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The Canonical and Alternate Duals of a Wavelet Frame

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Abstract

In the paper [3] we explore the relationship between canonical and alternate dual frames of a wavelet frame. We show that there exists a frame wavelet ψ with fast decay in the time domain and compact support in the frequency domain generating a wavelet system whose canonical dual frame is not a wavelet system generated by an arbitrary number of functions. On the other hand, there exists infinitely many alternate duals of ψ generated by a single function. Our argument closes a gap in the original proof of this fact by Daubechies and Han [7]

Introduction

Suppose ψ is an orthonormal wavelet and let $\theta(x) = \psi(x) + \varepsilon 2^{1/2}\psi(2x)$ for $0 < \varepsilon < 1$. Then θ generates a wavelet Riesz basis whose (canonical) dual is not of the form $\{2^{j/2}\phi(2^j x - k) : j, k \in \mathbb{Z}, \phi \in \Phi\}$ for any finite set Φ of generators ([5, 6]), *i.e.*, the period of θ is infinite . Obviously, the Riesz wavelet has no alternate wavelet duals either, so one might ask:

Does the existence of a dual wavelet frame imply wavelet structure of the canonical dual?

This was negatively answered by Daubechies and Han [7]:

Theorem 1. *There exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:*
(i) $\hat{\psi}$ is C^∞ and compactly supported,
(ii) its canonical dual frame is not a wavelet system generated by a single function,
(iii) there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual frame wavelets. □

Issue: A mistake of a simple change of sign invalidates the original proof in [7] to the extent that an easy remedy appears to be doubtful.

Therefore, there is a need to provide an alternative proof of Theorem 1. We will use a completely different approach motivated by [4]. Instead of trying to work directly with the frame operator as in [7], we will use a less direct approach using the following result of Bownik and Weber [4]:

Fact: *If the canonical dual is a wavelet system generated by one function, then the space of negative dilates is $\mathbb{Z}$ shift invariant.*

Idea of new proof: Construct a nice frame wavelet with an alternate wavelet dual and a non $\mathbb{Z}$ shift invariant space of negative dilates. Use the negation of the above fact to conclude that the canonical dual does not have wavelet structure.

Notation

A frame for a separable Hilbert space $\mathcal{H}$ is a collection of vectors $\{f_j\}_{j\in\mathbb{N}}$ such that

$$\exists C_1, C_2 > 0 \qquad C_1 \|f\|^2 \le \sum_{j\in\mathbb{N}} |\langle f, f_j\rangle|^2 \le C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If the upper bound holds in the above inequality, then $\{f_j\}$ is said to be a Bessel sequence. Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be *dual frames* if $f = \sum_{j\in\mathbb{N}} \langle f, g_j\rangle f_j$ for all $f \in \mathcal{H}$. At least one dual always exists, it is given by $\{S^{-1}f_j\}$ and called the canonical dual, where the frame operator of $\{f_j\}$ is given by $S\colon \mathcal{H} \to \mathcal{H}, Sf = \sum_{j\in\mathbb{N}} \langle f, f_j\rangle f_j$. Redundant frames have several duals; a dual which is not the canonical dual is called an *alternate* dual.

A closed subspace $W \subset L^2(\mathbb{R})$ is said to be $M\mathbb{Z}$ shift invariant if $T_{Mz}W \subset W$ for all $z \in \mathbb{Z}$. The wavelet system generated by $\Psi = \{\psi_1, \dots, \psi_L\}$, is defined as $\{\psi_{j,k} : j, k \in \mathbb{Z}, \psi \in \Psi\}$ where $\psi_{j,k} := \psi(a^j \cdot -k) \equiv D_a^j T_k \psi$. Given a frame wavelet Ψ, the *space of negative dilates* $V(\Psi)$ is:

$$V(\Psi) = \overline{\operatorname{span}} \bigcup_{j<0} W_j(\Psi), \qquad W_j(\Psi) = \overline{\operatorname{span}}\left\{\psi_{j,k} : k \in \mathbb{Z}, \psi \in \Psi\right\}, \quad j \in \mathbb{Z}.$$

The Period of a Frame Wavelet

Definition 1. Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R})$ is a frame wavelet associated with an integer dilation factor a, $|a| \ge 2$. The *period* of Ψ is the smallest integer $p \ge 1$ such that for all $f \in \overline{\operatorname{span}}\{T_k\psi : k \in \mathbb{Z}, \psi \in \Psi\}$,

$$T_{pk}S^{-1}f = S^{-1}T_{pk}f \qquad \text{for all } k \in \mathbb{Z},$$

where S is the frame operator of the wavelet frame generated by Ψ. If there is no such p, we say that the period of Ψ is ∞. □

Proposition 1 ([4]). *Let $M \in \mathbb{N}$. If Ψ is a frame wavelet and the period of Ψ divides M, then $V(\Psi)$ is shift invariant by the lattice $M\mathbb{Z}$. In addition, if Ψ is a Riesz wavelet, then the period of Ψ divides M if and only if $V(\Psi)$ is shift invariant by the lattice $M\mathbb{Z}$.* □

The canonical dual of Ψ has the wavelet structure generated by $|\Psi|$ functions if, and only if the period of Ψ is one. Moreover, we have:

Proposition 2 ([3]). *Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R})$ is a frame wavelet. For any nonnegative integer $M \in \mathbb{N}$, the following statements are equivalent:*
(i) $P(\Psi) \mid M$, i.e., the period of Ψ, denoted $P(\Psi)$, divides M.
(ii) There exist ML functions $\Phi = \{\phi_1, \dots, \phi_{ML}\}$ such that $\{D_a^j T_{Mk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ is the canonical dual of $\{D_a^j T_k\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi} = \{D_a^j T_{Mk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi_M}$, where

$$\Psi_M := \{T_m\psi : m = 0, \dots, M-1, \psi \in \Psi\}.$$ □

Hence, if the period $P(\Psi)$ of a frame wavelet Ψ is finite, then the canonical dual frame is a wavelet system generated by $P(\Psi) \cdot |\Psi|$ functions, and this is the least number of generators.

The Main Theorem

We will prove the follwing extension of Theorem 1:

Theorem 2.
For all $J \in \mathbb{N}$, there exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:
(i) $\hat{\psi}$ is C^∞ and compactly supported,
(ii) its canonical dual frame is not a wavelet system generated by fewer than 2^J functions,
(iii) there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual wavelet frames.

Proof (Sketch). Fix $J \in \mathbb{N}$. Construct a smooth frame wavelet $\psi = \psi^0 + \varepsilon\psi^1$ as in Lemma 1 and Figure 1 with $N = J + 3$ and with a space of negative dilates $V(\psi)$ not shift invariant under $M\mathbb{Z}$ for any $M = 1, 2, \dots, 2^J$ (see Lemma 2). By Proposition 1 the period of such a framelet ψ will be at least 2^J. Therefore, by Proposition 2, the canonical dual is at least generated by 2^J functions. Finally, use the characteristic equations for dual wavelets frames to explicitly show that ψ_0 is an alternate dual wavelet of ψ, see Figure 1. ■

Lemma 1. *For every $N \ge 4$ and $0 < \delta < 2^{-N}$, there exists a frame wavelet ψ such that $\hat{\psi} \in C_0^\infty(\mathbb{R})$ and*

$$\text{(1)} \qquad \hat{\psi}(\xi) \neq 0 \iff \xi \in (-1/2, -1/4) \cup (1/2, 3/4) \cup \left(-2^{-N+1} - \delta, -2^{-N} + \delta\right) \cup \left(2^{-N} - \delta, 2^{-N+1} + \delta\right)$$

$$\text{(2)} \qquad \hat{\psi}(\xi) = \hat{\psi}(\xi - 1) \neq 0 \qquad \text{for } \xi \in (1/2, 3/4).$$ □

Proof. Let $\psi^0 \in L^2(\mathbb{R})$ be a frame wavelet such that $\hat{\psi}^0 \in C_0^\infty(\mathbb{R})$ and $\hat{\psi}^0(\xi) \neq 0$ if and only if $\xi \in (-2^{-N+1} - \delta, -2^{-N} + \delta) \cup (2^{-N} - \delta, 2^{-N+1} + \delta)$, where $N \ge 4$ and $0 < \delta < 2^{-N}$ as in the assumption. Let $\psi^1 \in L^2(\mathbb{R})$ be such that $\hat{\psi}^1 \in C_0^\infty(\mathbb{R})$ has support in $[-1/2, -1, 4] \cup [1/2, 3/4]$ and $\hat{\psi}^1(\xi) = \hat{\psi}^1(\xi - 1) \neq 0$ whenever $\xi \in (1/2, 3/4)$. For any such $\psi^1 \in L^2(\mathbb{R})$ the sequence $\{D^j T_k \psi^1\}$ generates a Bessel sequence.

Define $\psi \in L^2(\mathbb{R})$ by $\psi = \psi^0 + \varepsilon\psi^1$, where $\varepsilon\psi^1$ acts as a perturbation on the wavelet frame generated by ψ^0 and ensures that ψ satisfies (2), see also Figure 1. The function $\varepsilon\psi^1$ generates a Bessel sequence, hence, for sufficiently small $\varepsilon > 0$, the function ψ generates a wavelet frame. ■

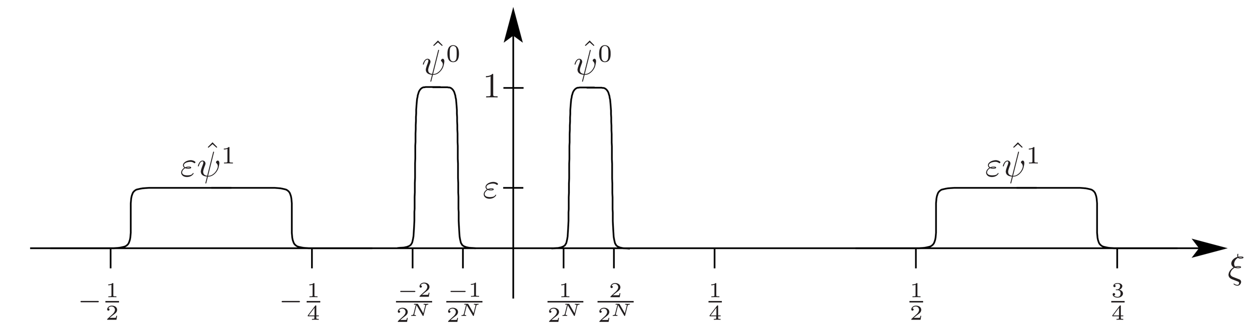

Figure 1: Sketch of the graph of $\hat{\psi} = \hat{\psi}^0 + \varepsilon\hat{\psi}^1$; ψ and ψ^0 are (alternate) duals.

Lemma 2. *Suppose that a function $\psi \in L^2(\mathbb{R})$ satisfies (1) and (2) for some $N \ge 4$ and $0 < \delta < 2^{-N}$. Then the space of negative dilates $V(\psi)$ is **not** $p\mathbb{Z}$-SI for any $p < 2^{N-3}$, $p \in \mathbb{N}$.* □

Proof (Sketch). Use that $V(\psi) = \overline{\operatorname{span}} \bigcup_{j<0} W_j(\psi)$, that $f \in W_0(\psi)$ if and only if $f(2^j \cdot) \in W_j(\psi)$, and that the pricipal shift-invariant subspace $W_0(\psi)$ can be described as

$$W_0(\psi) = \{f \in L^2(\mathbb{R}) : \hat{f} = \hat{\psi} m \quad \text{for some measurable, 1-periodic } m\}.$$

Note that $f \in V(\psi)$ if $\operatorname{supp} \hat{f} \subset [-2^{-N}, 2^{-N}]$. Show that $T_p f \notin V(\psi)$ for $p < 2^{N-3}$ and $\hat{f} = \chi_{[2^{-N+2}, 3/2 \cdot 2^{-N+2}] \cup [-2^{-N+2}, -1/2 \cdot 2^{-N+2}]} \in \hat{V}(\psi)$. See [3, Lemma 2] for the technical details. ■

The Result in Another Perspective

Auscher [1] proved that every "regular" orthonormal wavelet $\psi \in L^2(\mathbb{R})$ is associated with an MRA. "Regular" means that $|\hat{\psi}|$ is continuous and $\hat{\psi}(\xi) = \mathcal{O}(|\xi|^{-1/2-\delta})$ as $|\xi| \to \infty$ for some $\delta > 0$. This fact does not hold for tight frame wavelets. In fact, Baggett et al. [2] constructed a non-MRA C^r tight frame wavelet with rapid decay for any $r \in \mathbb{N}$. Once we allow non-tight frame wavelets we might lose even the GMRA property. Indeed, the frame wavelet from Theorem 2 is an example of a non-GMRA C^∞ frame wavelet with rapid decay.

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