}} \sinh(\sqrt{-b}\, r) & \text{if } b < 0. \end{cases}$$

Note that for $b > 0$ the function $Q_b(r)$ admits a smooth extension to $r = \pi/\sqrt{b}$.

Proposition 3.11 (See e.g. [21])**.** *Let $M_w^m = B^1 \times_w \mathbb{S}_1^{m-1}$ be a $w-$model. Let r_0 and r denote two points in B^1. Then the geodesic distance from every $x \in \pi^{-1}(r)$ to $\pi^{-1}(r_0)$ is $|r - r_0|$.*

Proposition 3.12 (See [21] p. 206)**.** *Let M_w^m be a $w-$model with warping function $w(r)$ and center o_w. The distance sphere of radius r and center o_w in M_w^m is the fiber $\pi^{-1}(r)$. This distance sphere has the following constant mean curvature vector in M_w^m*

$$H_{\pi^{-1}(r)} = -\eta_w(r)\, \nabla^M \pi = -\eta_w(r)\, \nabla^M r,$$

where the mean curvature function $\eta_w(r)$ is defined by

$$\eta_w(r) = \frac{w'(r)}{w(r)} = \frac{d}{dr} \log(w(r)).$$

In particular, we have for the constant curvature space forms $\mathbb{K}^m(b)$:

$$\eta_{Q_b}(r) = \begin{cases} \sqrt{b} \cot(\sqrt{b}\, r) & \text{if } b > 0 \\ \quad 1/r & \text{if } b = 0 \\ \sqrt{-b} \coth(\sqrt{-b}\, r) & \text{if } b < 0\,. \end{cases}$$

The radial curvature in model spaces is given by the following result

Proposition 3.13 (See [5] and [6])**.** *Let M_w^m be a $w-$model with center point o_w. Then the o_w-radial sectional curvatures of M_w^m at every $x \in \pi^{-1}(r)$ (for $r > 0$) are all identical and determined by*

$$K_{o_w, M_w}(\sigma_x) = -\frac{w''(r)}{w(r)}.$$

3.3. **Hessian and Laplacian comparison analysis.** Concerning the second order analysis of the distance function r we need firstly and foremost the Hessian comparison theorem for manifolds with a pole:

Theorem 3.14 (See [5], Theorem A)**.** *Let $N = N^n$ be a manifold with a pole o, let $M = M_w^m$ denote a $w-$model with center o_w, and $m \leq n$. Suppose* *that every o-radial sectional curvature at $x \in N \setminus \{o\}$ is bounded from above by the o_w-radial sectional curvatures in M^m_w as follows:*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}$$

for every radial two-plane $\sigma_x \in T_xN$ at distance $r = r(x) = \operatorname{dist}_N(o,x)$ from o in N. Then the Hessian of the distance function in N satisfies

$$\begin{aligned} \operatorname{Hess}^N(r(x))(X,X) &\geq \operatorname{Hess}^M(r(y))(Y,Y) \\ &= \eta_w(r)\left(1 - \langle \nabla^M r(y), Y\rangle_M^2\right) \\ &= \eta_w(r)\left(1 - \langle \nabla^N r(x), X\rangle_N^2\right) \end{aligned} \tag{3.3}$$

for every unit vector X in T_xN and for every unit vector Y in T_yM with $r(y) = r(x) = r$ and $\langle \nabla^M r(y), Y\rangle_M = \langle \nabla^N r(x), X\rangle_N$.

Remark 3.15. In [5, Theorem A, p. 19], the Hessian of r_M is less or equal to the Hessian of r_N provided that the radial curvatures of N are bounded from above by the radial curvatures of M and provided that $\dim M \geq \dim N$. This latter dimension condition is *not* satisfied in our setting. However, since (M^m, g) is a $w-$model space it has an $n-$dimensional $w-$model space companion with the same radial curvatures and the same Hessian of radial functions as (M^m, g). In effect, therefore, applying [5, Theorem A, p. 19] to the high-dimensional comparison space gives the low-dimensional comparison inequality as stated.

If $\mu\colon N \to \mathbb{R}$ denotes a smooth function on the ambient space N, then the restriction $\tilde{\mu} = \mu_{|S}$ is a smooth function on the submanifold S and the respective Hessian tensors, $\operatorname{Hess}^N(\mu)$ and $\operatorname{Hess}^S(\tilde{\mu})$, are related as follows:

Proposition 3.16 ([13])**.**

$$\operatorname{Hess}^S(\tilde{\mu})(X,Y) = \operatorname{Hess}^N(\mu)(X,Y) + \langle \nabla^N(\mu), \alpha_x(X,Y)\rangle \tag{3.4}$$

for all tangent vectors $X, Y \in T_xS^m \subset T_xN^n$, where α_x is the second fundamental form of S at x in N.

If we compose μ with a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ we then get:

Corollary 3.17 ([13])**.**

$$\begin{aligned} \operatorname{Hess}^S(f \circ \tilde{\mu})(X,X) &= f''(\mu)\langle \nabla^N(\mu), X\rangle^2 \\ &+ f'(\mu)\big(\operatorname{Hess}^N(\mu)(X,X) + \langle \nabla^N(\mu), \alpha_x(X,X)\rangle\big) \end{aligned}$$

for all $X \in T_xS^m$.

Combining the estimate (3.3) with Corollary 3.17 and tracing the resulting Hessian comparison statement in an orthonormal basis of T_xS^m, we obtain the following instrumental inequality for the Laplacian of (extrinsic) radial functions restricted to the submanifold S:

Proposition 3.18. *Suppose that the assumptions of Theorem 3.14 are satisfied. Then we have for every smooth real-valued function $f \circ r$ with $f' \geq 0$ the following inequality for the standard Laplacian:*

$$\Delta^S(f \circ r) \geq \left(f''(r) - f'(r)\eta_w(r)\right)\|\nabla^S r\|^2 + mf'(r)\left(\eta_w(r) + \langle \nabla^N r, H_S\rangle\right),$$

where H_S denoted the mean curvature vector of S in N.

4. Main results

Applying the notion of a comparison constellation as defined in the previous section, we now formulate our main p-hyperbolicity result. The proofs are developed through the following sections.

Theorem 4.1. *Consider a comparison constellation $\{N^n, S^m, M^m_w\}$ on the interval $[\,0,\infty[\,$. Assume further that the functions $h(r)$ and $\lambda(r)$ are* balanced *with respect to the warping function $w(r)$ by the following inequality:*

$$\mathcal{M}(r) := (m+p-2)\,\eta_w(r) - m\,h(r) - (p-2)\lambda(r) \geq 0. \tag{4.1}$$

Let $\Lambda(r)$ denote the function

$$\Lambda(r) = w(r)\exp\left(-\int_1^r \frac{\mathcal{M}(t)}{(p-1)g^2(t)}\,dt\right).$$

Suppose finally that $p \geq 2$ and that

$$\lim_{R\to\infty}\int_\rho^R \Lambda(t)\,dt < \infty. \tag{4.2}$$

Then S^m is p-hyperbolic.

We observe the following corollaries; they will be proved in Section 8.

Corollary 4.2. *Suppose (in Theorem 4.1) that we can choose $w(r) = Q_b(r) = \sinh(\sqrt{-b}\,r)/\sqrt{-b}$ for some $b < 0$, i.e. we apply the negatively curved space form $\mathbb{K}^m(b)$ to play the role of a model space in the comparison constellation. Suppose that there exist constants λ_0 and h_0 such that*

$$\mathcal{B}(x) \leq \lambda_0 \text{ and}$$
$$\mathcal{C}(x) \leq h_0 \text{ for all } x \in S^m.$$

Suppose further that for some $\tilde{p} \geq 2$ we have

$$m\,h_0 + (\tilde{p}-2)\lambda_0 < (m-1)\sqrt{-b}. \tag{4.3}$$

Then S^m is p-hyperbolic for all p in the range $2 \leq p \leq \tilde{p}$.

Corollary 4.3. *Consider a purely intrinsic setting and comparison constellation: $S^n = N^n = M^n_w$. Then S^n is p-hyperbolic if and only if*

$$\int_\rho^\infty \frac{1}{w(t)^{\frac{n-1}{p-1}}}\,dr < \infty.$$

This observation is originally due to M. Troyanov, see [26, Corollary 5.4].

Corollary 4.4. *Let (M^m, g) denote a complete manifold with intrinsic concentric metric balls $B_r(o)$ centered at $o \in M$. Suppose that for some $p \geq 2$ and for some $\rho > 0$ we have*

$$\int_\rho^\infty \frac{1}{\operatorname{Vol}(\partial B_r(o))^{\frac{1}{p-1}}}\,dr = \infty, \tag{4.4}$$

and suppose that there are constants $\lambda_0 > 0$ and $b < 0$ so that

$$(p-2)\lambda_0 < (m-1)\sqrt{-b}. \tag{4.5}$$

Then (M, g) does not admit a minimal isometric immersion with bounded second fundamental form $\|\alpha\| \leq \lambda_0$ into any Hadamard–Cartan manifold N^n, $n \geq m$, with sectional curvatures bounded from above by b.

Proof. Condition (4.4) implies that the manifold (M^m, g) is p-parabolic according to [26, Corollary 5.4], whereas the condition (4.5) implies p-hyperbolicity of (M^m, g) according to Corollary 4.2 of the present work - upon observing that $\mathcal{C}(x) \equiv 0$ by the minimality assumption and that $\|\alpha_x\| \leq \lambda_0$ implies $\mathcal{B}(x) \leq \lambda_0$.

□

5. Drifted 2-capacity of model spaces

Definition 5.1. Let (M, g) denote a Riemannian manifold with Laplace operator Δ^M, and let V denote a continuous vector field on M. The drifted Brownian motion on M with the drift vector field V is then generated by the modified Laplacian L

$$\mathrm{L}\, f = \Delta^M f + \langle \nabla^M f, V \rangle$$

for every smooth function f on M.

We consider, in particular, the drift vector field

$$V = \mathcal{V}(r) \nabla^M r$$

with

$$\mathcal{V}(r) = \frac{\mathcal{M}(r)}{(p-1)g^2(r)} - m\, \eta_w(r)$$

on model spaces $M = M^m_w$, so that the modified Laplacian then reads as

$$\mathrm{L}\, \psi(x) = \Delta^M \psi(x) + \psi'(r(x)) \mathcal{V}(r(x))$$

for smooth functions ψ on M^m_w. For purely *radial* functions $\psi(r)$ we get

Lemma 5.2. *Let $\psi = \psi(r)$ denote a function on the w-model space $M = M^m_w$ which only depends on the radial distance r to the center o_w. Then*

$$\mathrm{L}\, \psi(r) = \psi''(r) + \psi'(r) \left(\frac{\mathcal{M}(r)}{(p-1)\, g^2(r)} - \eta_w(r) \right).$$

The Dirichlet problem associated to L defined on so-called *extrinsic annuli* is defined as follows:

First, the annular domains in the model space are denoted by

$$A^w_{\rho,R} = \{x \in M^n_w \colon \pi(x) \in [\rho, R]\} = \pi^{-1}([\rho, R]),$$

and the corresponding boundaries are denoted by $\partial D^w_\rho = \pi^{-1}(\rho)$ and $\partial D^w_R = \pi^{-1}(R)$, respectively. We consider the unique radial function $\psi_{\rho,R}(r)$ which solves the one-dimensional Laplace-Dirichlet problem on the model space annulus $A^w_{\rho,R}$:

$$\begin{cases} \mathrm{L}\, \psi & = 0 \text{ on } A^w_{\rho,R} \\ \psi & = 0 \text{ on } \partial D^w_\rho \\ \psi & = 1 \text{ on } \partial D^w_R. \end{cases} \tag{5.1}$$

The explicit solution to the Dirichlet problem (5.1) is given in the following Proposition, with a focus towards the corresponding expression for the drifted annular capacity in the model space; see [20], [19], and Section 9 below.

Proposition 5.3. *The solution to the Dirichlet problem (5.1) only depends on r and is given explicitly - via the function $\Lambda(r)$ introduced in Theorem 4.1, by:*

$$\psi_{\rho,R}(r) = \frac{\int_\rho^r \Lambda(t)\,dt}{\int_\rho^R \Lambda(t)\,dt}. \tag{5.2}$$

The corresponding 'drifted' 2-capacity is

$$\begin{aligned} \mathrm{Cap}_{\mathrm{L}}(A^w_{\rho,R}) &= \int_{\partial D^w_\rho} \langle \nabla^M \psi_{\rho,R}, \nu\rangle \, dA \\ &= \mathrm{Vol}(\partial D^w_\rho)\Lambda(\rho)\left(\int_\rho^R \Lambda(t)\,dt\right)^{-1}. \end{aligned} \tag{5.3}$$

6. p-Laplacian comparison

Let us consider comparison constellations $\{N^n, S^m, M^m_w\}$ on intervals $[0,R]$ for $R > 0$. Since the o-radial mean convexity of S has an upper bound

$$\mathcal{C}(x) = -\langle \nabla^N r(x), H_S(x)\rangle \leq h(r(x)),$$

we obtain the following estimate using Proposition 3.18

$$\Delta^S(f\circ r) \geq \big(f''(r) - f'(r)\eta_w(r)\big)\,\|\nabla^S r\|^2 + m\, f'(r)\,(\eta_w(r) - h(r))\,. \tag{6.1}$$

In what follows we use shorthand $F(x) = f'(r(x))\|\nabla^S r(x)\|$ for all $x \in S$ to simplify the notation. To get estimates for the p-Laplacian of $f\circ r$ we first compute

$$\begin{aligned} &\Delta^S_p f(r(x)) = \mathrm{div}^S\left(\|\nabla^S f(r(x))\|^{p-2}\,\nabla^S f(r(x))\right) \\ &= \|\nabla^S f(r(x))\|^{p-2}\,\Delta^S f(r(x)) + \left\langle \nabla^S\|\nabla^S f(r(x))\|^{p-2}, \nabla^S f(r(x))\right\rangle \\ &= F^{p-2}(x)\Delta^S f(r(x)) + \left\langle \nabla^S F^{p-2}(x), f'(r(x))\nabla^S r(x)\right\rangle \\ &= F^{p-2}(x)\Delta^S f(r(x)) \\ &\qquad + \Big\langle (p-2)F^{p-3}(x)\left(f''(r(x))\|\nabla^S r(x)\|\nabla^S r(x) + f'(r(x))\nabla^S\|\nabla^S r(x)\|\right), \\ &\qquad\quad f'(r(x))\nabla^S r(x)\Big\rangle \\ &= F^{p-2}(x)\Big((p-2)\Big(f''(r(x))\|\nabla^S r(x)\|^2 + f'(r(x))\frac{\langle \nabla^S r(x), \nabla^S\|\nabla^S r(x)\|\rangle}{\|\nabla^S r(x)\|}\Big) \\ &\qquad + \Delta^S f(r(x))\Big). \end{aligned}$$

This partial 'isolation' of the factor $(p-2)$ is the reason behind the general assumption $p \geq 2$ in this work. The factor on $(p-2)$ is controlled via the following observation, which introduces the bound $\lambda(r)$ into this setting:

Lemma 6.1. *Let $\{N^n, S^m, M^m_w\}$ be a comparison constellation on $[0,R]$ for $R > 0$. Suppose that the o-radial component of the second fundamental form of S (see Definition 3.3) has an upper bound*

$$\mathcal{B}(x) \leq \lambda(r(x)).$$

Then

$$\begin{aligned}
&\frac{\left\langle \nabla^S r(x), \nabla^S \|\nabla^S r(x)\| \right\rangle}{\|\nabla^S r(x)\|} \\
&\qquad\qquad = \mathrm{Hess}^S(r(x))\,(U_r, U_r) \\
&\qquad\qquad = \mathrm{Hess}^N(r(x))\,(U_r, U_r) + \left\langle \nabla^N r(x), \alpha_x\,(U_r, U_r) \right\rangle \\
&\qquad\qquad \geq \eta_w(r(x))\left(1 - \|\nabla^S r(x)\|^2\right) - \lambda(r(x)).
\end{aligned} \tag{6.2}$$

Proof. By definition of the Hessian via the induced connection D^S in S we have directly for the first equality in (6.2):

$$\begin{aligned}
\mathrm{Hess}^S(r)\left(\nabla^S r, \nabla^S r\right) &= \left\langle \mathrm{D}^S_{\nabla^S r}\,\nabla^S r, \nabla^S r \right\rangle \\
&= \tfrac{1}{2}\,\mathrm{D}^S_{\nabla^S r}\left\langle \nabla^S r, \nabla^S r \right\rangle \\
&= \tfrac{1}{2}\nabla^S r\left\langle \nabla^S r, \nabla^S r \right\rangle \\
&= \tfrac{1}{2}\left\langle \nabla^S \|\nabla^S r\|^2, \nabla^S r \right\rangle \\
&= \|\nabla^S r\|\left\langle \nabla^S \|\nabla^S r\|, \nabla^S r \right\rangle,
\end{aligned}$$

so that

$$\begin{aligned}
\mathrm{Hess}^S(r(x))\,(U_r, U_r) &= \frac{\mathrm{Hess}^S(r)\left(\nabla^S r, \nabla^S r\right)}{\|\nabla^S r\|^2} \\
&= \frac{\left\langle \nabla^S r(x), \nabla^S \|\nabla^S r(x)\| \right\rangle}{\|\nabla^S r(x)\|}.
\end{aligned}$$

The other (in)equalities in (6.2) follow from (3.4) and (3.3), respectively. □

The following result relates the p-Laplacian of a radial function $f(r)$ with its 2-drifted Laplacian, as defined in Section 5.

Lemma 6.2. *Let $\{N^n, S^m, M^m_w\}$ be a comparison constellation on $[0, R]$ for $R > 0$. Let $f \circ r$ be a smooth real-valued function with $f' \geq 0$, and suppose now that $f(r)$ satisfies the following condition (to be molded shortly from the balance condition (4.1)):*

$$f''(r) - f'(r)\eta_w(r) \leq 0. \tag{6.3}$$

Then, for all $x \in S$,

$$\Delta^S_p f(r(x)) \geq (p-1)F^{p-2}(x)g^2(r(x))\,\mathrm{L}(f(r(x))),$$

where L is the modified 2-Laplacian defined in Lemma 5.2.

Proof. By using the assumption $p \geq 2$ together with the comparison constellation assumptions (3.2) we obtain from (6.1) and (6.2) that

$$\begin{aligned}
&\Delta^S_p(f(r(x))) \\
&\geq F^{p-2}(x)(p-2)\left(f''(r)\|\nabla^S(r)\|^2 + f'(r)\,\mathrm{Hess}^S(r)\,(U_r, U_r)\right) \\
&\quad + F^{p-2}(x)\left(f''(r)\|\nabla^S(r)\|^2 - f'(r)\eta_w(r)\|\nabla^S(r)\|^2 + m\,f'(r)\,(\eta_w(r) - h(r))\right) \\
&\geq F^{p-2}(x)(p-1)\|\nabla^S(r)\|^2\left(f''(r) - f'(r)\eta_w(r)\right) \\
&\quad + F^{p-2}(x)f'(r)\,((p-2+m)\eta_w(r) - (p-2)\lambda(r) - m\,h(r)) \\
&= F^{p-2}(x)\left(\left(f''(r) - f'(r)\eta_w(r)\right)(p-1)\|\nabla^S(r)\|^2 + f'(r)\mathcal{M}(r)\right).
\end{aligned}$$

Since $f(r)$ satisfies inequality (6.3), we have, via $\|\nabla^S(r)\| \le g(r)$, that:

$$
\begin{aligned}
&\Delta_p^S(f(r(x)))\\
&\ge F^{p-2}(x)\left(\left(f''(r) - f'(r)\eta_w(r)\right)(p-1)g^2(r) + f'(r)\mathcal{M}(r)\right)\\
&= (p-1)F^{p-2}(x)g^2(r)\left(f''(r) - f'(r)\eta_w(r) + f'(r)\frac{\mathcal{M}(r)}{(p-1)g^2(r)}\right)\\
&= (p-1)F^{p-2}(x)g^2(r)\left(f''(r) + f'(r)\left(\frac{\mathcal{M}(r)}{(p-1)g^2(r)} - \eta_w(r)\right)\right)\\
&= (p-1)F^{p-2}(x)g^2(r)\,\mathrm{L}(f(r)),
\end{aligned}
$$

as claimed in the lemma. □

7. First proof of Theorem 4.1

Next we show that (4.2) is also a sufficient condition for p-hyperbolicity of S^m. First we transplant the model space solutions $\psi_{\rho,R}(r)$ of equation (5.1) into the extrinsic annulus $A_{\rho,R} = D_R(o) \setminus \bar{D}_\rho(o)$ in S by defining

$$\Psi_{\rho,R}\colon A_{\rho,R} \to \mathbb{R}, \quad \Psi_{\rho,R}(x) = \psi_{\rho,R}(r(x)).$$

Here the extrinsic ball $D_\rho(o)$ is as in Definition 3.6 and $D_R(o)$ is that component of $B_R(o) \cap S$ which contains $D_\rho(o)$. Next we extend $\Psi_{\rho,R}$ to $S \cap \bar{B}_\rho(o)$ by setting $\Psi_{\rho,R}(x) = 0$ for $x \in S \cap \bar{B}_\rho(o)$.

Using $w'(r) = \eta_w(r)w(r)$ and the balance condition (4.1) it is straightforward to check that

$$\psi''_{\rho,R}(r) - \psi'_{\rho,R}(r)\eta_w(r) \le 0.$$

Since $\psi'_{\rho,R}(r) \ge 0$ and $\mathrm{L}\,\psi_{\rho,R} = 0$ in $A^w_{\rho,R}$, we obtain from Lemma 6.2 that

$$\Delta_p^S \Psi_{\rho,R} \ge 0 \quad \text{in} \quad D_R(o) \setminus \bar{B}_\rho(o).$$

Thus $\Psi_{\rho,R}$ is a p-subsolution in $D_R(o)\setminus\bar{B}_\rho(o)$. In fact, $\Psi_{\rho,R}$ is a p-subsolution in the whole extrinsic ball $D_R(o)$ since $\Psi_{\rho,R}(x) = 0$ for $x \in S \cap \bar{B}_\rho(o)$; see [7, Theorem 7.25, Lemma 7.28]. Furthermore, for fixed ρ and fixed $x \in S$, $\Psi_{\rho,R}(x)$ is defined for sufficiently large R and it is decreasing as a function of R, see equation (5.2). Hence the limit function

$$\Psi_\rho := \lim_{R\to\infty} \Psi_{\rho,R}$$

exists in S and, moreover, it is positive in $S\setminus\bar{B}_\rho(o)$ by (4.2). By [7, Theorem 3.75], Ψ_ρ is a p-subsolution in S. Hence $1 - \Psi_\rho$ is a non-negative, nonconstant p-supersolution in S, and therefore S is p-hyperbolic. This proves Theorem 4.1.

8. Proof of Corollaries

Proof of Corollary 4.2. The balance condition (4.1) is clearly satisfied by (4.3). Thus we only need to check the p-hyperbolicity condition (4.2). Since $g(r) \le 1$, we have

$$\frac{\mathcal{M}(r)}{(p-1)g^2(r)} > (1+c)\sqrt{-b}$$

for some positive constant c by (4.3). Hence

$$\Lambda(r) \leq \frac{\sinh(\sqrt{-b}\,r)}{\sqrt{-b}} \exp\left(-\int_1^r (1+c)\sqrt{-b}\,dt\right)$$

and therefore it is straightforward to check that

$$\lim_{R\to\infty} \int_\rho^R \Lambda(t)\,dt < \infty,$$

which concludes the proof. □

Proof of Corollary 4.3. The assumptions amount to $g(r) \equiv 1$, $h(r) \equiv 0$, and $\lambda(r) \equiv 0$ and the only 'free' function is $w(r)$. In this intrinsic setting we therefore have

$$\mathcal{M}(r) = (m+p-2)\eta_w(r),$$

so that with $g(r) = 1$ we get

$$\begin{aligned}\int_\rho^r \frac{\mathcal{M}(t)}{(p-1)g^2(t)}\,dt &= \tfrac{m+p-2}{p-1} \int_\rho^r \frac{w'(t)\,dt}{w(t)} \\ &= \tfrac{m+p-2}{p-1} \log \frac{w(r)}{w(\rho)},\end{aligned}$$

and hence

$$\begin{aligned}\Lambda(r) &= w(r) \exp\left(-\tfrac{m+p-2}{p-1} \log \frac{w(r)}{w(\rho)}\right) \\ &= w(r)^{1-\frac{m+p-2}{p-1}} w(\rho)^{-\frac{m+p-2}{p-1}} \\ &= w(r)^{-\frac{m-1}{p-1}}\, c(\rho),\end{aligned}$$

where $c(\rho)$ is a constant depending on the fixed inner radius of the annuli used in the proof of the p-hyperbolicity. Then $\Lambda(r)$ has bounded integral precisely if

$$\lim_{R\to\infty} \int_\rho^R \frac{1}{w(r)^{\frac{m-1}{p-1}}}\,dt < \infty,$$

as claimed. □

9. p-capacity bounds

In this section we give lower bounds on the p-capacity of closed (compact) extrinsic balls relative to S^m. Let $G \subset S^m$ be a precompact open set such that $\bar{D}_\rho(o) \subset G$. We recall from the introduction that the p-capacity of $\bar{D}_\rho(o)$ relative to G is defined by

$$\mathrm{Cap}_p(\bar{D}_\rho(o), G) = \inf_v \int_G \|\nabla^S v\|^p\,d\mu,$$

where the infimum is taken over all real-valued functions $v \in C_0^\infty(G)$, with $v \geq 1$ in $\bar{D}_\rho(o)$. If ∂G is regular for the Dirichlet problem for p-harmonic functions, then there exists a unique function $u \in C(\bar{G})$ which is p-harmonic in $G \setminus \bar{D}_\rho(o)$ such that $u = 0$ in $\bar{D}_\rho(o)$, $u = 1$ in ∂G, and that

$$\mathrm{Cap}_p(\bar{D}_\rho(o), G) = \int_G \|\nabla^S u\|^p\,d\mu.$$

We refer to [7, Chapter 6] for the boundary regularity. For our purposes it is enough to know that every open set can be exhausted by open sets with regular boundaries.

Since u is p-harmonic in $G \setminus \bar{D}_\rho(o)$, we have

$$\operatorname{Cap}_p(\bar{D}_\rho(o), G) = \int_G \langle \|\nabla^S u\|^{p-2} \nabla^S u, \nabla^S \varphi \rangle \, d\mu \tag{9.1}$$

for every function $\varphi \in W^{1,p}(G)$ which is continuous in $\bar{G}$ with values $\varphi = 0$ in $\bar{D}_\rho(o)$ and $\varphi = 1$ in ∂G. In particular, (9.1) holds for all $0 \le t < s \le 1$ with the function

$$\varphi(x) = \begin{cases} 0 & \text{if } u(x) \le t \\ \frac{u(x)-t}{s-t} & \text{if } t < u(x) < s \\ 1 & \text{if } u(x) \ge s. \end{cases}$$

Applying the co-area formula ([23], [4, 3.2.12, 3.2.46], [29]) we obtain

$$\operatorname{Cap}_p(\bar{D}_\rho(o), G) = \frac{1}{s-t} \int_t^s \left(\int_{u^{-1}(\tau)} \|\nabla^S u\|^{p-1} \, d\mathcal{H}^{m-1} \right) d\tau.$$

Letting $s \to t$ we finally get

$$\operatorname{Cap}_p(\bar{D}_\rho(o), G) = \int_{u^{-1}(t)} \|\nabla^S u\|^{p-1} \, d\mathcal{H}^{m-1} \tag{9.2}$$

for a.e. $t \in [0, 1]$. We will use the equation (9.2) to get lower bounds on the p-capacity $\operatorname{Cap}_p(\bar{D}_\rho(o), D_R(o))$ in terms of the corresponding drifted 2-capacity in the model space.

Our main comparison estimate for the p-capacity now reads as follows:

Theorem 9.1. *Let $\{N^n, S^m, M^m_w\}$ denote a comparison constellation on $[0, R]$, $R > \rho$, in the sense of Definition 3.6. Then*

$$\operatorname{Cap}_p\big(\bar{D}_\rho(o), D_R(o)\big) \ge \left(\frac{\operatorname{Cap}_{\mathrm{L}}(A^w_{\rho,R})}{\operatorname{Vol}(\partial D^w_\rho)} \right)^{p-1} \int_{\partial D_\rho} \|\nabla^S r\|^{p-1} \, d\mathcal{H}^{m-1}. \tag{9.3}$$

Proof. Let $G \subset D_R(o)$ be a precompact open set with regular boundary such that $\bar{D}_\rho(o) \subset G$. Let $u \in C(\bar{G})$ be p-harmonic in $G \setminus \bar{D}_\rho(o)$ with $u = 0$ in $\bar{D}_\rho(o)$ and $u = 1$ in ∂G. Furthermore, let $\Psi_{\rho,R}$ be the p-subsolution in $D_R(o)$ defined in Section 7. By the comparison principle,

$$u(x) \ge \Psi_{\rho,R}(x)$$

for all $x \in D_R(o)$. Since $\nabla^S u$ is Hölder-continuous up to the boundary $\partial D_\rho(o)$ by [15] and $u(x) = \Psi_{\rho,R}(x) = 0$ for all $x \in \bar{D}_\rho(o)$, we obtain

$$\|\nabla^S u(x)\| \ge \|\nabla^S \Psi_{\rho,R}(x)\| \tag{9.4}$$

for all $x \in \partial D_\rho(o)$. Combining (9.2) and (9.4), we arrive at

$$\begin{aligned}
\mathrm{Cap}_p(\bar{D}_\rho(o), G) &\geq \int_{\partial D_\rho} \|\nabla^S \Psi_{\rho,R}\|^{p-1}\, d\mathcal{H}^{n-1} \\
&= \left(\psi'_{\rho,R}(\rho)\right)^{p-1} \int_{\partial D_\rho} \|\nabla^S r\|^{p-1}\, d\mathcal{H}^{m-1} \\
&= \left(\frac{\mathrm{Cap}_{\mathrm{L}}(A^w_{\rho,R})}{\mathrm{Vol}(\partial D^w_\rho)}\right)^{p-1} \int_{\partial D_\rho} \|\nabla^S r\|^{p-1}\, d\mathcal{H}^{m-1}.
\end{aligned}$$

The desired estimate (9.3) now follows since

$$\mathrm{Cap}_p\big(\bar{D}_\rho(o), D_R(o)\big) = \inf_G \mathrm{Cap}_p(\bar{D}_\rho(o), G),$$

where $G \subset D_R(o)$ is a precompact open set with regular boundary. □

9.1. **Second Proof of Theorem 4.1.** Using the explicit capacity comparison obtained in Theorem 9.1 we finally observe the following direct proof of the main theorem.

Let $\{N^n, S^m, M^m_w\}$ denote a comparison constellation on $[0,\infty]$ in the sense of Definition 3.6. By assumption $D_\rho(o)$ is precompact with a smooth boundary and thence, in equation (9.3) we have

$$\int_{\partial D_\rho} \|\nabla^S r\|^{p-1}\, d\mathcal{H}^{m-1} > 0.$$

From (5.3) and the assumption (4.2) we also have

$$\lim_{R\to\infty} \mathrm{Cap}_{\mathrm{L}}(A^w_{\rho,R}) > 0,$$

so that Theorem 9.1 implies:

$$\mathrm{Cap}_p\big(\bar{D}_\rho(o), S^m\big) = \lim_{R\to\infty} \mathrm{Cap}_p\big(\bar{D}_\rho(o), D_R(o)\big) > 0.$$

Thus $\bar{D}_\rho(o)$ is a compact subset with positive p-capacity in S^m, and p-hyperbolicity of that submanifold follows again.

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P.O. Box 68 (Gustaf Hällströmin katu 2b), FIN-00014 University of Helsinki, Finland.
E-mail address: ilkka.holopainen@helsinki.fi

Department of Mathematics, Technical University of Denmark.
E-mail address: S.Markvorsen@mat.dtu.dk

Departament de Matemàtiques, Universitat Jaume I, Castellon, Spain.
E-mail address: palmer@mat.uji.es