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Electromagnetic Sources in a Moving Conducting Medium

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The problem of an arbitrary source distribution in a uniformly moving, homogeneous, isotropic, nondispersive, conducting medium is solved. The technique used is to solve the problem in the rest system of the medium and then write the result in an appropriate four-dimensional, covariant form which is valid in any inertial system.

I. INTRODUCTION

Recently Besieris and Compton[1] solved the problem of electromagnetic radiation by an arbitrary source in a uniformly moving, homogeneous, isotropic, nondispersive, conducting medium by making use of a relation between the fundamental solution of a radiation problem and that of a corresponding Cauchy initial value problem. An alternative method was provided by Chen and Yen,[2] who applied judiciously chosen affine transformations to the pertinent differential equation.

It is the purpose of this paper to solve the same problem but in a different way. The most essential feature of the technique used in the present paper is that the problem is handled in the rest system K' of the medium "as long as possible," because the pertinent differential equations are much simpler in K'. In fact the whole problem is solved in K' by making use of the known fundamental solution of the Klein–Gordon differential equation; the result is then transformed to an arbitrary inertial system K by means of an appropriate tensor formulation.

We use Cartesian tensor notation as in Ref. 3. By a tensor we understand a tensor defined on the Lorentz transformation group. Latin subscripts run from 1 to 4, Greek subscripts run from 1 to 3. The coordinate $x_4 \equiv ict$, where t is the time and c the speed of light in vacuum; therefore, the metric tensor in 4-space is equal to the Kronecker symbol δ_{ij} (when Cartesian spatial coordinates are used) and we do not distinguish between contravariant and covariant tensors. Repeated subscripts obey the summation convention, and commas in subscripts denote partial differentiation with respect to coordinates (or covariant differentiation since the metric tensor is independent of the coordinates).

II. THE POTENTIAL TENSOR (4-VECTOR)

In any inertial system Maxwell's equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_f + \mathbf{J},$$

$$\nabla \cdot \mathbf{D} = \rho_f + \rho,$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1}$$

where **E**, **H**, **D**, **B** are familiar symbols for the field quantities, $\mathbf{J}_f$, ρ_f denote the free current and charge densities, and $\mathbf{J}$, ρ the externally applied current and charge densities.

In the rest system K' of the medium, the following constitutive relations are assumed to be valid:

$$\mathbf{D}' = \epsilon \mathbf{E}',\quad \mathbf{B}' = \mu \mathbf{H}',\quad \mathbf{J}_f' = \sigma \mathbf{E}', \tag{2}$$

where ϵ, μ, σ are the dielectric constant, the permeability and the conductivity, respectively.

It is well known[4] that vector and scalar potentials $\mathbf{A}'$, Φ' can be introduced in K', satisfying the equations

$$\nabla^2 \mathbf{A}' - \mu\epsilon \frac{\partial^2 \mathbf{A}'}{\partial t'^2} - \sigma\mu \frac{\partial \mathbf{A}'}{\partial t'} = -\mu \mathbf{J}',$$

$$\nabla^2 \Phi' - \mu\epsilon \frac{\partial^2 \Phi'}{\partial t'^2} - \sigma\mu \frac{\partial \Phi'}{\partial t'} = -\frac{1}{\epsilon}\rho', \tag{3}$$

where we have assumed that $\rho_f' = 0$ because of the brevity of usual relaxation times.

$\mathbf{A}'$ and Φ' are connected by the gauge condition

$$\nabla \cdot \mathbf{A}' + \mu\epsilon \frac{\partial \Phi'}{\partial t'} + \sigma\mu\Phi' = 0. \tag{4}$$

The translation of (3) into tensor language is given in Ref. 5 for the case $\sigma = 0$. It is not difficult to show that if $\sigma \neq 0$, a single term has to be added so that the tensor wave equation for the potential tensor A_i (consult Ref. 6) which is valid in any inertial system K may be written as

$$A_{i,nn} - \kappa A_{i,rs} U_r U_s - \sigma\mu A_{i,r} U_r = -S_i, \tag{5}$$

where $\kappa \equiv (n^2 - 1)/c^2$, $n \equiv c/c'$, $c' \equiv (\mu\epsilon)^{-\frac{1}{2}}$, $S_i \equiv \mu(J_i + \kappa/n^2 J_r U_r U_i)$, U_r is the velocity 4-vector, and finally J_i is the current density 4-vector (consult Ref. 6) of the external source.

The tensor equation for the gauge condition turns out to be

$$A_{r,r} - \kappa A_{r,s} U_r U_s - \sigma\mu A_r U_r = 0. \tag{6}$$

III. INTEGRATION OF THE TENSOR EQUATION FOR DAMPED WAVES

The first-order term in (5) may be eliminated. Let k_i denote a constant 4-vector (i.e., independent of the space–time coordinates x_r). Also, tensor functions B_i and T_i are defined by

$$B_i \equiv A_i e^{-k_r x_r}, \quad T_i \equiv S_i e^{-k_r x_r}. \tag{7}$$

From (5) and (7) we derive

$$B_{i,nn} - \kappa B_{i,rs} U_r U_s + l^2 B_i = -T_i, \tag{8}$$

where

$$l \equiv [k_r k_r - \kappa (k_r U_r)^2 - \sigma\mu k_r U_r]^{\frac{1}{2}} \tag{9}$$

and k_r is subjected to the condition

$$2k_i - (2\kappa k_r U_r + \sigma\mu) U_i = 0. \tag{10}$$

Since $U_i' = (0, 0, 0, ic)$, (10) is satisfied in K' if we define

$$k_i' \equiv \left(0, 0, 0, i\frac{\sigma\mu c}{2n^2}\right). \tag{11}$$

Because (10) is a tensor equation, it holds in any system of inertia K since it holds in K'. [In K we can get k_r from (11) by means of the tensor transformation law.]

l is defined by (9) and transforms like an invariant under a Lorentz transformation. It is easily shown (in K') that

$$l = \tfrac{1}{2}\sigma(\mu/\epsilon)^{\frac{1}{2}}. \tag{12}$$

In K' (8) reduces to

$$\left(\nabla'^2 + n^2 \frac{\partial^2}{\partial x_4'^2} + l^2\right) B_i' = -T_i'. \tag{8'}$$

In preparation for the integration of this equation, consider

$$\left(\nabla'^2 - (in)^2 \frac{\partial^2}{\partial x_4'^2} - (il)^2\right) G' = -4\pi\delta(u_r'), \tag{13}$$

where $u_r' \equiv x_r' - z_r'$; z_r' are parameters and $\delta(u_r') \equiv \delta(u_1')\delta(u_2')\delta(u_3')\delta^\star(u_4')$. $\delta^\star(u_4')$ is a delta-function with purely imaginary argument, i.e., $\int_{-i\infty}^{i\infty} f(z)\delta^\star(z)\,dz = f(0)$ for a great class of functions f.

Equation (13) is the Klein–Gordon equation for the time-dependent Green's function G'. The solution of (13) for the whole space is given in Ref. 7 for real constants (in) and (il). It is readily seen that the solution also holds when (in) and (il) are purely imaginary; therefore

$$G'(x_r', z_r') = \begin{cases} \dfrac{\delta^\star(u_4' - inr')}{r'} - i\dfrac{l}{n}\dfrac{J_1(lR')}{R'} 1_+^\star(u_4' - inr'), & \dfrac{u_4'}{i} > 0 \\ 0, & \dfrac{u_4'}{i} < 0 \end{cases}, \tag{14}$$

where

$$R' \equiv [r'^2 + (u_4'/n)^2]^{\frac{1}{2}}, \quad r' \equiv (u_\nu' u_\nu')^{\frac{1}{2}}. \tag{15}$$

J_1 is the Bessel function of first kind and first order, and $1_+^\star$ denotes the unit step function with purely imaginary argument, i.e.,

$$1_+^\star(x) = \begin{cases} 1, & x/i \geq 0 \\ 0, & x/i < 0 \end{cases}.$$

By means of G' we are able to write down an integral representation for the potentials A_i' connected with B_i' by (7):

$$A_i'(x_r') = \frac{1}{4\pi} \iiint_{-\infty}^{\infty}\int_{-i\infty}^{i\infty} e^{k_r' u_r'} G'(x_r', z_r') \times S_i'(z_r')\, du_1'\, du_2'\, du_3'\, du_4'. \tag{16}$$

IV. TRANSFORMATION TO AN ARBITRARY INERTIAL SYSTEM

Let a_{ji} be the transformation matrix for a proper Lorentz transformation, i.e., $x_r = a_{rs} x_s'$. Multiplying (16) by a_{ji}, we see that the left side is equal to $A_j(x_r)$ because A_j is a tensor. a_{ji} may be taken under the integral and, since S_i is a tensor, $a_{ji} S_i'(z_r') = S_j(z_r)$ if the Lorentz transformation is also applied to the integration variables, i.e., if $z_r = a_{rs} z_s'$ which implies $u_r = a_{rs} u_s'$. Furthermore, k_r is a tensor so that $e^{k_r' u_r'} = e^{k_r u_r}$.

Next we investigate how the Green's function G' is transformed. Without loss of generality, we choose a_{rs} so that $x_1 = x_1'$, $x_2 = x_2'$, $x_3 = \gamma(x_3' + i\beta x_4')$, $x_4 = \gamma(x_4' - i\beta x_3')$, where $\gamma \equiv (1 - \beta^2)^{-\frac{1}{2}}$, $\beta \equiv v/c$, and v is the velocity of K relative to K'.

Consider a three-dimensional hypersurface in Minkowski space, which in K' is given by $R'^2 = 0$, $u_4'/i > 0$ (Fig. 1) (cf. Ref. 3). In order to express the surface independently of the inertial system, we define

$$R \equiv \left[u_r u_r + \kappa\left(\frac{u_r U_r}{n}\right)^2\right]^{\frac{1}{2}}. \tag{17}$$

Obviously R is an invariant function of u_r, and it is easily seen that $R = R'$ in K'. Therefore, the

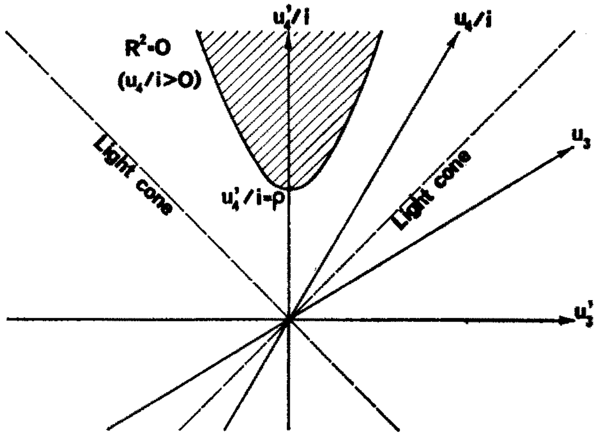

FIG. 1. Location of zeros of the function R^2 in Minkowski 4-space for $u_4/i > 0$.

hypersurface is also given by $R^2 = 0$, $u_4/i > 0$ because it is located entirely in that part of Minkowski space where the condition $u_4/i = (x_4 - z_4)/i > 0$ is valid in any inertial system K.

u_4 is purely imaginary so R^2 may be negative, which is the case inside the hatched domain in Fig. 1. The roots of $R^2 = 0$ are given by (cf. Ref. 3)

$$\frac{u_4}{i} = \tau_\pm \equiv \frac{n\beta}{1-(n\beta)^2}\left[\left(n - \frac{1}{n}\right)u_3 \pm \left(\frac{1}{\beta} - \beta\right)(u_3^2 + a\rho^2)^{\frac{1}{2}}\right], \qquad (18)$$

where $a \equiv [1-(n\beta)^2]/(1-\beta^2)$, $\rho \equiv (u_1^2 + u_2^2)^{\frac{1}{2}}$.

It is seen that $n\beta < 1$ implies $\tau_+ > 0$, $\tau_- < 0$. In the domain $u_4/i > 0$ the equation $R^2 = 0$ defines a one to one correspondence between u_3 and u_4 (for given ρ). This is not the case for $n\beta > 1$ (Čerenkov region) because $\tau_\mp > 0$ for $u_4/i < -\rho\,|a|^{\frac{1}{2}}$, and both roots are complex or do not belong to the domain $u_4/i > 0$ for $u_4/i > -\rho\,|a|^{\frac{1}{2}}$.

From the preceding remarks we conclude that the step function in (14) may be written in covariant form as $1_+(-R^2)$, $u_4/i > 0$.

As to the δ function in (14), we have

$$\frac{\delta^\star(u_4' - inr')}{-(dR^2/du_4')_{u_4'=inr'}} = \frac{\delta^\star(u_4' - inr')}{i(2/n)r'} = \delta(-R^2), \qquad u_4/i > 0. \qquad (19)$$

Finally we observe that the limits of integration in (16) remain unchanged because a Lorentz transformation is a one-to-one mapping of the Minkowski space on itself.

We are now able to write down the covariant forms of (14) and (16) valid in an arbitrary inertial system:

$$A_i(x_r) = \frac{1}{4\pi}\iiint_{-\infty}^{\infty}\int_{-i\infty}^{i\infty} e^{k_r u_r} G(x_r, z_r) \times S_i(z_r)\, du_1\, du_2\, du_3\, du_4, \qquad (20)$$

$$G(x_r, z_r) = \begin{cases} i\dfrac{2}{n}\delta(-R^2) - i\dfrac{l}{n}\dfrac{J_1(lR)}{R}1_+(-R^2), & \dfrac{u_4}{i} > 0 \\ 0, & \dfrac{u_4}{i} < 0 \end{cases}. \qquad (21)$$

When $n\beta < 1$,

$$i\frac{2}{n}\delta(-R^2) = i\frac{2}{n}\frac{\delta^\star(u_4 - i\tau_+)}{-(dR^2/du_4)_{u_4=i\tau_+}} = \frac{\delta^\star(u_4 - i\tau_+)}{(u_3^2 + a\rho^2)^{\frac{1}{2}}}, \qquad u_4/i > 0. \qquad (22)$$

When $n\beta > 1$, both roots $\tau_\pm$ play a part as pointed out before. In this case it turns out that

$$i\frac{2}{n}\delta(-R^2) = \frac{\delta^\star(u_4 - i\tau_+) + \delta^\star(u_4 - i\tau_-)}{(u_3^2 + a\rho^2)^{\frac{1}{2}}}, \qquad u_4/i > 0. \qquad (22')$$

If the medium is nonconductive, i.e., $\sigma = 0$, then $k_r u_r = 0$ and $l = 0$ [cf. (11) and (12)], furthermore, the second term in (21) evidently vanishes. This problem has been investigated previously by Compton,[8] Lee and Papas,[9] Tai,[10,11] and the author.[3] As pointed out by Tai,[11] the first term in (21) is equivalent to the corresponding expression found by Compton.[8]

As to the general case ($\sigma \neq 0$), the result given by Besieris and Compton[1,12] is in error[13] due to miscalculation, and there is a formal error in Ref. 2,[14] so the author hopes deeply that he is right in asserting that the results in Refs. 1 and 2 can be brought into agreement with the results given here.

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[14] In Eq. (21) etc., Ref. 2, the argument of the Bessel function is in error.