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Formalizing Implicational Axiomatics for Classical First-Order Logic with Functions in Isabelle/HOL

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Abstract. We present a concise formalization in the proof assistant Isabelle/HOL of classical first-order logic with functions. Our standalone formalization is based on entries in the Isabelle Archive of Formal Proofs and has recently been used for teaching logic and automated reasoning. The soundness and completeness theorems hold for languages of arbitrary cardinalities. The axioms and rules start from classical implicational logic and add falsity and universal quantification. Relatively simple functional programs are provided for substitution and other operations. We consider the axiomatics a stepping stone for learning about higher-order logic and structured natural deduction proofs in Isabelle/HOL.

1 Introduction

We have several formalizations in the proof assistant Isabelle/HOL [11] of first-order logic [2,4]. However, in Isabelle or other proof assistants, the only axiomatic system for first-order logic [1] is by From [5]. In that formalization the soundness theorem holds for languages of arbitrary cardinalities but the completeness theorem holds for countable languages only. Following newer work by From [6] we here present soundness and completeness theorems for languages of arbitrary cardinalities, with the requirement that the type of functions have cardinality at least that of the type of predicates and is infinite.

Our standalone formalization has recently been used for teaching logic and automated reasoning [9, 10, 12]:

`https://hol.compute.dtu.dk/Quantify/`

The Isabelle file consists of just 1750 lines and only imports the Main theory. Relatively simple functional programs are provided for operations used in the inductive definition of the axiomatic system. The key operations are substitution and freshness of constants (seen as functions with no arguments). We consider the axiomatics a stepping stone for learning about higher-order logic and structured natural deduction proofs in Isabelle. We have explained the details and experiences with our approach elsewhere [10].

2 Implicational Axiomatics

In the implicational axiomatics, the axioms and rules start from classical implicational logic and then falsity and first-order universal quantification are added. We use the so-called Bernays-Tarski axiom system for classical implicational logic [7,8,13], except that instead of Peirce's Law as an axiom we formulate it as Peirce's rule (PR). In implicational logic, using the modus ponens rule, Peirce's Law as an axiom immediately gives us PR but it is not obvious how to obtain Peirce's Law from PR. In the formalization we prove it using Isabelle's powerful automation, using some lemmas to obtain the result in a fraction of a second. So our starting point is the axioms by Wajsberg 1937 [3]:

$$p \to (q \to p)$$
$$((p \to q) \to p) \to p$$
$$(p \to q) \to ((q \to r) \to (p \to r))$$
$$\bot \to p$$

In the work by From [5] there are no axioms for propositional logic. Instead all tautologies with respect to the first-order language are added. However, in the formalization it suffices with the following axioms by Wajsberg 1939 [3]:

$$p \to (q \to p)$$
$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$
$$((p \to \bot) \to \bot) \to p$$

We include a screenshot from our formalization showing all 4 axioms and 3 rules. Substitution is $\langle \ldots \rangle$ where $\star$ turns a function (name) into a constant (term). The function `new` tests for freshness (relative to a formula). The double arrow $\Longrightarrow$ builds up the inductive definition (rules and their side conditions).

```
AK: ⊢ p ⟶ q ⟶ p
AT: ⊢ (q ⟶ r) ⟶ (r ⟶ p) ⟶ q ⟶ p
AX: ⊢ ⊥ ⟶ p
AY: ⊢ ∀p ⟶ ⟨t⟩p
MP: ⊢ q ⟶ p ⟹ ⊢ q ⟹ ⊢ p
GR: ⊢ q ⟶ ⟨⋆a⟩p ⟹ new a p ⟹ new a q ⟹ ⊢ q ⟶ ∀p
PR: ⊢ (p ⟶ q) ⟶ p ⟹ ⊢ p
```

Note that we introduce the metavariables p, q, ... from right to left in axioms and rules in order to emphasize the goal-directed view using rules bottom-up.

3 Conclusions

We have formalized soundness and completeness for a succinct axiom system for classical first-order logic with functions. We import only the Main theory in Isabelle/HOL and handle languages of arbitrary cardinalities. At the end of our formalization we also instantiate the theorems for particular cardinalities, e.g. natural numbers *nat* (for functions and predicates, respectively, in formulas *fm*):

$$\textit{proposition} \quad \vdash p \longleftrightarrow \models p \quad \textit{for } p :: (\textit{nat}, \textit{nat})\ \textit{fm}$$

But we can replace *nat* with *nat set*, *nat set set*, *nat set set*, etc.

References

1. Mordechai Ben-Ari. *Mathematical Logic for Computer Science*. Springer, 2012. doi:10.1007/978-1-4471-4129-7.
2. Jasmin Christian Blanchette. Formalizing the metatheory of logical calculi and automatic provers in Isabelle/HOL (invited talk). In Assia Mahboubi and Magnus O. Myreen, editors, *Proceedings of the 8th ACM SIGPLAN International Conference on Certified Programs and Proofs, CPP 2019*, pages 1–13. ACM, 2019.
3. Alonzo Church. *Introduction to Mathematical Logic*. Princeton University Press, 1956.
4. Asta Halkjær From, Anders Schlichtkrull, and Jørgen Villadsen. A sequent calculus for first-order logic formalized in Isabelle/HOL. *Journal of Logic and Computation*, 33(4):818–836, 2023.
5. Asta Halkjær From. Soundness and completeness of an axiomatic system for first-order logic. *Archive of Formal Proofs*, September 2021. https://isa-afp.org/entries/FOL_Axiomatic.html, Formal proof development.
6. Asta Halkjær From. Synthetic completeness. *Archive of Formal Proofs*, January 2023. https://isa-afp.org/entries/Synthetic_Completeness.html, Formal proof development.
7. Asta Halkjær From, Agnes Moesgård Eschen, and Jørgen Villadsen. Formalizing axiomatic systems for propositional logic in Isabelle/HOL. In Fairouz Kamareddine and Claudio Sacerdoti Coen, editors, *Intelligent Computer Mathematics - 14th International Conference, CICM 2021, Timisoara, Romania, July 26-31, 2021, Proceedings*, volume 12833 of *Lecture Notes in Artificial Intelligence*, pages 32–46. Springer, 2021.
8. Asta Halkjær From and Jørgen Villadsen. Soundness and completeness of implicational logic. *Archive of Formal Proofs*, September 2022. https://isa-afp.org/entries/Implicational_Logic.html, Formal proof development.
9. Frederik Krogsdal Jacobsen and Jørgen Villadsen. On exams with the Isabelle proof assistant. In Pedro Quaresma, João Marcos, and Walther Neuper, editors, *Proceedings 11th International Workshop on Theorem Proving Components for Educational Software, ThEdu@FLoC 2022, Haifa, Israel, 11 August 2022*, volume 375 of *EPTCS*, pages 63–76. Open Publishing Association, 2022. doi:10.4204/EPTCS.375.6.
10. Simon Tobias Lund and Jørgen Villadsen. Teaching higher-order logic using Isabelle. In Julien Narboux, Walther Neuper, and Pedro Quaresma, editors, *Proceedings 12th International Workshop on Theorem proving components for Educational software, Rome, Italy, 5th July 2023*, volume 400 of *Electronic Proceedings in Theoretical Computer Science*, pages 59–78. Open Publishing Association, 2024. doi:10.4204/EPTCS.400.5.
11. Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL - A Proof Assistant for Higher-Order Logic*, volume 2283 of *Lecture Notes in Computer Science*. Springer, 2002.
12. Jørgen Villadsen and Frederik Krogsdal Jacobsen. Using Isabelle in two courses on logic and automated reasoning. In João F. Ferreira, Alexandra Mendes, and Claudio Menghi, editors, *Formal Methods Teaching*, pages 117–132. Springer, 2021. doi:10.1007/978-3-030-91550-6_9.
13. Jan Łukasiewicz. The shortest axiom of the implicational calculus of propositions. *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, 52:25–33, 1948.