Nonlinear sine-Gordon soliton waves and acoustic shock waves

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Ref. Gorria, C., Gaididei, Y. B., Sørensen, M. P., Christiansen, P. L., and Caputo, J. G. (2004). Kink propagation and trapping in a two-dimensional curved Josephson junction. Physical Review B, 69(13), 134506.

Ref.: A.R. Rasmussen, et al., Analytical and numerical modeling of front propagation and interaction of fronts in nonlinear thermoviscous fluids including dissipation. arXiv:0806.0105v2, vol: physics.flu-dyn, p. 1-11 (2008).

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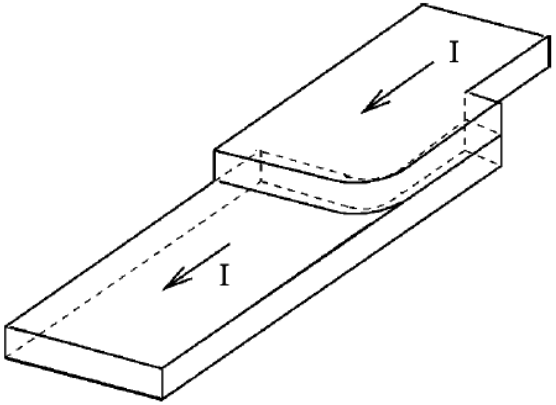
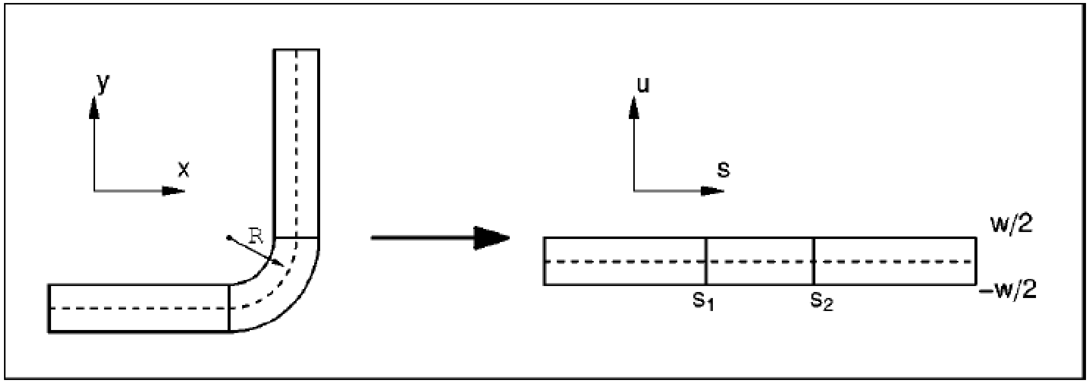
- Generalized travelling wave solutions, kinks or shock waves
- Oscillating shock waves

Ref. Gaididei, Y., Rasmussen, A. R., Christiansen, P. L., and Sørensen, M. P. (2016). Oscillating nonlinear acoustic shock waves. Evolution Equations & Control Theory, 5(3), 367.

The curved Josephson junction





Sine-Gordon model in 2D

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} + \alpha \frac{\partial \varphi}{\partial t} + \sin(\varphi) = 0$$

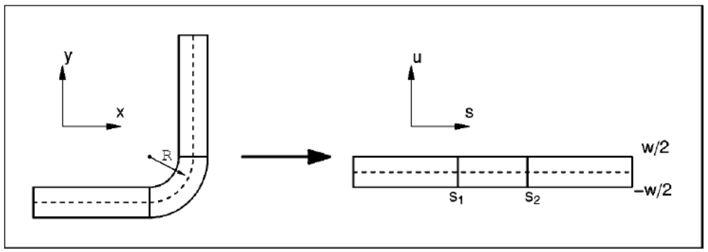
Boundary conditions (BC) at $y = \pm \frac{w}{2}$ $\quad n \cdot \nabla \varphi = \pm \frac{I}{2\ell}$

Ref. Gorria, C., Gaididei, Y. B., Sørensen, M. P., Christiansen, P. L., and Caputo, J. G. (2004). Kink propagation and trapping in a two-dimensional curved Josephson junction. Physical Review B, 69(13), 134506.

The curved Josephson junction

Coordinate transformation

$$\vec{r} = \vec{\rho}(s) + u\vec{n}(s)$$



Curvature and the metric tensor

$$\kappa(s)$$

$$g_{ss} = [1 - u\,\kappa(s)]^2$$

$$g_{uu} = 1$$

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{1}{g}\frac{\partial}{\partial s}\left(\frac{1}{g}\,\frac{\partial\varphi}{\partial s}\right) - \frac{1}{g}\frac{\partial}{\partial u}\left(\frac{1}{g}\,\frac{\partial\varphi}{\partial u}\right) + \alpha\frac{\partial\varphi}{\partial t} + \sin(\varphi) = 0$$

where $g = \sqrt{g_{ss}\,g_{uu}} = 1 - u\,\kappa(s)$

Boundary conditions at $u = \pm\frac{w}{2}$ $\qquad \frac{\partial\varphi}{\partial u} = \pm\frac{I}{2\ell}$

A collective coordinate approach

Solution ansatz: $\varphi(s,u,t) = \Phi_0(u) + \psi\,(s,u,t)$

$$\frac{d^2\Phi_0}{du^2} - \sin(\Phi_0) = 0$$

BC: $\frac{d\Phi_0}{du} = \pm\frac{I}{2\ell}$ **at** $u = \pm\frac{w}{2}$

$$\frac{\partial^2\psi}{\partial t^2} - \frac{1}{g}\frac{\partial}{\partial s}\left(\frac{1}{g}\ \frac{\partial\psi}{\partial s}\right) - \frac{1}{g}\frac{\partial}{\partial u}\left(g\ \frac{\partial\psi}{\partial u}\right) + \alpha\frac{\partial\psi}{\partial t} + \sin(\Phi_0+\psi) - \sin(\Phi_0) - \frac{1}{g}\frac{\partial g}{\partial u}\frac{d\Phi_0}{du} = 0$$

BC: $\frac{\partial\psi}{\partial u} = 0$ **at** $u = \pm\frac{w}{2}$

A collective coordinate approach

Kink soliton with varying parameters:

$$\psi(s,u,t) = 4\tan^{-1}\left[exp\left(\frac{s - S(u,t)}{B(u,t)}\right)\right]$$

Further proceedure:

1. ***Determine the Langrangian density.***
2. ***Integrate over the fast variables.***
3. ***Fourier expansion of S and B.***
4. ***Derive the Euler Lagrange equations.***

Fourier expansion:

$$S(u,t) = S_0(t) + S_1(t)\sin\left(\frac{\pi}{w}u\right)$$

$$B(u,t) = 1 + B_1(t)\sin\left(\frac{\pi}{w}u\right)$$

Assume $\dot{S}_0 \ll 1$

A collective coordinate approach

The Lagrangian $L = T - V$ **is composed of**

$$T = 4\left(\dot{S}^2 + \frac{\pi^2}{24}\dot{B}^2\right)$$

with $S(t) = S_0(t)$

$S_1(t)$ **neglected**

and

$$V = 4\left\{\left(\frac{\pi^4}{24w^2} + \frac{1}{2}\right)B^2 + a_0 U - \left[\left(\frac{2w}{\pi^2 R} - a_1\right)F + \left(\frac{2w}{\pi^2 R} + a_1\right)U\right]B\right\}$$

with $B(t) = B_1(t)$

A collective coordinate approach

Potential V

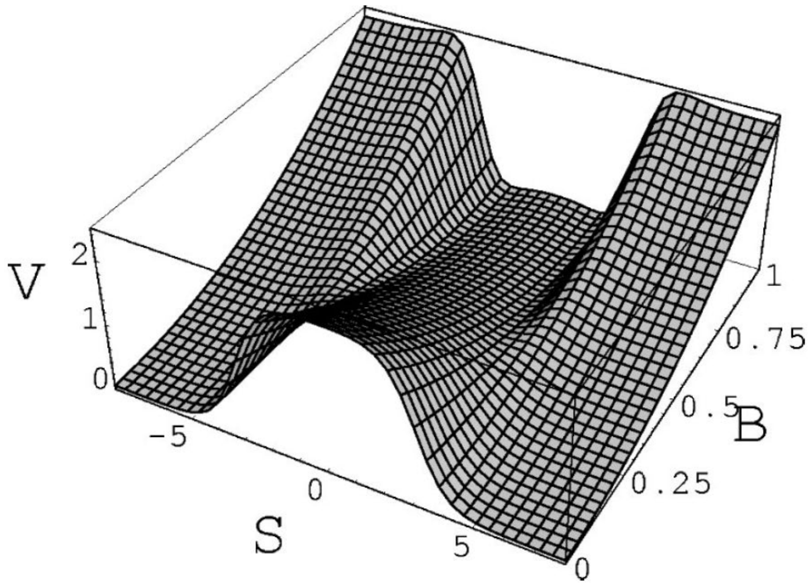


FIG. 4. The effective potential V given by Eq. (27) for an equivalent mechanical system with $w=6, R=4, s_2=-s_1=\pi$.

The Euler Lagrange equations read

$$8\,\ddot{S}_0 = -\frac{\partial V}{\partial S_0} \qquad \frac{\pi^2}{3}\,\ddot{B}_1 = -\frac{\partial V}{\partial B_1}$$

A collective coordinate approach

Potential V

$B_1 = 0$

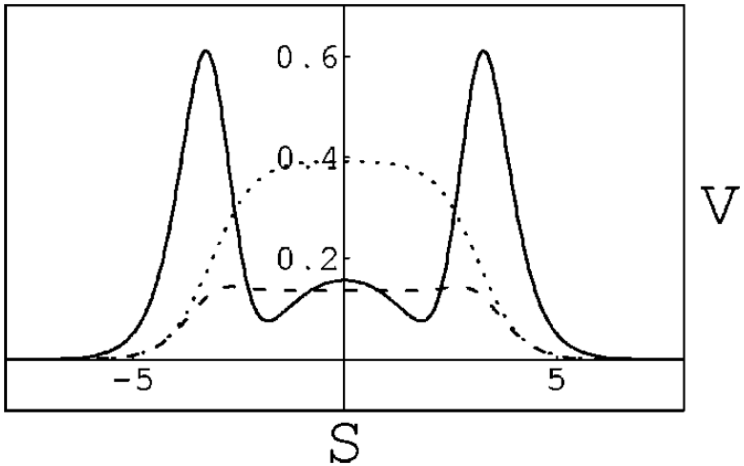


FIG. 5. Effective potential projection along the path $B=0$ for $w=4, R=4, s_2=-s_1=\pi$ (doted line) and along the path $B=B_{st}$ for $w=6, R=4, s_2=-s_1=\pi$ (continuous line) and for $w=4, R=4, s_2=-s_1=\pi$ (dashed line).

Numerical simulation results

Reflection $v = 0.17$

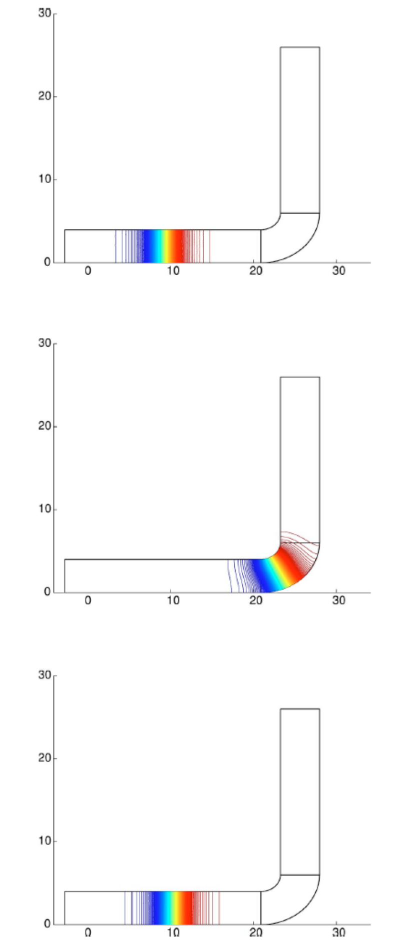


FIG. 6. The wave is reflected for $R=4$, $w=4$, and $v_0=0.17$, while $v_{cr}\approx 0.19$.

Transmission $v = 0.18$

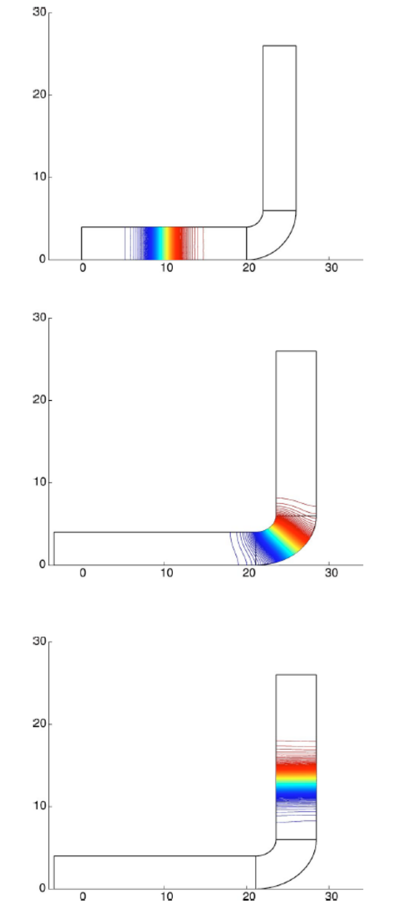


FIG. 7. The wave is transmitted for $R=4$, $w=4$, and $v_0=0.18$, while $v_{cr}\approx 0.19$.

Numerical simulation results

Phase potrait

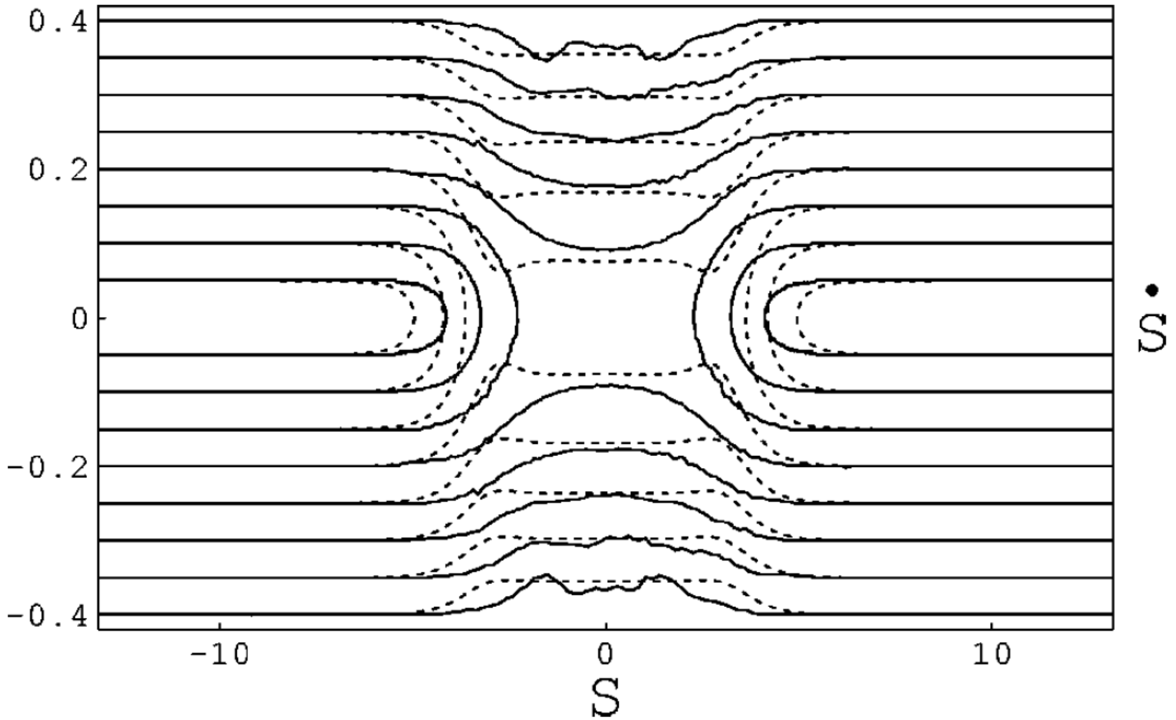


FIG. 8. Phase portrait $(S,\dot{S})$ using $w=4, R=4$ for the simulations (continuous line) and the collective variable equations (37) (dashed line). The critical velocity for transmission is $v_{cr} \approx 0.18$.

Nonlinear oscillating acoustic waves in a tube

Weakly nonlinear acoustic wave equation for thermo-viscous fluids

Derived from:

- **Equation of motion (Navier Stokes equations)**
- **Equation of continuity**
- **Heat transfer equation**
- **Equation of state**

$$\psi_{tt} - \psi_{xx} = \psi_t \, \psi_{xx} + (\gamma - 2)\psi_{tt}\psi_t + 2\psi_{xt} \, \psi_x + b\psi_{xxt} \quad \textbf{(1)}$$

ψ **is the fluid flow velocity potential**

$0 \leq x \leq \ell$ $\qquad$ $0 \leq t$

Dissipation parameter $b = 5 \cdot 10^{-4}$

$$\gamma = \frac{C_p}{C_v} = 1.4$$

Ref.: P.M. Jordan, Phys. Lett. A. 326(1-2), p77 (2004).

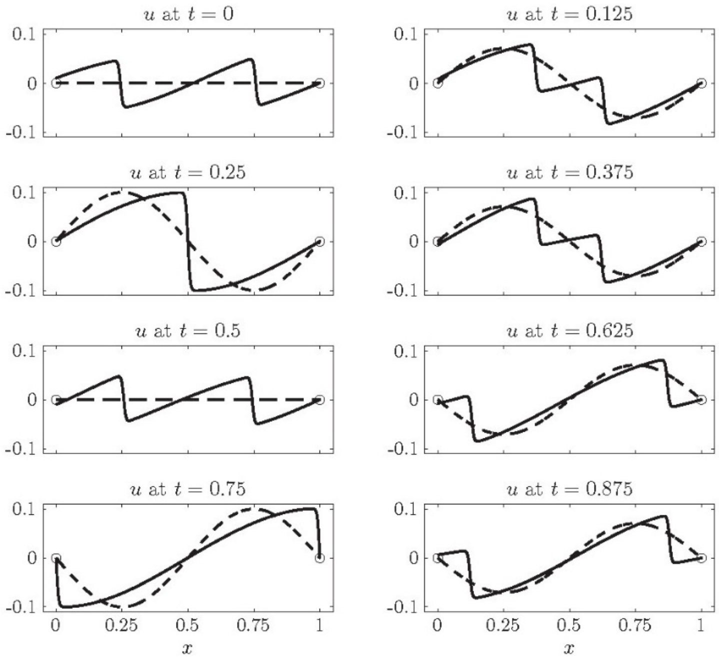
Nonlinear oscillating acoustic waves in a tube

Fluid velocity $u = -\psi_x$ **Pressure** $p = \psi_t$

Boundary conditions

$$u(0,t) = D\sin(\omega t) \qquad u(\ell,t) = 0$$

Numerical results



$\omega = 2\pi$

Nonlinear oscillating acoustic waves in a tube

Generalized travelling wave ansatz

$$\psi(x,t) = \Psi(\xi) + a_1 x + a_2 t + a_3 x t + a_4 \frac{1}{2} x^2 + a_5 \frac{1}{2} t^2$$

$$\xi = x - x_0 - vt$$

Introduce $\Phi = -\Psi'$ $\quad u = -\psi_x = \Phi(\xi) - a_1 - a_3 t - a_4 x$

Insertion into the weakly nonlinear PDE (1) gives the ODE

$$b\Phi'' - \frac{8}{3}\Phi'\Phi + \left(2a_1 + v - \frac{1}{v} - \frac{2a_2}{3v}\right)\Phi' - \frac{8a_5}{9v^2}\Phi - \frac{a_5}{v} + \frac{2a_1 a_5}{3v^2} - \frac{(2a_2 - 1)a_5}{9v^3} = 0 \qquad \mathbf{(2)}$$

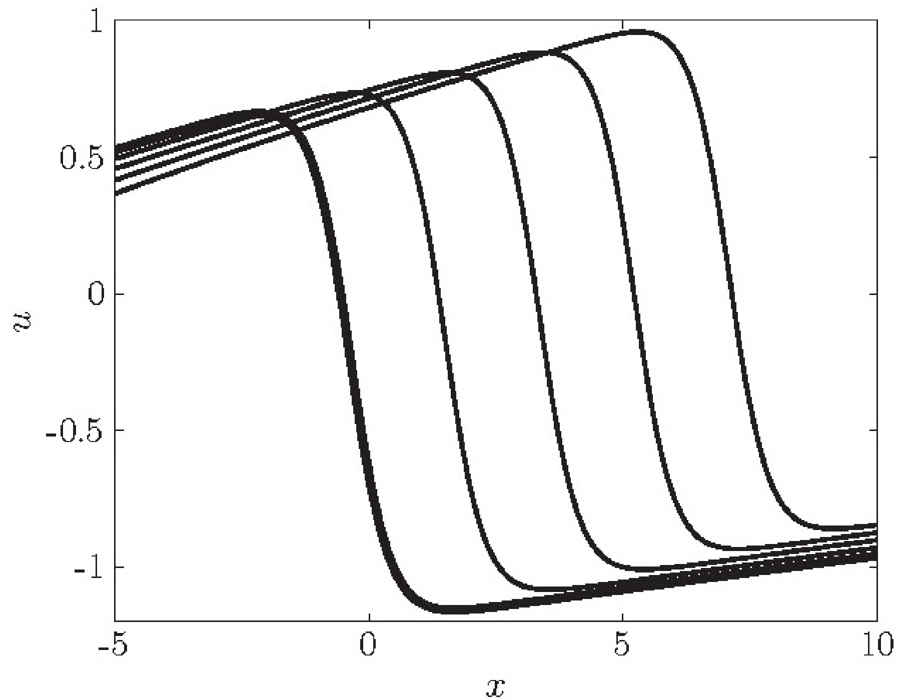
Nonlinear oscillating acoustic waves in a tube

Numerical results

$$v = \frac{1}{\sqrt{3(2-\gamma)}} = 0.7454$$

Numerical solution of the full PDE eq. (1).

IC found from the above ODE equation (2)



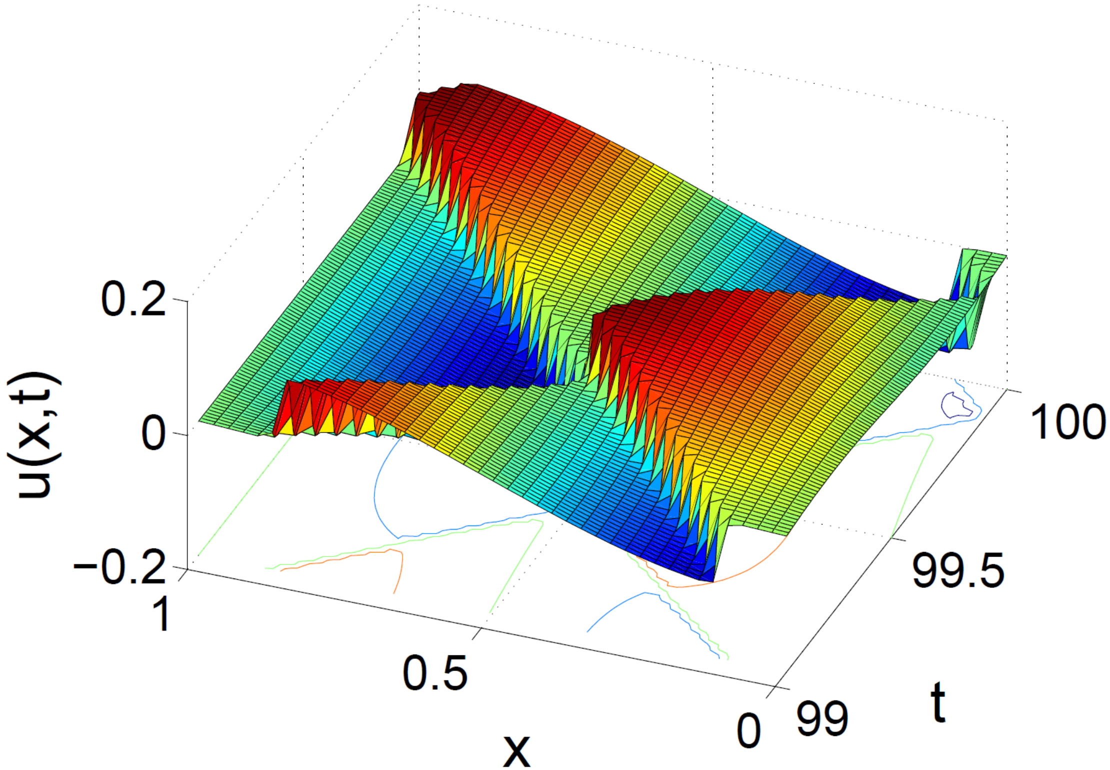
Ref.: A.R. Rasmussen, et al., Interacting wave fronts and rarefaction waves in a second order model of nonlinear thermo-viscous fluids.
Acta Applicandae Mathematica. DOI 10.1007/s10440-010-9581-7. 2010, pp. 1-19.

Nonlinear oscillating acoustic waves in a tube

Further numerical results

$\omega = 2\pi$

$D = 0.01$



Ref. Gaididei, Y., Rasmussen, A. R., Christiansen, P. L., and Sørensen, M. P. (2016). Oscillating nonlinear acoustic shock waves. Evolution Equations & Control Theory, 5(3), 367.

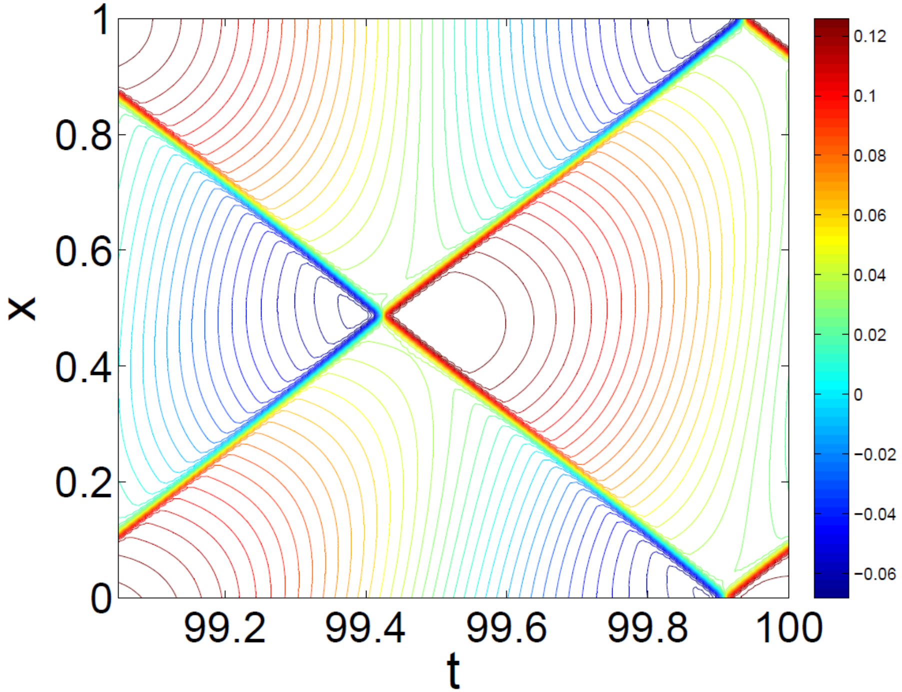
Nonlinear oscillating acoustic waves in a tube

Further numerical results

$\omega = 2\pi$

$D = 0.01$

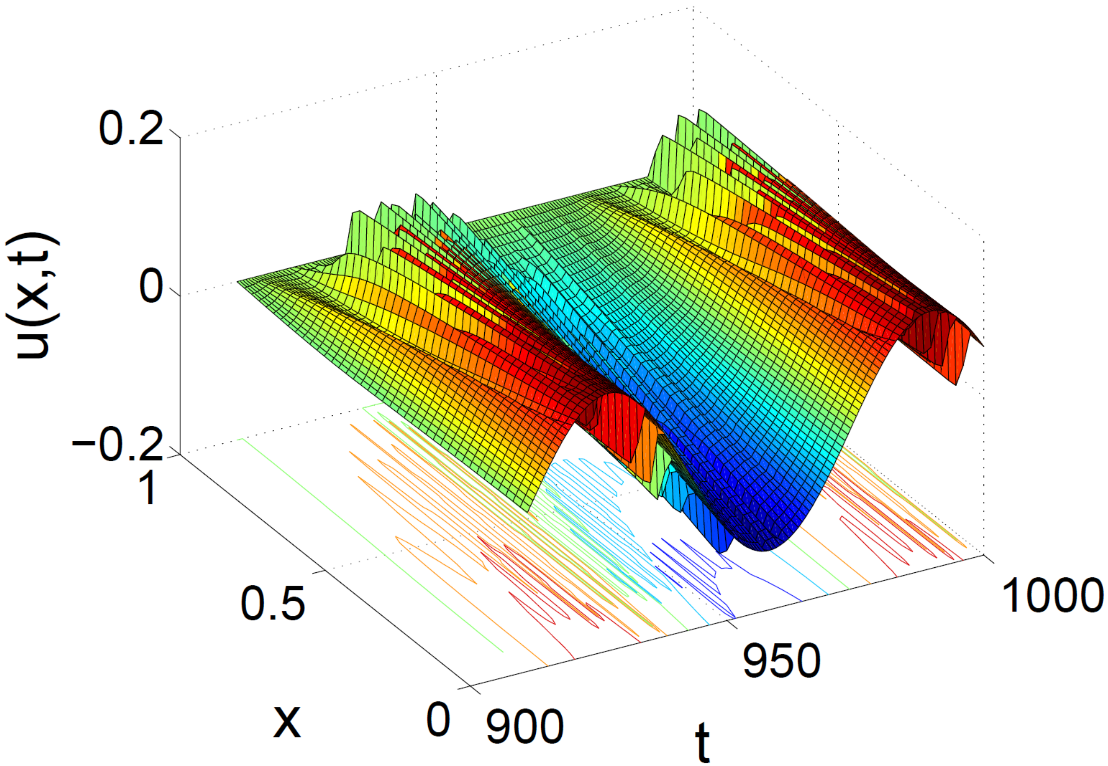
$p = \psi_t$



Nonlinear oscillating acoustic waves in a tube

Further numerical results

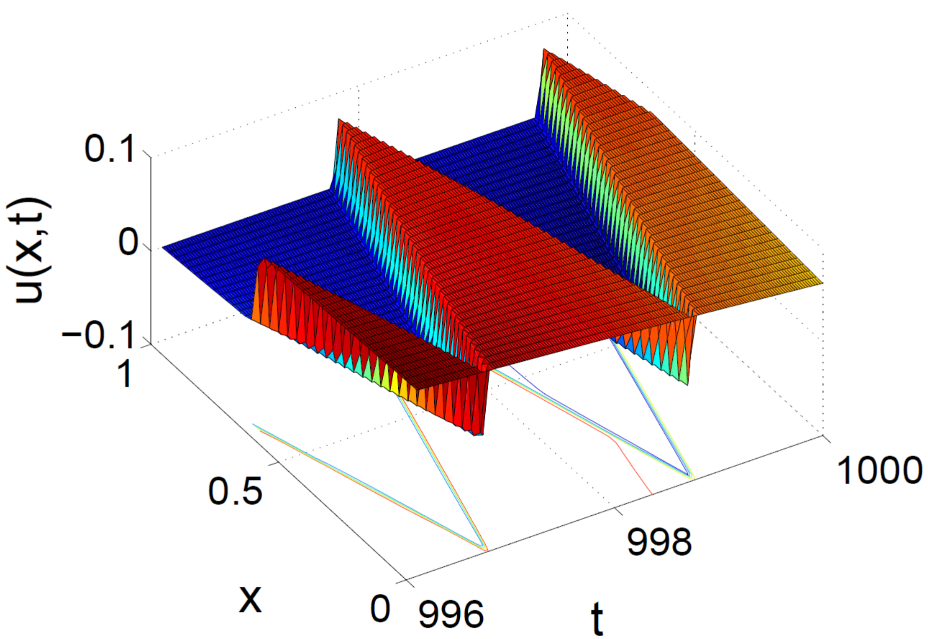
$\omega = 0.1 \qquad D = -0.125$



Nonlinear oscillating acoustic waves in a tube

Further numerical results

$\omega = 0.1 \qquad D = -0.125$



Summary

The dynamics of coherent sine-Gordon kinks in curved Josephson junctions can be analysed relatively accurate by a collective coordinate approach utilizing the coherent properties of solitons / quasi solitons.

The dynamics of shock waves in a thermo-viscous fluid are likewise a coherent travelling wave, which can be analysed relatively accurate by a collective coordinate approach utilizing the coherent properties of the shocks.

The validity and limits of the collective coordinate abroach have been verified by full numerical simulations.

Yuri Gaididei was a great master of the collective coordinate approach.

Les Houches, France, 2003