Diffraction by a plane angular sector, a new derivation

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Thus, other than the null series result, the current in the shadow region does not have an asymptotic series representation. This is an important result because it is the first time (to the author's knowledge) that the failure of an L-K series in the shadow region has been both demonstrated and explained.

While the L-K series does not lead to an exact result for the surface current density, it still holds the potential for providing a tractable improvement to a pure geometrical optics solution. This fact is demonstrated by Ansorge’s [5] calculations for scattering by a dielectric sphere using the ray optics field approach. The attractiveness of the current approach as developed here is due in large part to the fact that the complete L-K representation can be developed entirely from an integral of known functions, i.e.,

\[ \int \nabla G(\Delta r) \exp(\pm jk_0 \Delta r) d^2 \sigma. \]

While a complete L-K series development for this integral is prohibitive, it may be possible to obtain the terms up to and including \( k_0^2 \) [7]. One of the primary advantages of obtaining the \( k_0^2 \) and \( k_0^3 \) corrections to the \( k_0^2 \) asymptotic expansion of this integral is the recovery of some of the cross-polarizing properties of a rough surface in the high frequency (but not optical) limit.

The results obtained in this communication raise an interesting question. It should be noted that \( \nabla G(\Delta r) \) has essentially been computed by Chaloupka and Meckelburg [8] in a very clever application of integration by parts. Their results should provide the basic ingredients to study the question raised in the previous paragraph.

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Diffraction by a Plane Angular Sector, A New Derivation

THORKILD B. HANSEN

Abstract—The purpose of the present communication is to present an alternative derivation of the exact solution to the scattering problem in which a Hertz dipole illuminates a perfectly conducting plane angular sector.

I. Introduction

The geometrical theory of diffraction (GTD) is one of the most widely used methods for calculating scattered electromagnetic fields. In this theory, the scattered field is approximated by a series in which each term is a contribution from a certain point on the scatterer. These contributions to the scattered field are called critical points by Van Kampen [1]. To calculate such contributions, diffraction coefficients belonging to each critical point are needed. Diffraction coefficients are found from exact solutions to so-called canonical scattering problems. As an example, the edge diffraction coefficient for the perfectly conducting edge is found from the exact solution to a canonical scattering problem that contains a corner.

The total electric field is written as a series expansion of vector wave functions. The expansion coefficients are found by integration over a surface which is a sphere, with center at the tip of the plane angular sector, although a corner diffraction coefficient has not been found from it. However, progress has recently been made by Smyshlyaev [2], [3] in deriving a corner diffraction coefficient from this solution.

A plane angular sector is a section of a plane bounded by two half-lines which have common endpoints. As an example, the plane angular sector is a quarter-plane if the half-lines are orthogonal to each other. The vector solution to this scattering problem was first found by Satterwhite [4]. This solution has been used for numerical calculations in several papers e.g. [5]–[7]. Corrections to some misprints in the vector wave function \( \tilde{N} \) appearing in these references are given in Section III. Because Satterwhite’s solution is the only one existing for a scattering problem where the scatterer contains a corner, it would be valuable to verify it. The purpose of this communication is to present a new derivation of Satterwhite’s solution. No attempt is made to derive a corner diffraction coefficient from it.

The method used by Satterwhite [4] can be described as follows. The total electric field is written as a series expansion of vector wave functions. The expansion coefficients are found by integration over a surface which is a sphere, with center at the tip of the plane angular sector, except for an infinitesimal cut around the plane angular sector. During integration, use is made of Green’s second identity for vectors, the divergence theorem, and orthogonal-
The plane angular sector shown in Fig. 1 is given by the equation

$$\theta = \pi$$

if the ellipticity parameter $k$ associated with the coordinate system is equal to $\cos \beta$. For a detailed discussion of the spheroidal coordinate system, see [9] or [11]. By inserting $\Psi = R(r)\Theta(\theta)\Phi(\phi)$ in the scalar Helmholtz equation

$$\nabla^2 \Psi + \alpha^2 \Psi = 0, \quad \alpha \in \mathbb{R}^+$$

performing the separation with two separation constants denoted by $\mu$ and $\nu$ ($\nu + 1$), one finds that the radial functions $R(r)$ are spherical Bessel functions and that the angular functions $\Theta(\theta)$ and $\Phi(\phi)$ are Lamé functions. The following notation for the scalar wave functions $\Psi(\bar{r}, \alpha)$ is useful.

$$\Psi_{\alpha m}(\bar{r}, \alpha), \quad A \in \{I, II\}, \quad s \in \{e, o\}, \quad m \in \mathbb{N} \text{ and } i \in \{1, 2\}$$

where $A = I$ when $R(\bar{r})$ is a spherical Bessel function of the first kind and $A = II$ when $R(\bar{r})$ is a spherical Hankel function of the second kind. Index $s = e$ when $\Theta(\theta)$ and $\Phi(\phi)$ are even, and $s = o$ when $\Theta(\theta)$ and $\Phi(\phi)$ are odd. Index $i$ is equal to one or two when $\Psi$ satisfies the Dirichlet or Neumann condition on the plane angular sector, respectively. The eigenvalue pairs $(\mu, \nu)$ belonging to each of the values of $(i, s)$ are numbered, and $(\mu_{\alpha m}, \nu_{\alpha m})$ denotes the $m$th of these. The corresponding subscript $m$ on $\Psi_{\alpha m}$ implies that $\Psi_{\alpha m}$ is the eigenfunction belonging to this eigenvalue pair.

Vector wave functions satisfying the vector Helmholtz equation are constructed in the following way:

$$\overline{M}_{\alpha m}^A(\bar{r}, \alpha) = \nabla \times \left( \nabla \times \overline{G}(\bar{r}, \phi) \right)$$

and

$$\overline{N}_{\alpha m}^A(\bar{r}, \alpha) = \frac{1}{\alpha} \nabla \times \nabla \times \left( \nabla \times \overline{G}(\bar{r}, \phi) \right).$$

The correct form of $\overline{N}_{\alpha m}^A(\bar{r}, \alpha)$ is

$$\overline{N}_{\alpha m}^A(\bar{r}, \alpha) = \frac{d}{dr} \left[ \sqrt{\frac{1}{r^2 \sin^2 \theta} + k^2 \sin^2 \phi} \right] \left[ \sqrt{1 - k^2 \cos^2 \phi} \Theta_{\alpha m}(\theta) \Phi_{\alpha m}(\phi) \right] \frac{d}{dr} \left[ \sqrt{\frac{1}{r^2 \sin^2 \theta} + k^2 \sin^2 \phi} \right]$$

Writing down the explicit expressions for $\overline{M}_{\alpha m}^A(\bar{r}, \alpha)$ and $\overline{N}_{\alpha m}^A(\bar{r}, \alpha)$ one finds that only $\overline{M}_{\alpha m}^I(\bar{r}, \alpha)$ and $\overline{N}_{\alpha m}^I(\bar{r}, \alpha)$ have zero tangential component on the plane angular sector. We can therefore write the electric dyadic Green's function as

$$\overline{G}(\bar{r}, \phi) = \int_0^\infty \left[ \sum_{\alpha m} \overline{M}_{\alpha m}^I(\bar{r}, \alpha) \overline{A}_{\alpha m}(\bar{r}, \alpha) + \overline{N}_{\alpha m}^I(\bar{r}, \alpha) \overline{B}_{\alpha m}(\bar{r}, \alpha) \right] d \alpha.$$

By imposing the finite-energy corner condition one finds that $A = I$. By inserting (4) into (2) and using that the vector wave functions satisfy the vector Helmholtz equation one obtains

$$\int_0^\infty \left[ (\alpha^2 - k^2) \sum_{\alpha m} \overline{M}_{\alpha m}^I(\bar{r}, \alpha) \overline{A}_{\alpha m}(\bar{r}, \alpha) + \overline{N}_{\alpha m}^I(\bar{r}, \alpha) \overline{B}_{\alpha m}(\bar{r}, \alpha) \right] d \alpha = -\hat{I}(\bar{r}, \phi).$$

Using the orthogonality relations (9)-(11) the following expressions
for the expansion coefficients \( \tilde{A}_{sm} \) and \( \tilde{B}_{sm} \) are obtained

\[
\tilde{A}_{sm}(\rho, \alpha) = \frac{2}{\pi \alpha^2} \frac{1}{R_{sm1}^2(\rho, \alpha)} \quad (6)
\]

and

\[
\tilde{B}_{sm}(\rho, \alpha) = \frac{2}{\pi \alpha^2} \frac{1}{R_{sm1}^2(\rho, \alpha)} \quad (7)
\]

Insertion of (6) and (7) into (4) and integration with respect to \( \alpha \), [9] gives our final expressions for the electric dyadic Green's function:

\[
\tilde{G}(\rho, \alpha) = J_0 \sum_{s,m} \left[ M_{sm1}^H(\rho, \alpha) \right] \frac{1}{R_{sm1}^2(\rho, \alpha)} + N_{sm1}^H(\rho, \alpha) \frac{1}{R_{sm1}^2(\rho, \alpha)}, \quad \text{when } |\rho| > |\rho'|
\]

and the same expression with \( \rho \) and \( \rho' \) interchanged when \( |\rho| < |\rho'| \).

This solution, which agrees with that in [4], is shown in [4] and [9] to satisfy the radiation and the edge condition. Explicit expressions for \( M_{sm1} \) and \( N_{sm1} \) may be found in [4] and [9].

Note that when fields are to be calculated within the source region an additional term must be added to the expression in (8) [12], [13].

In conclusion, the solution obtained by Satterwhite [4] is critically confirmed by using the Ohm–Rayleigh method of solution to determine the electric dyadic Green's function for the perfectly conducting plane angular sector.

**APPENDIX**

**ORTHOGONALITY RELATIONS**

The formulas shown in this Appendix are proved in [9] and, to the author's knowledge, have not previously been published.

\[
\int_{R^3} \tilde{M}_{sm1}^H(\rho, \alpha) \cdot \tilde{N}_{sm1}^H(\rho', \alpha') \, dV = 0 \quad (9)
\]

\[
\int_{R^3} \tilde{N}_{sm1}^H(\rho, \alpha) \cdot \tilde{M}_{sm1}^H(\rho', \alpha') \, dV = 0
\]

\[
- \Lambda_{sm2} \frac{\pi}{2} \delta(\alpha - \alpha') \frac{\delta_{sm}}{\alpha^2}
\]

\[
\int_{R^3} \tilde{N}_{sm1}^H(\rho, \alpha) \cdot \tilde{N}_{sm1}^H(\rho', \alpha') \, dV
\]

\[
= \Lambda_{sm1} \frac{\pi}{2} \delta(\alpha - \alpha') \frac{\delta_{sm}}{\alpha^2}
\]

where \( \Lambda_{sm1} \) are normalization constants and \( \delta_{pq} = 0 \) when \( p \neq q \) and \( \delta_{pp} = 1 \).

**PROOF OF (10)**

The integral in (10) can be written as

\[
\int_0^\infty j_{sm1}(\alpha r) j_{sm1}(\alpha' r') d\rho \int_0^\pi m_{sm1}(\theta, \phi) \cdot \tilde{M}_{sm1}^H(\theta, \phi) \cdot \tilde{N}_{sm1}^H(\theta, \phi) \, d\theta d\phi
\]

where \( j_0(z) \) is the spherical Bessel function and \( m_{sm1}(\theta, \phi) \) is the angular dependent part of the vector wave function \( \tilde{M}_{sm1}(\rho, \alpha) \).

Furthermore, \( h_1 h_2 h_3 = r^2 \phi(\theta, \phi) \) where \( h_1, h_2, \) and \( h_3 \) are the metric coefficients in the spherocylindrical coordinate system [11]. Note that the double integral is independent of \( \alpha, \alpha' \), and \( r \). In order to evaluate the double integral in (12) we note that

\[
\mathbf{v} \cdot (\tilde{N}_{sm1}(\rho, \alpha) \mathbf{v} \tilde{M}_{sm1}(\rho, \alpha)) = - \frac{\alpha}{\alpha'} \tilde{N}_{sm1}(\rho, \alpha) \mathbf{v} \tilde{M}_{sm1}(\rho, \alpha)
\]

By integrating this identity over a spherical volume with center at \((0, 0, 0)\) using the divergence theorem and the orthogonality relations for Lamé functions [11, appendix II], one finds that the double integral in (12) is zero when \( s \neq s' \) or \( m \neq m' \). By using [8, p. 11, eq. (26)] and denoting the double integral in (12) by \( \Lambda_{sm2} \) when \( s = s' \) and \( m = m' \), the formula (10) is proved.

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**The Electromagnetic Field of a Slotted Elliptic Cone**

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**Abstract—** The exact solution for the electromagnetic field diffracted by a perfectly conducting slotted elliptic cone is determined. The solution is presented in the form of a dyadic Green's function, which is the most general form of a possible solution. Some numerical results are

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