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Zahle, Torben U.

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On Coding the States of Sequential Machines with the Use of Partition Pairs

TORBEN U. ZAHLE

Abstract—This article introduces a new technique of making state assignment for sequential machines. The technique is in line with the approach used by Hartmanis [1], Searns and Hartmanis [3], and Curtis [4]. It parallels the work of Dolotta and McCluskey [7], although it was developed independently. The paper describes a procedure for making assignments based on partition pairs with successive choice of the partition.

The author is with the Laboratory for Pulse and Digital Techniques, Technical University of Denmark, Copenhagen 2, Denmark.

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REFERENCES


INTRODUCTION

It will be assumed that the reader is familiar with the works named in the abstract. Some usual assumptions are made, namely that realization is based on two-level AND-OR logic and delay elements, variables and their complements are available without additional cost, the input is coded in advance, and the output is to be handled separately. The fundamental idea of the technique is based on the following observations.

1) Any assignment whatsoever has to be built up of two-block partitions corresponding to each secondary variable.

2) A two-block partition corresponding to a $y$-variable fixes the placement of ones and zeros in the $Y$-diagram when we keep the same ordering of states.

Example

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>d</td>
<td>c</td>
<td>$a \to 0$</td>
<td>0</td>
<td>1</td>
<td>1 1 1 1 1 1</td>
</tr>
<tr>
<td>b</td>
<td>f</td>
<td>c</td>
<td>$b \to 0$</td>
<td>1</td>
<td>0</td>
<td>0 0 1 1 1 1</td>
</tr>
<tr>
<td>c</td>
<td>e</td>
<td>b</td>
<td>$c \to 1$</td>
<td>1</td>
<td>0</td>
<td>0 1 0 1 1 1</td>
</tr>
<tr>
<td>d</td>
<td>b</td>
<td>e</td>
<td>$d \to 1$</td>
<td>1</td>
<td>0</td>
<td>0 1 1 1 1 1</td>
</tr>
<tr>
<td>e</td>
<td>a</td>
<td>d</td>
<td>$e \to 0$</td>
<td>1</td>
<td>0</td>
<td>0 0 1 1 1 1</td>
</tr>
<tr>
<td>f</td>
<td>c</td>
<td>d</td>
<td>$f \to 0$</td>
<td>1</td>
<td>1</td>
<td>1 1 1 1 1 1</td>
</tr>
</tbody>
</table>

Machine A:

$Y_1$ $Y_2$ $Y_3$

or for $y_i$:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
</tr>
<tr>
<td>e</td>
<td>0</td>
</tr>
<tr>
<td>f</td>
<td>0</td>
</tr>
</tbody>
</table>

The $Y_i$-diagram shown will be the same no matter how we choose the other variables. The partition made by ones and zeros in the 0-column is called $C_0$. In general, the $C$-partition has as index the input of the column. The notation $P$ and $C$-partition is in accordance with the definition of Curtis [4], p. 336. The logical product of the $C$-partitions corresponding to each column is the same as Curtis’ $C$-partition.

3) The aim of the assignment procedure is to choose $P$-partitions so that the $Y$-functions can be realized with as few and as large subcubes as possible, or equivalently, so that the blocks of the $C$-partitions can easily be realized by the $P$-partitions.

Definition of Potentially Useful $P$-Partitions

$\lambda$: It will be potentially useful if two columns with neighboring input codes have identical $C$-partitions, because we then can realize the two columns at the same time. To find $P$-partitions with this property we construct a partition $\lambda$ (which as index has the input variables which are common to the two columns) by taking the two neighbor columns and identifying states of the same row. Any $P \geq \lambda$ will then have the property that the two corresponding $C$-partitions are identical. This corresponds to Dolotta and McCluskey’s identity in adjacent columns. If there are only two inputs, the $\lambda$-partition is identical with the partition Hartmanis introduced in [2].
I: We get another potentially useful partition if a whole column is entirely zeros or ones. These partitions we find by putting all states of a column in the same group I. Any \( P \) which does not split the group \( I \) will then have the desired property. This corresponds to Dolotta and McCluskey's all-zero or all-one entry.

C: We get a potentially useful partition if one of the \( C \)-partitions is identical with the \( P \)-partition. This can happen when the group of ones in the \( P \)-partition is identical with the group of ones in the \( C \)-partition. To find \( P \)-partitions where this is the case we construct a partition "\( C \)" by identifying every state with the state it goes to for the column in question. Any \( P \geq C \) will then have the desired property. This corresponds to Dolotta and McCluskey's "identity with base entry."

\( \bar{C} \): It would be equally desirable if the group of ones in the \( P \)-partition was identical with the group of zeros of the \( C \)-partition. To find all \( P \)'s where this is the case we must construct a grouping "\( \bar{C} \)" by placing each state and the state it goes to in opposing groups. Any \( P \) which keeps the opposing groups, in the two different blocks of the \( P \)-partition, will have the desired property. This corresponds to Dolotta and McCluskey's "identity with complement of base entry."

S: Any \( C \)-partition of a particular column is \( \geq \) the partition "\( S \)" made by identifying states which go to "the same state in the particular column." Therefore, any \( P \)-partition which should have a possibility of realizing a \( C \)-partition of the column must be \( \geq S \). This corresponds to Dolotta and McCluskey's consideration of identity with already chosen column.

Example

\[ \begin{array}{c|c|c|c}
\text{Column} & \text{States} & \text{Counts} \\
\hline
1 & a & 0 \\
2 & b & 0 \\
3 & c & 0 \\
4 & d & 0 \\
\end{array} \]

Machine A:

\[ \begin{array}{c|c|c|c|c|c}
\text{Column} & \text{States} & \text{Counts} & \text{Counts} \\
\hline
1 & a & 0 & 0 \\
2 & b & 0 & 0 \\
3 & c & 0 & 0 \\
4 & d & 0 & 0 \\
\end{array} \]

Thus we have: \( C = P \)

C) For column "1" we get \( a \rightarrow 1 \) \( b \rightarrow 1 \) \( c \rightarrow 1 \) \( d \rightarrow 0 \) \( e \rightarrow 0 \) \( f \rightarrow 0 \)

\[ \begin{array}{c|c|c|c|c|c}
\text{Column} & \text{States} & \text{Counts} & \text{Counts} \\
\hline
1 & a & 1 & 1 \\
2 & b & 0 & 0 \\
3 & c & 0 & 0 \\
4 & d & 0 & 0 \\
\end{array} \]

Therefore, \( P = \text{abc def} \) which gives \( P = \text{abc def} \) because the group \( ab \) can combine with \( d \) as well as \( ef \).

S) For column "1" we get \( S = \text{ab cd ef} \). We notice that all \( C \)'s above are \( \geq S \).

Assignment Procedure

For a given machine we now construct the \( \lambda \)-partitions, and for each column of the machine the group \( \lambda \), the grouping \( C \), and the partitions \( C \) and \( S \). These are placed in the upper half of a diagram called \( PCC \).

\[ \begin{array}{c|c|c|c|c|c}
\text{Column} & \text{States} & \text{Counts} & \text{Counts} \\
\hline
1 & a & 0 & 0 \\
2 & b & 0 & 0 \\
3 & c & 0 & 0 \\
4 & d & 0 & 0 \\
\end{array} \]

Thus we have: \( C = P \)

C) For column "1" we get \( a \rightarrow 1 \) \( b \rightarrow 1 \) \( c \rightarrow 1 \) \( d \rightarrow 0 \) \( e \rightarrow 0 \) \( f \rightarrow 0 \)

\[ \begin{array}{c|c|c|c|c|c}
\text{Column} & \text{States} & \text{Counts} & \text{Counts} \\
\hline
1 & a & 1 & 1 \\
2 & b & 0 & 0 \\
3 & c & 0 & 0 \\
4 & d & 0 & 0 \\
\end{array} \]

Therefore, \( P = \text{abc def} \) which gives \( P = \text{abc def} \) because the group \( ab \) can combine with \( d \) as well as \( ef \).
All the potentially useful P-partitions are found, as described in
the previous section, and listed in the lower half of the diagram. As an
example the partition $S_1$: ab c d e f have five partitions $\geq S_1$; that is,
$P = ab cdef, P = cd abef, P = ebc def, P = adef bc$, and $P = abcd ef$.

For each P-partition, the corresponding C-partitions are con-
structed as illustrated in the Introduction.

The ones and zeros in the middle of the diagram indicate that we
always put the group which corresponds to the ones first.

When the problem is worked by hand it is not necessary to list all
potentially useful P-partitions. We can limit ourselves to listing
the most useful P-partitions, that is, partitions which satisfy several
criteria at the same time. In our example we have not listed all the
partitions to which $S_1$ leads because this would be the set of all possible
P-partitions; thus, we might say that $S_1$ does not help us in finding
particularly useful P-partitions.

In the diagram we now underline the C-partitions which can
be easily realized, i.e., where one of the simplifications which are
described in the previous section occur. If the simplification occurs
because $C_1 = C_2$, we underline both C's with a dotted line; this counts as
only one underlining.

The procedure now is to choose P-partitions one at a time so that
at each step we secure the maximum number of easy C-partitions.

1) Choose the P-partition with the maximum number of under-
linings. If two P's have the same number of underlinings preference is
put on the $\lambda$ and I simplifications because they can be realized with
fewer diodes than the other simplifications.

2) Put a minus by all the P's that are not compatible with the
chosen P.

3) Underline

a) all C's that are identical with the chosen P

b) all C's that are identical with the chosen C in the same
column, provided this simplification does not spoil a
$\lambda$-simplification, and

c) every P with one line for each of the now chosen C's that it
realizes.

Note that each C-partition can only be underlined once.

The procedure continues until all P's are chosen. If among the
potentially useful P-partitions we cannot find all the P-partitions we
need, we choose the last P-partitions arbitrarily.

By carrying out this procedure we end up with the following diagram:

<table>
<thead>
<tr>
<th>P</th>
<th>C_0</th>
<th>C_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0</td>
<td>1 0</td>
<td>1 0</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$bc acdf$</td>
<td>$cd abef$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$af bcde$</td>
<td>$be acdf$</td>
</tr>
<tr>
<td>$y_4$</td>
<td>$abc def$</td>
<td>$def abc$</td>
</tr>
<tr>
<td>$ab cef$</td>
<td>$ade bcf$</td>
<td>$def abed$</td>
</tr>
<tr>
<td>$cd abef$</td>
<td>$af bcde$</td>
<td>$abef cd$</td>
</tr>
<tr>
<td>$ab cdef$</td>
<td>$de abef$</td>
<td>$abef cd$</td>
</tr>
<tr>
<td>$ef abed$</td>
<td>$bc adef$</td>
<td>$d abed$</td>
</tr>
</tbody>
</table>

Having thus found $y_1, y_2, y_3$, we now have the application
equations:

$$y_1 = \bar{x}_3 + x_1$$

$$y_2 = \bar{x}_3$$

$$y_3 = \bar{x}_3$$

One may notice that by comparison of the three selected y-
variables in the PCC-diagram with the $Y_1, Y_2$, and $Y_3$ tables, the
application equations may be found directly from the PCC diagram.
Whether it is advantageous to complement any of the variables can
also be seen as easily from the PCC diagram as from the Y tables.

The method can also handle “don't cares.” This is done by
omitting the states which are “don't cares” in the C-partitions of
a particular column. These “don't cares” are, in the course of the
procedure, associated either with the block of ones or the block of zeros
of each C-partition, depending on what is most convenient.

**Results**

The author has tried the method on all the machines of the
articles [1]–[6]. In nearly all cases we found a result that required
the same number of diodes or less. In the cases where we did not, it
was because by choosing the P-partitions one at a time, we did not
find a set of P-partitions which realized each other's C-partitions
and, therefore, the simplification could only be seen if the set of P-
partitions was taken as a whole. This is a difficulty which is in-
trinsically connected with choosing the P-partitions one at a time.
However, when the method is worked by hand we do not have to
stick strictly to this procedure. Curtis, who examines all possible
sets of P-partitions in his (C, P)-method, overcomes this difficulty
in principle. His aim, however, is not directly to minimize diode
count, and so he finds a utilization with 55 diodes (or 42 with optimal
choice of complementing variables and of subcubes) while this much
shorter method finds a utilization with 38 diodes. To demonstrate
our method for a bigger machine than that used above, this example
is shown in the Appendix.

The procedure is offered with incomplete evaluation of its useful-
ness. There are many unanswered questions: a) Will the procedure
be equally efficient on larger machines? b) Does the procedure have
any value for problems for which no simple assignment exists? c) How
does it compare with earlier procedures when there are many
“don't cares”? The method has not been programmed on a computer, but it is
believed to be suited for programation because it is rigorous and
consists of a large number of simple steps.

**Discussion**

A drawback of the method is that it is only concerned with maxi-
mizing the number of very easily realizable C's, while it leaves to
chance whether the remaining C's are realized with few or many
subcubes. This again is connected with the selection of the P-parti-
tions one at a time; it is of little value to see whether a C-partition
can be realized by a set of a few chosen P's, because we don't know if
later we will get a \( P \)-partition which can realize the \( C \)-partition in an even more simple way.

The use of a criterion which evaluated the ability to get subcubes with common input variables was considered at a certain point. This showed extraordinary little efficiency. The explanation must be found in the fact that the inputs are coded in advance, and we therefore have too little freedom of choice to make larger subcubes with common input variables.

**Conclusion**

The advantages of the method are that it surveys the problem directly in the \( PCC \) diagram; it is relatively short; it goes to the center of the problem, which is to obtain a small number of large subcubes; it utilizes common subcubes; it is sufficiently straightforward to be programmed on a computer; if the problem is worked by hand, the \( Y \)-functions can be written down directly from the \( PCC \) diagram; the method can handle "don't cares"; and, it is particularly efficient if a very simple solution exists.

**Appendix**

**\( PCC \) procedure on machine B in Curtis [4]**

\[ Y_1 = \overline{x_1}x_2y_1 + \overline{x_1}x_2y_1 + x_1x_2y_2 \]
\[ Y_2 = \overline{x_1}x_2y_3 + \overline{x_1}x_2y_3 + x_1x_2y_3 + x_1x_2y_3 + \overline{x_1}y_1 \]

\[ Y_3 = \overline{x_1}x_2y_4 + \overline{x_1}x_2y_4 + \overline{x_1}y_2 + x_1x_2y_1 \]

With optimal choice of complementing variables and of subcubes it gives 42 diodes

---

* We stop at once when we see that \( c \) should be in both blocks.
The procedure can arbitrarily start with $P = abc$ or $P = df$ abce.
The second choice would have led to a solution with 47 diodes, but
in such a situation one would normally try both cases.

REFERENCES

A Design for $(d, k)$ Graphs
HENRY D. FRIEDMAN, SENIOR MEMBER, IEEE

A $(d, k)$ graph is defined as a linear graph of diameter $k$ whose
nodes are each of degree $d$ or less. The degree of a node is the number
of branches incident at that node. The diameter of a graph is the
max-min distance (measured in branches) between all pairs of nodes.
Elspas [1] has been interested in finding $(d, k)$ graphs for which the
number of nodes $n(d, k)$ is a maximum. Akers [2] gave an elegant
method for constructing $(d, d-1)$ graphs of degree $d$ and diameter
d-1, which results in a graph with

$$
\binom{2d-1}{d}
$$

nodes. Akers’ graphs, though not demonstrably maximal, are equal
or superior to any graphs reported in Elspas.

In this note we construct $(d, k)$ graphs that establish a general
lower bound on $n(d, k)$ when $d$ is sufficiently large. We first construct
a simple hierarchical tree with $\lceil k/2 + 1 \rceil$ levels of nodes (using $\lceil \cdot \rceil$ to indicate "greatest integer in $\cdot$"), in which all nodes have degree $d$
except for the lowest level, where all nodes have degree $1$. (When
$d = 5$ and $k = 4$, we would have $[6/2] = 3$ hierarchical levels. (See Fig. 1.)
To construct the desired $(d, k)$ graph we take $d$ identical hierarchical
trees and use them to share, in common, their lowest-level nodes
(see Fig. 2). Thus, the full $(d, k)$ graph consists of $d$ trees joined at
these nodes without changing the order of the branches.

It is obvious that each node of the resulting graph has degree $d$.
To show that the graph has diameter $k$, we note that, within any tree,
the shortest distance between any two lowest-level nodes is at most $k$
branches. Therefore, any two given nodes of the full graph lie on a
loop of $2k$ or fewer branches; hence, they cannot be farther apart than
$k$ branches.

The number of nodes at the different levels of a hierarchical tree are
$1, d, d(d-1)^2, \ldots, d(d-1)^{k/2-1}$. By counting each level $d$ times, except for the lowest level, which is counted
only once, we easily find that the number of nodes in the full $(d, k)$
graph is $(d > 2)$

$$
N = \frac{2d}{d-2} (d-1)^{\lceil k/2 \rceil}. 
$$

This expression is not particularly useful when $d$ is small (compare
with Table I in Elspas [1]). However, when $d$ is sufficiently
large, $N$ can be used as a general lower bound on $n(d, k)$. It is inter-
esting, also, to compare the expression for $N$ with the Moore graph
upper-bound expression [1]

$$
n_M(d, k) = \frac{d(d-1)^h-2}{d-2}. 
$$

When $k = d - 1$, comparing $N$ with Akers’ number

$$
\left( \frac{2d-1}{d} \right)
$$

Fig. 1. Hierarchic tree for $d=5, k=4$.

Fig. 2. Construction of $(5, 4)$ graph. Each plane contains a tree.