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Published in:
International Journal of Heat and Mass Transfer

Link to article, DOI:
10.1016/j.ijheatmasstransfer.2022.122694

Publication date:
2022

Document Version
Early version, also known as pre-print

Link back to DTU Orbit

Citation (APA):
Harmonic analysis of temperature profiles of active caloric regenerators

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Abstract: The design of a thermal system is initially carried out by considering the fundamental influencing variables. For novel solid-state cooling systems using active caloric regenerators, the non-linearity of coupled phenomena of functional material properties, heat transfer, and hydraulic flow can complicate the interpretation of experimental and simulated results. We elucidate the operation of active regenerators by deriving easy-to-manage analytical expressions for the temperature transients of caloric materials and heat transfer fluid. An internal temperature measurement system for packed bed regenerators is developed for validation. The derived expressions have a reasonable accuracy relative to the experimental and numerical results for temperature profiles in both magnitude and sensitivity. Useful figures of merit are post-calculated using the derived temperature profiles. We found that the average temperature profiles are linear for passive regenerators and nonlinear for active regenerators, and their transients are nonlinear functions of the configuration and operating parameters.

Keywords: Active caloric regenerator; Heat transfer; Analytical model; Harmonic analysis; Caloric effect

<table>
<thead>
<tr>
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<td>1D</td>
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<td>$L$</td>
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<td>$m$</td>
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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>( \dot{m} )</td>
<td>Mass flow rate, ([\text{kg s}^{-1}])</td>
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<tr>
<td>( NTU )</td>
<td>Number of transfer units, [-]</td>
</tr>
<tr>
<td>( Q_x )</td>
<td>Net enthalpy flux, ([\text{W}])</td>
</tr>
<tr>
<td>( S_g )</td>
<td>Entropy generation, ([\text{J K}^{-1}])</td>
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<tr>
<td>( T )</td>
<td>Temperature, ([\text{K}])</td>
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<tr>
<td>( T_{ax} )</td>
<td>Axial temperature measurement, ([\text{K}])</td>
</tr>
<tr>
<td>( T_{ch} )</td>
<td>Horizontal temperature measurement at cold side plane, ([\text{K}])</td>
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<tr>
<td>( T_{cv} )</td>
<td>Vertical temperature measurement at cold side plane, ([\text{K}])</td>
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<tr>
<td>( T_{mh} )</td>
<td>Horizontal temperature measurement at middle plane, ([\text{K}])</td>
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<tr>
<td>( T_{mv} )</td>
<td>Vertical temperature measurement at middle plane, ([\text{K}])</td>
</tr>
<tr>
<td>( T_{hh} )</td>
<td>Horizontal temperature measurement at hot side plane, ([\text{K}])</td>
</tr>
<tr>
<td>( T_{hv} )</td>
<td>Vertical temperature measurement at hot side plane, ([\text{K}])</td>
</tr>
<tr>
<td>( t )</td>
<td>Time, ([\text{s}])</td>
</tr>
<tr>
<td>( U )</td>
<td>Utilization, [-]</td>
</tr>
<tr>
<td>( V )</td>
<td>Volume, ([\text{m}^3])</td>
</tr>
<tr>
<td>( V^* )</td>
<td>Relative fluid displacement, [-]</td>
</tr>
<tr>
<td>( W )</td>
<td>Power, ([\text{W}])</td>
</tr>
<tr>
<td>( x )</td>
<td>Axial position coordinate, ([\text{m}])</td>
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<tr>
<td>( X )</td>
<td>Unspecified local variable</td>
</tr>
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**Greek symbols**

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<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>( \alpha )</td>
<td>Specific surface area, ([\text{m}^{-1}])</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Normalized flow rate weighted average of the specific heat, [-]</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Dimensionless parameter for analytical model, [-]</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>Difference</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>Porosity, [-]</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>Heat transfer effectiveness, [-]</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Dimensionless temperature, [-]</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Dimensionless fluid mass flow rate, [-]</td>
</tr>
<tr>
<td>( \xi )</td>
<td>Dimensionless axial position coordinate, [-]</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Density, ([\text{kg m}^{-3}])</td>
</tr>
<tr>
<td>( \tau )</td>
<td>Period time, ([\text{s}])</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Dimensionless time, [-]</td>
</tr>
<tr>
<td>( \psi )</td>
<td>Phase angle of dimensionless adiabatic temperature change, ([\text{rad}])</td>
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</tbody>
</table>

**Subscripts**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>( a )</td>
<td>Binding agent</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>Variable amplitude in first-order expansion</td>
</tr>
<tr>
<td>( ad )</td>
<td>Adiabatic</td>
</tr>
<tr>
<td>( app )</td>
<td>Applied field</td>
</tr>
<tr>
<td>( avg )</td>
<td>Area-weighted average</td>
</tr>
<tr>
<td>( c )</td>
<td>Cold end or cold-to-hot blow</td>
</tr>
<tr>
<td>( Curie )</td>
<td>Curie point</td>
</tr>
<tr>
<td>( disp )</td>
<td>Dispersion</td>
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</table>
1. Introduction

Thermal regenerators are heat exchangers that also store heat for alternate indirect exchange of heat between hot and cold reservoirs. As a component, a regenerator is critical in implementing renewable energy generation, energy storage, and energy-efficient technologies toward carbon-neutral societies [1,2]. Regenerators have been widely applied in a variety of industries: 1) air treatment and engine waste heat recovery [3–5], 2) turbine applications [6,7], 3) pulse-tube cryocoolers and thermoacoustic engines [8–10], and 4) novel caloric solid-state heating and cooling [11–13]. Operating frequencies (from near 0 to 500 Hz [14,15]), working types (rotary or fixed-bed), and matrices (passive or functional materials) vary greatly for a range of applications. Periodic-flow fixed-bed regenerators, which are also in the scope of this study, have been commonly employed in the aforementioned industries with shared: 1) control equation architectures and modelling methodologies [16,17], 2) irreversible elements [18–20], and 3) thermodynamic cycles [21–23]. Design concepts and characteristic analyses of regenerators can be inherited from different applications.

An active caloric regenerator (ACR) is a thermal regenerator made of a caloric material that is subjected to a time-varying field. By carefully designing the cycle, an ACR can be used in caloric heating and cooling devices as well as waste heat power generation [24]. Such devices are regarded as promising non-vapor-compression alternatives from energy-saving and environmental perspectives [25,26]. The change in thermal energy in the material is caused by an externally applied field that drives the caloric effect. Depending on the applied field, the effect may be magnetocaloric [26–28], elastocaloric [29–31], barocaloric/mechanocaloric [32–34], or electrocaloric [35–37]. There is a wide range of caloric materials available [38], and each undergoes a temperature change induced by a varying relevant applied field. The maximum temperature change is in the vicinity of an ordering transition temperature. As the caloric effect is often insufficiently strong to meet the demands of cooling devices at room temperature, constructing an ACR by combining heat regeneration with the caloric effect allows the temperature span of the device to greatly exceed the adiabatic temperature change of caloric materials [39]. The ACR consists of a porous medium of caloric materials with voids through which an oscillating heat transfer fluid (HTF) flow synchronized with the periodically changing field.

A representative thermodynamic cycle for an ACR operation, e.g., a Brayton cycle, is depicted in Fig. 1. After a number of operating cycles, a cyclic steady state with a longitudinal temperature gradient builds up
within both the caloric material and the HTF. Generally, passive regeneration allows the HTF to span the temperature gap, while active regeneration causes a temperature change in response to a varying applied field, allowing cooling loads to be accepted at the cold end and subsequently rejected at the hot end. Consequently, active regenerators exhibit certain characteristics different from those of conventional passive regenerators: 1) temperature-and field-dependent specific heat [40,41] and possibly multi-layering with different caloric materials all operating near their transition temperatures [42,43], and 2) dynamic internal heat source caused by the caloric effect or even multi-caloric integration [11,44].

Fig. 1 Illustration of the thermodynamic cycle for magnetocaloric heating and cooling systems. The cycle consists of four individual steps: 1) magnetization, 2) cold-to-hot blow – transferring heat to heat sink, 3) demagnetization; 4) hot-to-cold blow – absorbing heat from heat source. The magnetic field is analogous to uniaxial stress, hydrostatic pressure, or electric field for elastocaloric, barocaloric, and electrocaloric systems, respectively.

The design of ACRs requires not only general heat transfer theory, such as $\varepsilon$-$NTU$ [45–47] or $\Lambda$-$II$ [48,49], but also knowledge of complex interactions of solid-fluid convection (conjugate heat transfer) [50,51], caloric material properties [52–54], matrix geometries [55–57] and dimensions [58,59], operating parameters [60,61], as well as applied field and hydraulic flow waveform synchronization [62–66]. The fluid and matrix temperature profiles, influenced by the aforementioned physical interactions, are closely linked to regeneration efficiency and irreversibilities [67]. Apart from simply imposing the linear temperature profile assumption [68], accurate temperature profiles were investigated both experimentally and numerically to better understand the nuances of ACR systems. He et al. [69] experimentally compared the time evolution of temperature distributions in packed bed magnetocaloric regenerators subjected to series, parallel, and cascade cycle modes. Nonlinear temperature profiles were identified due to the nonlinear term for magnetocaloric heating, which is associated with the temperature- and field-dependent properties of caloric materials. Borbolla [70] also observed nonlinear temperature profiles of fluid and solid phases along the flow direction for magnetocaloric regenerators. However, the internal temperature dynamics of regenerators are often difficult to capture experimentally (particularly for microchannel monoliths), and detailed numerical models may be opaque, making it difficult to define figures of merit. An analytical solution would advance the fundamental understanding of the underlying physics and
working principles of regenerators, as well as provide a powerful tool for system design. The temperature profile can be modelled analytically using energy equations together with a few simplifying assumptions. The strategy of regenerator analytical models focuses on reducing the temporal and spatial dimensions of fluid and matrix temperatures as follows:

(1) Local thermal equilibrium assumption: solid and fluid temperatures are weighted averaged for reducing the two energy equations to only one partial differential equation [71,72].
(2) Thermal non-equilibrium equivalence: when the phase difference between fluid and solid is critical [73,74], an equivalent thermal conductivity can be defined to account for finite conjugate heat transfer and other parasitic losses without affecting the temperature profile solution [75].
(3) Scale analysis: certain terms that are several orders of magnitudes smaller than the others, e.g., the term of thermal mass rate of entrained gaseous HTF [76,77], may be omitted.
(4) Variable transformation: mathematical manipulations, such as the Taylor series for step-wise waveforms [78], Fourier series for sinusoidal waveforms [79], Lagrange frame of reference [80] or Laplace transform [81], may separate temporal and spatial variables.

Although analytical solutions to regenerator temperature profiles have been successfully implemented in various areas, only a few analytical studies have been devoted to active regenerators. In caloric heating and cooling, the additional term of the caloric effect is associated with the applied field waveforms that are generally regarded as nearly step-wise [63], sinusoidal [82], or rectified sinusoidal functions [83]. Rowe [71,72] used a local thermal equilibrium assumption based on a step-wise function of the magnetic field to derive explicit expressions for magnetic work and net heat flux, as well as simplified partial differential equations for the temperature profile. Burdyny and Rowe [84] improved this simplified model by including thermal non-equilibrium equivalence to account for finite conjugate heat transfer. Later, this simplified model was extended for detailed modelling of magnetic field waveforms [85] and multi-layer magnetocaloric materials [86]. Sebald et al. [87] developed an analytical model of the temperature profile for coplanar infinite layers made of natural elastocaloric rubber subjected to sinusoidal waveforms of applied load and fluid flow. The authors predicted the cooling power and temperature span for varying frequencies, fluid and material layer thickness, fluid displacement amplitudes, as well as material and fluid thermal conductivities. Although previous studies allow an almost systematic design approach, the scientific community still has several unanswered questions concerning the complex physical interactions within active regenerators. Developing easy-to-manage models is highly desirable to visualize the effect of each parameter on the temperature profile and ACR performance.

This analytical approach offers a systematic investigation of complex regenerative heat exchanger systems and a detailed understanding of the design parameters that are vital for developing caloric heating and cooling prototypes. Instead of solving all the transients simultaneously, a harmonic analysis, as inspired by Refs. [88,89], is applied to show the relationships between temperature average and amplitude using a complex notation. The solutions of the control equations in a sinusoidal waveform are produced by the assumed harmonics of temperature, magnetic field, and fluid flow. For cycle-average temperature profiles, a simplified linear ordinary equation and its explicit estimation are obtained using the definition of temperature effectiveness of active regenerators. By upgrading our passive regenerator tester, an embedded thermocouple grid has been implemented for detailed internal temperature measurement of packed bed regenerators. The analytical model data is validated and compared with the experimental and numerical results. Based on this analytical model, important figures of merit, such as the net enthalpy flux, applied
field work, and individual components of entropy generation, are determined for designing new system components.

2. Analytical model

The analytical model is derived from a one-dimensional (1D) numerical model, as described in [90], with a general caloric term adapted from a magnetocaloric heat source term. The following are the steps in the solution process:

(1) By performing a harmonic analysis on the energy equations, we reveal the quantitative and explicit mapping between local temperature amplitude and phasing, as well as the cycle-average temperature profile. Some simplified assumptions from passive regenerators are inherited, followed by post-corrections.

(2) By temporally integrating the energy equations, we simplify the equations for cycle-average temperature profiles and obtain explicit estimations based on caloric materials with weak temperature dependence.

To facilitate the following deduction, temporal and spatial integral averages are defined as:

\[
\overline{X} = \begin{cases} 
\int_0^1 Xdt, & \text{dimensional} \\
\int_0^{2\pi} Xd\phi, & \text{dimensionless}
\end{cases}
\]  
\[
\langle X \rangle = \begin{cases} 
\int_0^L Xdx, & \text{dimensional} \\
\int_0^1 Xd\xi, & \text{dimensionless}
\end{cases}
\]

where \(t, x, f, L_r\) are the time, position coordinate, cycle frequency, and regenerator length, respectively. \(X\) is a local variable representing temperature or material properties.

2.1. Dimensionless numerical model and harmonic analysis

The analytical model is simply an explicit estimation of a 1D model’s control equations. The energy equations for caloric material and HTF phases are derived in Eq. 3 and Eq. 4 by omitting the viscous dissipation term and applying non-dimensional parameters. The non-dimensional parameters are summarized in Table 1. The objective for applying a dimensionless analysis is to reveal the oscillating thermal energy transfer more readily. Furthermore, the time variation in the force field is transformed into a temperature variation.

\[
\frac{2\pi \theta_{\text{ad}}}{\partial \phi} = \frac{2\pi \theta_f}{\partial \phi} - k f \frac{\partial^2 \theta_f}{\partial \xi^2} - NTU(\theta_f - \theta_s)
\]  
\[
\begin{align*}
\text{Caloric heat} & \quad \text{Heat storage} & \quad \text{Heat conduction} & \quad \text{Heat convection} \\
2\pi \theta_f & = \frac{2\pi \theta_f}{\partial \phi} + k f \frac{\partial^2 \theta_f}{\partial \xi^2} - NTU(\theta_f - \theta_s)
\end{align*}
\]

In this study, \(x = 0\ (\xi = 0)\) and \(x = L_r\ (\xi = 1)\) denote the hot side and the cold side, respectively. The flow rate is negative during the cold-to-hot blow and positive during the hot-to-cold blow.
Table 1 Non-dimensional parameters for active caloric regenerators.

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<thead>
<tr>
<th>Items</th>
<th>Descriptions</th>
<th>Equations</th>
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<tbody>
<tr>
<td>Local utilization</td>
<td>A 1D and transient parameter that links the thermal capacity of fluid displaced via the regenerator to the temperature-varying thermal capacity of the solid matrix.</td>
<td>( U(\xi, \phi) = \frac{\rho_f V_f}{(1 - \epsilon) \rho_s \alpha_s(\xi, \phi) V_f} ) Eq. 5</td>
</tr>
<tr>
<td>Relative fluid displacement</td>
<td>The volumetric ratio of the displaced fluid to the entrained fluid [91,92].</td>
<td>( V^* = \frac{V_p \epsilon}{V_f} ) Eq. 6</td>
</tr>
<tr>
<td>Number of transfer unit (NTU)</td>
<td>The ratio of overall conductance to the fluid heat capacity rate. The effective heat transfer coefficient ( h_{eff} ) is applied considering boundary layer effects and longitudinally internal temperature gradient in the matrix [93]. Due to less variation in laminar flows, ( h_{eff} ) is averaged both temporally and spatially.</td>
<td>( NTU = \frac{(k_{eff}) \alpha V_r}{V_f \rho \rho_f} ) Eq. 7</td>
</tr>
<tr>
<td>Dimensionless solid thermal conductivity</td>
<td>Non-dimensional parameter of the heat conduction term in the solid energy equation. The static conductivity ( k_{stat} ) is applied considering matrix geometries and solid-fluid contact [94]. Thus, ( k_{stat} ) is more dependent on geometrical parameters than operational parameters [28] and can be assumed to be evenly distributed along an ACR.</td>
<td>( \tilde{k}<em>s = \frac{k</em>{stat} \alpha V_r}{V_p \rho_f V_f} ) Eq. 8</td>
</tr>
<tr>
<td>Dimensionless fluid thermal conductivity</td>
<td>The definition is similar to ( \tilde{k}<em>s ). The dispersion conductivity ( k</em>{disp} ) is applied considering the microscopic mixing effects. As ( k_{disp} ) is dependent on fluid velocity [94], a temporal average is applied.</td>
<td>( \tilde{k}<em>f = \frac{k</em>{disp} \alpha V_r}{V_p \rho_f V_f} ) Eq. 9</td>
</tr>
<tr>
<td>Dimensionless flow rate</td>
<td>The instant mass flow rate of the HTF normalized by the cycle-average mass flow rate.</td>
<td>( \lambda = \frac{\dot{m}_f}{\dot{m}_f</td>
</tr>
<tr>
<td>Dimensionless axial position</td>
<td>The axial coordinate normalized by the regenerator length.</td>
<td>( \xi = \frac{x}{L_r} ) Eq. 11</td>
</tr>
<tr>
<td>Dimensionless time</td>
<td>Conversion of time to phase angle in a cyclic waveform.</td>
<td>( \phi = 2\pi ft ) Eq. 12</td>
</tr>
<tr>
<td>Dimensionless temperature</td>
<td>Local temperature normalized by regenerator temperature span.</td>
<td>( \theta = \frac{T - T_c}{T_h - T_c} ) Eq. 13</td>
</tr>
<tr>
<td>Dimensionless adiabatic temperature change</td>
<td>Local adiabatic temperature transient of caloric material normalized by temperature span</td>
<td>( \Delta \theta_{ad} = \int_{T_h - T_c}^{\theta} \left( \frac{\partial h}{\partial T} \right) dT ) Eq. 14</td>
</tr>
</tbody>
</table>

The temperature field, as understood qualitatively, contains a time average value at each location, with a periodically time-varying component superimposed on it. We assume that the time-varying component of the temperature field is synchronized to the displacement fluid thermal capacity and 90 degrees phase offset from the displacement velocity. Thus, the dimensionless flow rate, temperatures, and adiabatic temperature change can be expanded using the first-order Fourier series shown below:

\[ \lambda = Re\left(\pi e^{i\phi}\right) \quad \text{Eq. 15} \]
\[ \theta_s(\xi, \phi) = \bar{\theta}_s(\xi) + \text{Re}[\theta_{s, a1}(\xi)e^{i\phi}] \]  
Eq. 16

\[ \theta_f(\xi, \phi) = \bar{\theta}_f(\xi) + \text{Re}[\theta_{f, a1}(\xi)e^{i\phi}] \]  
Eq. 17

\[ \theta_{ad}(\xi, \phi) = \text{Re}[\theta_{ad, a1}(\xi) + \frac{1 + e^{i\phi}}{2}] \]  
Eq. 18

Where \( i = \sqrt{-1} \), and subscript \( a1 \) denotes the amplitude of the first-order term in the Fourier series. The phase offsets of solid and fluid temperatures relative to the harmonic phase of fluid flow are embodied in the complex numbers of \( \theta_{s, a1}(\xi) \) and \( \theta_{f, a1}(\xi) \), respectively. As \( \theta_{ad} \) is theoretically in phase with the applied field, the complex number \( \theta_{ad, a1}(\xi) \) indicates the phase offset of the applied field relative to the harmonic phase of the temperature field. According to Eq. 14, the relations of \( \theta_{ad, a1} \) and \( \Delta T_{ad} \) are derived as follows:

\[ \Delta T_{ad}(F_i \to F_f, \bar{T}(\xi)) = |\theta_{f, a1}(\xi)| (T_h - T_c) \]  
Eq. 19

\[ \frac{\Delta T_{ad,c}(\Delta F(\phi = 0 \to \pi), \bar{T}(\xi))} = (T_h - T_c) \frac{\int_0^\pi \text{Re}[\theta_{f, a1}(\xi)e^{i\phi}]d\phi}{\pi} = (T_h - T_c)\theta_{f, a1,c} \]  
Eq. 20

where \( \Delta T_{ad} \) and \( \Delta T_{ad,c} \) define the adiabatic temperature change based on the maximum applied field variation and local temperature, as well as the adiabatic temperature change based on average applied field variation during a cold-to-hot blow (in high field) and local temperature, respectively. The schematics of the phasing (argument of complex number) of \( \theta_{s, a1}, \theta_{f, a1} \) and \( \theta_{ad, a1} \) are shown in Fig. 2.

Fig. 2 Schematics of the argument of the complex number for sinusoidal oscillations of non-dimensionless solid temperature, fluid temperature, and adiabatic temperature change.

Eq. 15- Eq. 18 are substituted into the energy equations Eq. 3 and Eq. 4 with the following rearrangement:
\[
e^{i\phi} \left( \frac{2\pi \theta_{s,at}}{U} - \frac{\pi \theta_{s,at}}{U} - k_s \frac{\partial^2 \theta_{s,at}}{\partial \xi^2} - NTU(\theta_{f,at1} - \theta_{s,at1}) \right)^2 = \bar{k}_s \frac{\partial^2 \bar{\theta}_s}{\partial \xi^2} + NTU(\bar{\theta}_f - \bar{\theta}_s) \quad \text{Term 1s}
\]

\[
e^{i\phi} \left( \frac{2\pi \theta_{f,at1}}{V^*} - k_f \frac{\partial^2 \theta_{f,at1}}{\partial \xi^2} + \pi \bar{\theta}_f \frac{\partial \bar{\theta}_f}{\partial \xi} + NTU(\theta_{f,at1} - \theta_{s,at1}) \right)^2 = \bar{k}_f \frac{\partial^2 \bar{\theta}_f}{\partial \xi^2} - NTU(\bar{\theta}_f - \bar{\theta}_s) + \bar{e}^{i\phi} \quad \text{Term 1f}
\]

The terms 1s, 2s, 1f, 2f, and 3f are functions of only \( \xi \). The only way to satisfy these equations is for all terms to equal zero. Note that the substitution of real number type Eq. 15 - Eq. 18 can be modified by its corresponding complex type in the solution of Eq. 21 and Eq. 22, as neither multiplication nor division operations of complex numbers were involved in the following solutions in Eq. 23 - Eq. 27.

Term 2s: \( \bar{k}_s \frac{\partial^2 \bar{\theta}_s}{\partial \xi^2} + NTU(\bar{\theta}_f - \bar{\theta}_s) = 0 \)  
Eq. 23

Term 3f: \( \bar{k}_f \frac{\partial^2 \bar{\theta}_f}{\partial \xi^2} - NTU(\bar{\theta}_f - \bar{\theta}_s) = 0 \)  
Eq. 24

Term 2f: \( \frac{\partial \theta_{f,at1}}{\partial \xi} = 0 \)  
Eq. 25

Term 1s: \( \frac{2\pi \theta_{s,at1}}{U} - \frac{\pi \theta_{s,at1}}{U} - k_s \frac{\partial^2 \theta_{s,at1}}{\partial \xi^2} - NTU(\theta_{f,at1} - \theta_{s,at1}) = 0 \)  
Eq. 26

Term 1f: \( \frac{2\pi \theta_{f,at1}}{V^*} - k_f \frac{\partial^2 \theta_{f,at1}}{\partial \xi^2} + \frac{\pi \bar{\theta}_f}{\partial \xi} + NTU(\theta_{f,at1} - \theta_{s,at1}) = 0 \)  
Eq. 27

Preliminary assumptions are made as follows to proceed with the harmonic analysis.

(a) Cycle-average local utilizations \( \bar{U}(\xi) \) are used in Eq. 21 and Eq. 26.
(b) Cycle-average temperatures of solid and fluid are identical [79],
\[
\bar{\theta}_s(\xi) = \bar{\theta}_f(\xi) = \bar{\theta}(\xi)
\]
Eq. 28
(c) Axial profiles for cycle-average temperatures are linear. The linear time-average temperature profiles commonly seen in passive regenerators resulted from direct assumption [95] or equation deduction [88]. The linear assumption in this model can be derived from the aforementioned assumption (b) combined with Eq. 23 and Eq. 24.
\[
\frac{\partial^2 \bar{\theta}_s}{\partial \xi^2} = \frac{\partial^2 \bar{\theta}_f}{\partial \xi^2} = 0
\]
Eq. 29
(d) \( \theta_{s,at1} \) and \( \theta_{f,at1} \) are weakly dependent on \( \xi \),
\[
\frac{\partial \theta_{s,at1}}{\partial \xi} = \frac{\partial \theta_{f,at1}}{\partial \xi} \approx 0
\]
Eq. 30
From Eq. 25 and Eq. 29, both \( \theta_{f,at1} \) and \( \frac{\partial \bar{\theta}_f}{\partial \xi} \) are independent of \( \xi \). Subsequently, \( \theta_{f,at1} \) and \( \theta_{s,at1} \) are independent of \( \xi \) in Eq. 27, therefore Eq. 30 is satisfied. In the following, we can preliminarily neglect \( \frac{\partial \theta_{s,at1}}{\partial \xi} \).
and \( \frac{\partial \theta_{s,a1}}{\partial \xi} \) during the harmonic analysis, and post correct the dependence of \( \theta_{s,a1} \) and \( \theta_{f,a1} \) on \( \xi \) in the final solutions. Thus, Eq. 26 and Eq. 27 can be written as:

\[
\frac{2\pi i \theta_{s,a1}}{\theta_s(\xi)} - \frac{\pi i \theta_{ad,a1}}{\theta_s(\xi)} - NTU(\theta_{f,a1} - \theta_{s,a1}) = 0 \quad \text{Eq. 31}
\]

\[
\frac{2\pi i \theta_{f,a1}}{\nu^*} + \frac{\pi i \theta_{ad,a1}}{\nu^*} + NTU(\theta_{f,a1} - \theta_{s,a1}) = 0 \quad \text{Eq. 32}
\]

The vanishing heat conduction terms imply that the axial heat conduction would violate the first-order harmonics. Eq. 31 and Eq. 32 are linear algebraic equations, which can be solved explicitly.

\[
\theta_{s,a1} = \frac{1}{2} \frac{\partial \overline{\theta}_s}{\partial \xi} + \frac{1}{2} \theta_s(0,0)
\]

\[
\theta_{f,a1} = \frac{1}{2} \frac{\partial \overline{\theta}_f}{\partial \xi} + \frac{1}{2} \theta_f(0,0)
\]

\[
\gamma_{s1} = \frac{V^* \cdot \overline{U} \cdot NTU}{NTU(V^* + \overline{U}) + 2\pi i}
\]

\[
\gamma_{s2} = \frac{V^* NTU}{NTU(V^* + \overline{U}) + 2\pi i}
\]

\[
\gamma_{f1} = \frac{V^* \overline{U} \cdot NTU}{NTU(V^* + \overline{U}) + 2\pi i}
\]

\[
\gamma_{f2} = \frac{V^* \cdot \overline{U} \cdot NTU}{NTU(V^* + \overline{U}) + 2\pi i}
\]

When Eq. 33 and Eq. 34 are substituted into Eq. 16 and Eq. 17, the relations between the solid and fluid temperatures, as well as the cycle-average temperature profile are explicit. The complex parameters \( \theta_{s,a1} \) and \( \theta_{f,a1} \), which indicate the oscillating amplitudes and phasing, are comprised of the effects of the passive terms \( (\gamma_{s1}, \gamma_{f1} \text{ and } \frac{\partial \overline{\theta}_s}{\partial \xi}) \) and the active terms \( (\gamma_{s2}, \gamma_{f2} \text{ and } \theta_{ad,a1}) \). Note that \( V^* \) and \( NTU \) are \( \xi \)-independent due to the definitions in Table 1, while \( \overline{U} \) and \( \theta_{ad,a1} \) are post corrected as \( \xi \)-dependent and so are the parameters \( \theta_{s,a1} \) and \( \theta_{f,a1} \).

**2.2. Cycle-average temperature profile**

From Eq. 33 and Eq. 34, the cycle-average temperature profile \( \overline{\theta} \) has not been solved by harmonic analysis, which requires the boundary conditions. The boundary values \( \overline{\theta}(0) \) and \( \overline{\theta}(1) \) can be estimated by referring to the heat transfer effectiveness. The regenerator effectiveness is defined in terms of the thermal ratios, specifically the actual heat exchange to the theoretical heat exchange. The effectiveness of an ACR can be defined as the actual cooling capacity relative to the maximum, with zero thermal capacity of fluid \( (U = 0) \) and negligible thermal losses [96]. The ACR effectiveness is redefined in a temperature-based form [47] to accommodate the solutions of \( \overline{\theta}(0) \) and \( \overline{\theta}(1) \).

\[
\epsilon_c \triangleq \frac{2\pi \int_0^{\pi/2} \overline{\theta}_{f,a1} dt - \epsilon_c}{(T_s - T_c) + \left| \frac{\partial \overline{\theta}_{ad,a1}}{\partial \phi} \right| (1 + (\overline{U}/V^*))} \quad \text{Eq. 39}
\]

\[
= \frac{1/\pi \int_0^{\pi/2} \theta_{f,a1} d\phi (\xi = 0,\phi) d\phi}{1 + \left| \frac{\partial \overline{\theta}_{ad,a1}}{\partial \phi} \right| (1 + (\overline{U}/V^*))}
\]
The aforementioned assumptions (a) and (c) are no longer applied in the analysis of the ACR. Applying the assumptions of balanced and symmetrical flows, was applied to approximate the temperature profile is derived as follows, which is consistent with the results in Ref. [78].

\[
\varepsilon_h = \frac{T_h - \frac{2}{\pi} \int_0^{\pi} T_f, dt}{(T_h - T_c) + \left| \frac{\Delta T_{ad,h}}{1 + \langle (\mathcal{U})/V \ast \rangle} \right| - 1 - \frac{1}{\pi} \int_0^{\pi} \theta(\xi = 1, \phi)d\phi} = \frac{1 - \frac{1}{\pi} \int_0^{\pi} \theta(\xi = 1, \phi)d\phi}{1 + \left| \frac{\theta_f,a_{1,h}}{1 + \langle (\mathcal{U})/V \ast \rangle} \right|}
\]

The denominators in the above temperature effectiveness \( \varepsilon_c \) (cold-to-hot blow) and \( \varepsilon_h \) (hot-to-cold blow) are related to the maximum heat exchange, which consists of the obtainable heat transfer associated with the caloric effect (dictated by \( \left| \frac{\Delta T_{ad}}{1 + \langle (\mathcal{U})/V \ast \rangle} \right| \) \[97\]) and the heat exchange between hot and cold reservoirs (dictated by \( (T_h - T_c) \)). The term \( \left| \frac{\Delta T_{ad}}{1 + \langle (\mathcal{U})/V \ast \rangle} \right| \) is based on the assumption that the HTF temperatures always follow the solid temperatures exactly with no thermal energy dissipation during the (de)magnetization processes. \( \langle \Delta T_{ad,c} \rangle \) and \( \langle \theta_f,a_{1,c} \rangle \) are spatial average values (Eq. 2) of \( \Delta T_{ad,c} \) and \( \theta_f,a_{1,c} \) defined in Eq. 19 and Eq. 20, respectively. Similar definitions are applied to \( \langle \Delta T_{ad,h} \rangle \) and \( \langle \theta_f,a_{1,h} \rangle \) during the hot-to-cold blow. From Eq. 39 and Eq. 40, the boundary values (\( \bar{\theta}(0) \) and \( \bar{\theta}(1) \)) and the temperature effectiveness (\( \varepsilon_c \) and \( \varepsilon_h \)) can be connected.

\[
\bar{\theta}(0) = \frac{1}{2\pi} \left( \int_0^\pi \theta(\xi = 0, \phi)d\phi + \int_\pi^{2\pi} \theta(\xi = 0, \phi)d\phi \right)
\]

\[
\varepsilon_c \left[ 1 + \left| \frac{\theta_f,a_{1,c}}{1 + \langle (\mathcal{U})/V \ast \rangle} \right| \right] + 1
\]

\[
= \frac{0.5 + NTU}{1 + NTU} + \frac{0.5 \cdot NTU}{1 + NTU} \cdot \left| \frac{\theta_f,a_{1,c}}{1 + \langle (\mathcal{U})/V \ast \rangle} \right|
\]

\[
\bar{\theta}(1) = \frac{1}{2\pi} \left( \int_0^\pi \theta(\xi = 1, \phi)d\phi + \int_\pi^{2\pi} \theta(\xi = 1, \phi)d\phi \right)
\]

\[
1 - \varepsilon_c \left[ 1 + \left| \frac{\theta_f,a_{1,h}}{1 + \langle (\mathcal{U})/V \ast \rangle} \right| \right]
\]

\[
= \frac{0.5}{1 + NTU} - \frac{0.5 \cdot NTU}{1 + NTU} \cdot \left| \frac{\theta_f,a_{1,h}}{1 + \langle (\mathcal{U})/V \ast \rangle} \right|
\]

Hot and cold side boundary conditions were set as follows. In a cold-to-hot blow with \( \phi = 0 \rightarrow \pi \), \( \theta_f \) was assumed to be unity at the inlet. In a hot-to-cold blow with \( \phi = \pi \rightarrow 2\pi \), the fluid entering the cold side was assigned to have zero \( \theta_f \). The empirical correlation of \( \varepsilon_c = \varepsilon_h = NTU/(1 + NTU) \) \[48\], which is based on the assumptions of balanced and symmetrical flows, was applied to approximate Eq. 41 and Eq. 42. For passive regenerators with \( \theta_{ad,a1} = 0 \) and linear profile (assumption (c) above, Eq. 29), the cycle-average temperature profile is derived as follows, which is consistent with the results in Ref. [78].

\[
\bar{\theta} = \frac{0.5 + NTU(1 - \ell)}{1 + NTU} \quad \text{Eq. 43}
\]

For active regenerators, the cycle-average temperature profile is not linear, as will be seen in Section 4. The aforementioned assumptions (a) and (c) are no longer applied in the analysis of \( \bar{\theta} \) of the ACR. Applying a cyclic integration shown in Appendix A, a linear second-order ordinary differential equation is derived for \( \bar{\theta} \).

\[
\frac{d^2 \bar{\theta}}{d\xi^2} - \gamma_1 \frac{d\bar{\theta}}{d\xi} + \gamma_2 \bar{\theta} = 0 \quad \text{Eq. 44}
\]
The explicit solution of $\bar{T}$ of active regenerators can be obtained by combining Eq. 41 - Eq. 42 and Eq. 44 - Eq. 46. For a caloric material with weak temperature dependence, a simplified expression for $\bar{T}$ is deduced in Appendix A. Consequently, the analytical model is fully defined by linking the expressions of $\bar{T}$ and the harmonic solutions for solid and fluid temperatures (Eq. 16, Eq. 17, Eq. 33 and Eq. 34). Compared to previous analytical models, this model features (1) a nonlinear cycle-average temperature profile, (2) various phasing synchronization for fluid flow, applied field, and temperatures of solid and fluid, as well as (3) temperature-dependent caloric effect and solid-phase specific heat. As the model is derived as a function of adiabatic temperature change and solid specific heat, it adapts to any ACR where hysteresis in the material is negligible.

2.3. Deduction for design configuration

With quantified knowledge of the regenerator temperature profile in relation to heat transfer, hydraulic flow, and caloric effect and their phasing, some design parameters are exemplified based on the analytical model. Similar to the deduction process in Eq. A6, net enthalpy flux (advection) through a cross-sectional plane at a location $x$ ($\dot{Q}_x$) can be derived, and therefore total work of applied field ($W_{app}$) is deduced.

$$\dot{Q}_x = \int \hat{f} \cdot \hat{\Psi} \cdot dh \cdot dt = \int_0^{2\pi} \int_0^{\infty} \frac{\rho \cdot V \cdot \hat{\Psi} \cdot (T_h - T)}{2\pi L} \cdot \hat{f} \cdot \hat{\Psi} \cdot d\phi \cdot \hat{f} \cdot \hat{\Psi} \cdot V \cdot \hat{\Psi} \cdot Re \cdot \left[i \gamma \cdot \int L \cdot \frac{\partial \hat{T}}{\partial x} - i \gamma \cdot \int \frac{\partial \hat{T}}{\partial x} \right]$$

$$W_{app} = \int \left( k_{stat} + k_{disp} \right) A \cdot c_r \cdot \frac{\partial \hat{T}}{\partial x} \right) dx$$

In Eq. 47 with a sinusoidal waveform of the applied field, the net enthalpy flux is a function of $U, V, \frac{\partial T}{\partial x}$ and $\Delta T_{ad}$, which is consistent with the expressions for a step-wise applied field in Ref. [71]. As finite conjugate heat transfer and various synchronizations between the applied field and fluid flow are assumed, $Q_x$ is also a function of NTU and $\psi = Arg(\theta_{ad,a1})$. Based on a local energy balance shown in Ref. [71], the work per unit length ($\frac{dW_{app}}{dx}$) is assumed as the derivative of net enthalpy flux, subtracting the axial conduction losses in Eq. 48. According to Eq. 47 and Eq. 48, optimizing the trade-off between heat transfer (NTU), flow configuration ($U$ and $V$), heat regeneration ($\frac{\partial T}{\partial x}$), caloric effect ($\Delta T_{ad}$) and phasing ($\psi$) to maximize $\dot{Q}_x$ and $W_{app}$, is one of the effective ways to design ACRs.

Combining Eq. 13, Eq. 33 and Eq. 34, as well as assuming infinite conjugate heat transfer, the local temperature difference ($\Delta T$) is simplified as follows.

$$\Delta T_{NTU} \rightarrow \infty = 2|\theta_{s,f,a1}|_{NTU} \rightarrow \infty = \frac{V^*}{V^* + V^* + V^* + V^*} \left( (\Delta T_{ad})^2 + (L \cdot \frac{\partial T}{\partial x})^2 \right)$$

According to Eq. 49, $\Delta T$ for active regenerators increases with the intensity of the caloric effect ($\Delta T_{ad}$), while $\Delta T$ for passive regenerators ($\Delta T_{ad} = 0$) increases with $U$. The local temperature difference is an important input for some design figures of merit, such as local entropy transfer rate [96] and normalized latent heat [98].

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Entropy generation minimization (EGM) is one of the design tools for regenerator efficiency [99,100], which is mathematically the minimization of the total entropy generation as the objective function [101]. Evaluating the irreversibility requires accessing the internal characteristics, which is experimentally challenging. Apart from numerical approaches [90,102,103], the analytical model presented above provides explicit expressions for entropy generation associated with conjugate heat transfer ($S_{g,ht}$) and axial conduction ($S_{g,cond}$).

\[
S_{g,ht} = \frac{V \rho c_p NTU}{2\pi} \int_0^{2\pi} \int_0^1 \frac{\left(\theta_s - \theta_f\right)^2}{\theta_s + T_c / (T_h - T_c)} \left[\theta_s + T_c / (T_h - T_c)\right] \, d\xi \, d\phi \tag{Eq. 50}
\]

\[
S_{g,hc} = \frac{\rho c_p \nu}{2\pi} \int_0^{2\pi} \int_0^1 \left[\frac{k_s (\partial \theta_s / \partial \xi)^2}{\theta_s + T_c / (T_h - T_c)} + \frac{k_f (\partial \theta_f / \partial \xi)^2}{\theta_f + T_c / (T_h - T_c)}\right] \, d\xi \, d\phi \tag{Eq. 51}
\]

Total entropy generation can be determined based on Eq. 50 and Eq. 51. Therefore, the second-law figures of merit, such as entropy generation number [104–106], will be available for the ACR design.

3. Experimental setup

The design of the passive regenerator tester has been previously described [51]. Figure 3 shows the instrumentation, which consists of 1) a regenerator assembly to accommodate the regenerator test specimen, 2) a heater and heat exchanger to adjust reservoir temperatures, 3) a piston displacer and motor assembly to generate the fluid flow, and 4) check valves to direct the fluid oscillation. The measurement of the temperature distribution of packed-bed regenerators subjected to a sinusoidal fluid waveform is novel to this study.

The basic techniques for temperature distribution measurements in a porous structure are mainly thermocouple thermometry and infrared thermography, each with its own set of constraints. In the present work, the materials used for the housing and insulation are not transparent enough for infrared imaging [107]. The use of thermocouples, which is regarded as an invasive instrumentation [108], may result in temperature [109] and velocity [110] field disturbance. In general, the induced changes decrease with the number of thermocouples and their diameters while both temporal and spatial resolutions are improved with increased numbers of thermocouples. By minimizing the thermocouple dimensions, even micron [111–113] and nanometer [114–116] resolutions have been achieved previously. Thus, thermocouple thermometry is still adaptable for regenerators with porous geometries. To monitor the interior temperatures, we implement an embedded thermocouple grid, as illustrated in Fig. 3, which provides a good resolution of the temperature profile. The thermocouples are mounted as a single strand across the regenerator housing where the two thermocouple leads are connected with a weld joint at the temperature measurement point inside the regenerator.

Commercial thermocouples (Omega, type E) with a diameter of about 0.25 mm (size for each lead with insulation) are used. The junction bead size is estimated as between 0.25 and 0.5 mm. The uncertainties of temperature measurement are estimated as ±0.3K based on previous work. As all the thermocouples have been calibrated with a thermal bath before measurements, the absolute uncertainties are reduced. Each of the 38 thermocouples is separated into two single wires and stretched out through the designed grooves to secure the junction at a defined location. These grooves are also intended for accommodating the glue. Fourteen points are arranged longitudinally to study the axial temperature transient.
Fig. 3 Schematics of the passive regenerator tester and the embedded thermocouple grid, which consists of 38 internal thermocouples in a customized 3D printed resin housing.

An oscillating flow in regenerators can produce apparent diffusion due to the interaction between bulk fluid flow with the boundary layer [117] when compared to a steady flow. To validate the 1D model, the
test conditions should be one-dimensional, which means that the fluid parameters can be averaged in the cross-section normal to the principal flow direction. As shown in Fig. 3, we selected six cross-sectional planes, which are at both ends and in the middle, inserting four more thermocouples for each to investigate the transverse temperature distribution. An area-weighted average dimensionless temperature ($\theta_{avg}$) is calculated in a cross-sectional area and compared to the corresponding normalized temperature in the center.

$$\theta_{avg} = \frac{\sum \theta_i A_i}{\sum A_i}$$

Eq. 52

where $\theta_i$ is a dimensionless temperature in a cross-sectional area and $A_i$ is the responsible area of the temperature measurement point. From the transverse temperature measurements, the horizontal temperatures in Fig. 4 (a) are evenly distributed, while the vertical temperatures in Fig. 4 (b) show an outlier with the topmost thermocouple ($\theta_{mv5}$) probably due to greater voids at the top caused by poor packing quality. In Fig. 4, the temperatures measured by the thermocouples at the center of the cross-sectional area ($\theta_{mv3}$ and $\theta_{mh3}$) are consistent with the corresponding $\theta_{avg}$, which verifies the 1D conditions of the test. The temperature measurements in the cross sections at both ends for different regenerators are shown in Appendix B. The uniform temperature profiles over the cross section validate the 1D assumption in the model.

As the individual $\theta_f$ and $\theta_s$ in the above analytical model cannot be measured with thermocouples, a volume-weighted average temperature $\theta_m$ was introduced to represent the temperature measurement from thermocouples.

$$\theta_m = (1 - \epsilon)\theta_s + \epsilon\theta_f$$

Eq. 53
4. Results and discussions

To validate the analytical model and study the characteristics of temperature profiles, the analytical results are compared with numerical results from Eq. 3 and Eq. 4 and experimental results. Three materials were experimentally tested: stainless steel, gadolinium, and Calorivac-HS, which is the trade name used by Vacuumschmelze GmbH for the magnetocaloric material of the type La(Fe,Mn,SI)₁₃H₇. The configuration of the three regenerator candidates is summarized in Table 2.

Table 2 Parameters of the packed particle bed regenerators for validation of the analytical model.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>REG-SS</th>
<th>REG-Gd</th>
<th>REG-HS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shape (D×L, mm×mm)</td>
<td>Ø20×64</td>
<td>Ø20×64</td>
<td>Ø20×64</td>
</tr>
<tr>
<td>Particle size (mm)†</td>
<td>1</td>
<td>0.5</td>
<td>0.52</td>
</tr>
<tr>
<td>Bulk porosity (ε₆)</td>
<td>0.48</td>
<td>0.37</td>
<td>0.37</td>
</tr>
<tr>
<td>Material</td>
<td>Stainless steel (SS)</td>
<td>Gadolinium (Gd)</td>
<td>Calorivac-HS</td>
</tr>
<tr>
<td>Mass (g)</td>
<td>93.8</td>
<td>100.8</td>
<td>88.4</td>
</tr>
<tr>
<td>Heat transfer fluid</td>
<td>Tap water</td>
<td>Tap water</td>
<td>The mixture of water and corrosion inhibitor‡</td>
</tr>
</tbody>
</table>

†Sauter mean diameter is applied

Fig. 4 Transverse temperature measurement in the middle planes with (a) horizontally and (b) vertically arranged thermocouples.
4.1. Temperature profiles for passive regenerators

For the passive regenerators, the temperature profiles of maximum ($\theta_{m,\text{max}}$), minimum ($\theta_{m,\text{min}}$) and cycle-average ($\theta_{m}$) are depicted experimentally, numerically, and analytically in Fig. 5. Within the operating temperature range (5 – 40 °C in this case), the specific heat values of the solid materials for REG-SS, REG-Gd, and REG-HS given in Table 2 exhibit nearly independence, weak dependence, and strong dependence on temperature, respectively (inset plot in Fig. 5). Additionally, different overall utilization factors (cycle- and spatial-average $\langle \overline{U} \rangle$) are applied in Fig. 5. All the $\overline{U}$ show a linear trend. The slight deviation in the linearity in measurements is probably due to the errors in (1) positioning the thermocouple junctions and (2) ambient heat leak. Thus, the cycle-average temperature profiles for passive regenerators can be assumed linear irrespective of whether the regenerator is charged with temperature-dependent or independent solid materials.

Regarding the oscillating amplitudes of local temperatures, constant amplitudes are observed for materials whose specific heat is weakly dependent on the temperature as shown in Fig. 5 (a-b). For the regenerator with a sharp-peak material shown in Fig. 5 (c), the variation of specific heat in the solid phase affects the amplitude of local temperature oscillation, the larger the specific heat (in the vicinity of phase transition), the smaller the amplitude. This phenomenon is consistent with the description in Eq. 49, where $\Delta T$ decreases with decreasing $\overline{U}$ caused by the local $c_s$ increment (phase transition). Note that the latent heat needed to overcome the phase transition was equivalently converted to the peak of specific heat in the vicinity of the Curie temperature.
Fig. 5 Passive temperature gradients for (a) REG-SS, (b) REG-Gd and (c) REG-HS. Due to the experimental limits, the operating frequencies are set to 0.5 Hz. The magnetic transition temperatures (Curie temperatures) are marked for REG-Gd and REG-HS. Note that in subfigure (b), the analytical model used the real measurement of zero-field specific heat of Gd in a differential scanning calorimetry (DSC) device, while the numerical model used...
the theoretical values of Gd properties. Thus, the analytical results predicted better than the numerical results in subfigure (b).

4.2. Temperature profiles for active regenerators

For active regenerators, analytical and numerical temperature profiles of REG-Gd under different frequencies, utilization factors, and phasing between the magnetic field and fluid flow are shown in Fig. 6. Under varying operating conditions, the profiles are no longer linear with the following characteristics: 1) the larger the caloric effect (approaching the transition temperature), the steeper slope the exhibits, and 2) the oscillating amplitudes in Fig. 6 roughly increase with the intensity of caloric effect (referring to Eq. 49), which is opposite to the passive results in Fig. 5.

Fig. 6 Active temperature profiles for REG-Gd operating at different (a-c) frequencies, (d-f) utilization factors, and (g-i) phase offset. The magnetic fields are set from 0 to 1.1 Tesla. The temperatures of the hot and cold reservoirs are set at 10 °C and 25 °C, respectively. The magnetic transition temperature (Curie temperature) is shown in subfigure (a). The captions “Analytical”, “Num.+Analy.” and “Numerical” denote the results of purely analytical model, numerical cycle plus analytical amplitudes (Eq. 33 and Eq. 34), and purely numerical model, respectively.

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4.3. Analytical model validation

Comparing the above analytical results of $\theta_m$ for passive and active regenerators to numerical results (Fig. 5 - Fig. 6) and experimental measurements (Fig. 5), the trends, magnitudes, and sensitivity are in relatively good agreement considering the range of operating conditions. In Fig. 6, the dotted lines denote the results of Eq. 16, Eq. 17, Eq. 33, and Eq. 34 where $\frac{\partial \bar{\theta}}{\partial \xi}$ was numerically solved from Eq. 3 and Eq. 4. The good agreement between dotted and solid lines indicates the validity of relations between cycle-average temperature gradient ($\partial \bar{\theta} / \partial \xi$) and oscillating amplitude ($\theta_{s,a1}$ and $\theta_{f,a1}$) shown in Eq. 33 and Eq. 34. The dashed lines in Fig. 6 represent the results making use of the derived Eq. 44 and boundary conditions Eq. 41 and Eq. 42, which validates the simplified Eq. 44 for $\bar{\theta}$. The primary discrepancies are observed at both ends of regenerators. One main reason is that the thermal penetration depth at both boundaries caused by the constant temperature inflow deviates from the harmonic assumptions of the temperature profiles.

For local temperature transients, the analytical results are compared to measurements and modelling for passive and active regenerators, respectively. As seen in Fig. 7, both experimental and numerical curves mainly display sinusoidal waveforms with consistent phasing to the analytical results. For the passive regenerator in Fig. 7 (a), the solid phase oscillations are damped compared to the fluid oscillations, which is consistent with published data [118]. In Fig. 7 (b), the fluid temperature lags behind the solid temperature, because the caloric heat is indirectly transferred to the fluid via the solid. The local temperature transients near both ends of the regenerator deviate from the harmonics due to the thermal boundaries and thermal penetration length effect [119,120]. In Fig. 7 (a), the analytical model was predicted with a maximum error of ~15% in magnitude. One possible reason for the discrepancies is the ambient heat leak experienced by the experiments that are not modelled. In Fig. 7 (b), the analytical model can essentially predict the phasing and magnitudes of individual solid and fluid temperatures relative to the numerical model. The large deviations are again captured near both ends of regenerators, which is also attributed to the constant thermal boundaries. The validation given above implies that the analytical solutions from the harmonic analysis and post-corrections can closely predict the temperature profiles, as well as solid and fluid temperature phasing. Thus, the figures of merit shown in Eq. 47, Eq. 48, Eq. 50 and Eq. 51 can be the potential design parameters for ACRs.
Fig. 7 Comparison of local temperature transients for (a) passive regenerator between experimental and analytical results, and (b) active regenerator between numerical and analytical results.
5. Conclusions

Explicit algebraic expressions for visualizing the temperature profile transient of active caloric regenerators subjected to sinusoidal field and flow waveforms were developed using harmonic approximations and post-correction. The proposed analytical model provides a quantitative understanding of the complex coupling of heat transfer, hydraulic flow, heat regeneration, and caloric effects. A passive regenerator tester was upgraded to improve the spatial resolution of temperature measurements by an advanced thermocouple arrangement. The analytical solutions were validated both experimentally and numerically for magnetocaloric regenerators. The results showed that the analytical solutions can accurately capture the major characteristics of temperature profile transients for both solid and fluid phases with comparable heat capacities. In addition, temperature profiles for active and passive regenerators were observed as follows.

(1) For passive regenerators, the cycle-average temperature is a linear function of the location irrespective of whether the solid properties are temperature-dependent or -independent. For caloric materials, the high heat capacity caused by thermally induced phase transition will result in locally smaller temperature amplitudes.

(2) For active regenerators, the cycle-average temperature is spatially nonlinear with a steeper slope and larger amplitude in the vicinity of the phase transition due to the caloric effect.

(3) The pattern of the temperature profile, solid and fluid amplitude, and phasing are nonlinear functions of the utilization, the number of transfer units, the caloric effect of the materials, and synchronization between fluid and applied field.

To facilitate the regenerator design, some important figures of merit, such as work, enthalpy flux, and internal entropy generation, were deduced and analyzed using this analytical model. The results obtained may be directly transferred to other regenerators with the coexistence of sinusoidal internal heat source and oscillating flow, such as active regenerators for thermochemical reactors and passive regenerators for Stirling engines.

Acknowledgments

This work was in part financed by the RES4Build project, which received funding from the European Union’s Horizon 2020 research and innovation program under grant agreement No. 814865. The authors wish to acknowledge Dr. Kaspar K. Nielsen for the valuable discussions of the experimental setup.

Appendix A: Estimation of cycle-average temperature profile for active regenerators

To explicitly solve the cycle-average temperature profile for active regenerators, we merged the energy equations Eq. 3 and Eq. 4 into a single partial differential equation.

\[
\frac{2\pi}{U} \frac{\partial (\theta_s - \theta_{ad})}{\partial \phi} + \frac{2\pi}{V} \frac{\partial \theta_f}{\partial \phi} - \tilde{k}_s \frac{\partial^2 \theta_s}{\partial \xi^2} - \tilde{k}_f \frac{\partial^2 \theta_f}{\partial \xi^2} + \lambda \frac{\partial \theta_f}{\partial \xi} = 0
\]

Eq. A1

Applying a cyclically temporal integration on each term in Eq. A1, the following expressions are derived.

\[
\int_0^{2\pi} \frac{2\pi}{U} \frac{\partial (\theta_s - \theta_{ad})}{\partial \phi} d\phi = \left( \frac{\theta_{s,ad1} - \theta_{ad,ad1}}{2} \right) \int_0^{2\pi} \frac{2\pi}{U(\xi,\phi)} \frac{\partial \theta_f}{\partial \phi} d\phi
\]

Eq. A2
\[ (2\theta_{s,a} - \theta_{ad,a})f_0^2\pi \frac{\lambda}{\tilde{u}(\xi, t)} d\phi \text{ via Eq. 15 - Eq. 17} \]

\[ \frac{2n\beta}{\tilde{u}(\xi)} (2\theta_{s,a} - \theta_{ad,a}) \] with a definition \( \beta \equiv \frac{\tilde{u}(\xi)}{2\pi} f_0^2\pi \frac{\lambda}{\tilde{u}(\xi, \phi)} d\phi = \frac{f}{m} \int_0^1 \frac{1}{f} \cdot \hat{m}_f c_s dt \)

where \( \beta \) is a parameter defining the flow rate weighted average of the specific heat for a thermodynamic cycle, which resembles the definition of effective applied field in Ref. [121]. For the regenerators with constant solid specific heat (e.g. passive regenerators), they yield \( \beta = 0 \).

As the parameter \( \frac{1}{\nu^*} \) in the fluid heat storage term is a constant, we arrive at

\[ \int_0^{2\pi} \frac{\partial^2 \theta_f}{\partial \xi^2} d\phi = 0 \text{ Eq. A3} \]

As both \( \theta_s \) and \( \theta_f \) are time-independent based on the definitions in Eq. 8 and Eq. 9, the temporal integration of heat conduction terms can be deduced as follows:

\[ \int_0^{2\pi} k_s \frac{\partial^2 \theta_s}{\partial \xi^2} d\phi = k_s \frac{\partial}{\partial \xi} \left[ \int_0^{2\pi} \frac{\partial \theta_s}{\partial \xi} d\phi \right] = 2\pi k_s \frac{\partial \theta_s}{\partial \xi} \approx 2\pi k_s \frac{\partial^2 \theta_s}{\partial \xi^2} \]

\[ \int_0^{2\pi} k_f \frac{\partial^2 \theta_f}{\partial \xi^2} d\phi \approx 2\pi k_f \frac{\partial^2 \theta_f}{\partial \xi^2} \text{ Eq. A5} \]

With regard to the integration of heat advection term, trigonometric operations are applied during the following deduction.

\[ \int_0^{2\pi} \frac{\partial \theta_f}{\partial \xi} d\phi = \pi \frac{\partial}{\partial \xi} \left\{ \int_0^{2\pi} \text{Re}(ie^{i\phi})[\bar{\theta}_f + \text{Re}(\theta_{f,a}e^{i\phi})] d\phi \right\} \text{ Eq. A6} \]

\[ \approx \pi \frac{\partial}{\partial \xi} \left[ \int_0^{2\pi} - \sin \phi \cdot \text{Re}(\theta_{f,a}e^{i\phi}) d\phi \right] \text{ via Eq. 15 and Eq. 17} \]

\[ \approx \frac{\partial}{\partial \xi} \left[ \theta_{f,a} \int_0^{2\pi} - \sin \phi \cdot \text{Re}(Ar \cdot \theta_{f,a}) d\phi + \sin \phi \cdot \sin(\text{Arg}(\theta_{f,a}))d\phi \right] \]

\[ \approx \frac{\partial}{\partial \xi} \left[ \theta_{f,a} \pi \sin(\text{Arg}(\theta_{f,a})) \right] \]

\[ \approx \frac{\pi^2}{2} \text{Re}(iy_{f,a}) \cdot \frac{d^2 \bar{\theta}}{d \xi^2} \text{ via Eq. 34} \]

For simplicity, the terms of \( \frac{\partial \bar{y}_{f,a}}{\partial \xi} \) and \( \frac{\partial (\bar{y}_{f,a} \theta_{ad,a})}{\partial \xi} \) are assumed negligible during the deduction of Eq. A6. Substituting Eq. A2 - Eq. A6 into Eq. A1, a second-order ordinary differential equation for \( \bar{\theta} \) is obtained.

\[ \frac{d^2 \bar{\theta}}{d \xi^2} - \gamma_1 \frac{d \bar{\theta}}{d \xi} + \gamma_2 \bar{\theta} = 0 \text{ Eq. A7} \]

\[ \gamma_1 \equiv \frac{4\beta \cdot \text{Re}(\bar{y}_{f,a})}{\pi \text{Re}(y_{f,a}) - 4\bar{\theta}(k_s + k_f)} \text{ Eq. A8} \]

\[ \gamma_2 \equiv \frac{4\beta \cdot \text{Re}(\bar{y}_{f,a} - 1) \theta_{ad,a}}{\pi \text{Re}(y_{f,a}) - 4\bar{\theta}(k_s + k_f)} \text{ Eq. A9} \]

Due to the involvement of division operation with complex numbers, the real parts of the complex variables were highlighted above. For the caloric materials with weak dependence on temperature, i.e. \( \gamma_1 \) and \( \gamma_2 \) are weakly dependent on \( \xi \), an explicit solution is derived based on Eq. A7 - Eq. A9.
\[
\bar{\theta} = \frac{(\gamma_1(\theta(0) - \theta(1)) + \gamma_2) e^{\gamma_1 \xi} - (\gamma_1 \theta(0) + \gamma_2 \xi) e^{\gamma_1 \xi} + \gamma_1 \theta(1) - \gamma_2 (1 - \xi)}{\gamma_1 (e^{\gamma_1 \xi} - 1)}\]

Eq. A10

Appendix B: Transverse temperature measurements in different planes

(a)

(b)
Fig. B Transverse temperature measurement in (a) the hot side and (b) the cold side. As some thermocouples were accidentally broken during the experiments, some temperature measurements are missing.

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